Basic functors of differential calculus over commutative algebras

Alexandre Vinogradov, Michael Vinogradov

A program of the course at the 4st Italian Diffiety School, Forino, July 17 — 29, 2000

1. The Diff-prolongations of differntial operators. The functor transformations
$$c_{s,l}\colon\ \mathrm{Diff}_s^+ \mathrm{Diff}_l^+ \to \mathrm{Diff}_{s+l}^+ .$$
2. The functor equivalence $\mathrm{Diff}_1 \simeq D \oplus \mathrm{id}$. The functor transformations
$$\kappa_1\colon\ \mathrm{Diff}_1^{(+)} \to D.$$
3. The kernel of the universal differential operator
$$\mathrm{D}_2^+\colon\ \mathrm{Diff}_2^+ P \to P,$$
and functors D_2 and $P_2^{(+)}$. The Spencer Diff-sequence of the second order.
4. The construction and properties of K-modules $D(P \subset Q)$ and $\mathrm{Diff}_1^{(+)}(P \subset Q)$.
5. Functors D_i and $P_i^{(+)}$. The functor equivalence $P_i \simeq D_i \oplus D_{i-1}$. The functor transformations
$$\kappa_i\colon\ P_i^{(+)} \to D_i.$$
6. The Spencer Diff-complexes.
7. The construction and properties of A-modules $P \otimes_{in} Q$ and $\mathrm{Hom}_A^{\bullet}(P, Q)$.
8. A-modules Λ^i's as representative objects of functors D_i.
9. Algebraic de Rham complexes. The exterior product, the Lie derivative and their properties.
10. The Spencer $\mathcal{J}$-complexes of an algebra A as as representative objects of the Spencer Diff-complexes.
11. The Spencer $\mathcal{J}$-complexes of A-modules and their properties.

Examination problems

To pass the exam by email one should solve 5 problems.

1. Let $\Phi_1,\ \Phi_2$ be functors on the category of all A-modules, $\varphi_1,\ \varphi_2$ be corresponding representative objects, that is
$$\Phi_1(P) = \mathrm{Hom}_A(\varphi_1, P), \quad \Phi_2(P) = \mathrm{Hom}_A(\varphi_2, P)$$

for any A-module P, and $F\colon \Phi_1 \to \Phi_2$ be a natural transformation. Set $P = \varphi_2$, and define the homomorphism $F^*\colon \varphi_2 \to \varphi_1$ by the formula

$$F^* = F(\mathrm{id}),$$

$\mathrm{id} \in \mathrm{Hom}_A(\varphi_1, \varphi_1) = \Phi_1(\varphi_1)$, $\quad F(\mathrm{id}) \in \mathrm{Hom}_A(\varphi_2, \varphi_1) = \Phi_2(\varphi_1)$.

Prove that for any A-module Q and any element $h \in \Phi_1(Q) = \mathrm{Hom}_A(\varphi_1, Q)$

$$F(h) = h \circ F^*$$

2. Prove that $\mathrm{Hom}_A(P^+ \otimes_{in} S, Q) = \mathrm{Hom}^{\bullet}_A(S, \mathrm{Hom}^+_A(P, Q))$.
3. Prove that $\mathrm{Diff}^+_s P = \mathrm{Hom}^+_A(\mathcal{J}^s, P)$.
4. Prove that

$$\mathcal{J}^s_+(P) = \mathcal{J}^s_+ \otimes P, \quad J^s(P) = \mathcal{J}^s \otimes_{in} P.$$

5. Using the algebraic definition of Lie derivation deduce all usual formulas for it.