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On the Geometry of Singular Lagrangians

F. Pugliese, A. M. Vinogradov

Abstract. The geometry of Lagrangian systems, whose Legendre map possesses generic singularities, is studied. On its basis the Transition Principle, prescribing the behaviour of phase trajectories on the singular hypersurface, is proposed. As a by-product, the notion of relative Hamiltonian field associated with an arbitrary Lagrangian is introduced.

1. Introduction

The Legendre map $\mathcal{L}$ of a *generic* Lagrangian is, according to a well-known fact of Singularity Theory, just a *local* diffeomorphism except for a singular hypersurface S along which it inevitably degenerates. We call *singular* the Lagrangians of this kind to distinguish them from those whose Legendre maps are everywhere degenerate. The latter are the *constrained* Lagrangians and were widely studied (the Dirac-Bergmann theory, [2], [9]) due to their fundamental importance for gauge theories, etc. On the contrary, singular lagrangians were studied just in a few works (see, for instance, [7]), in spite of the fact that they appear not infrequently in the literature, for instance, in various post-galilean models (see [3], [1], [12]). However, the principal question of the *singular lagrangian* dynamics has not yet, to our knowledge, been answered. Namely, what happens to a phase trajectory when it reaches the singular hypersurface S. It seems that this question cannot be resolved on the basis of the standard variational approach because for this purpose one must prescribe *ad hoc* from the very beginning the class of admissible curves-trajectories. As a rule Nature rejects such a human dictate. In this paper we study the natural geometry of singular Lagrangian systems, which suggests the Transition Principle (see subsection 4.2), a prescription for what a phase trajectory must do when reaching S.

More exactly, we formulate this principle only for *generic* singular Lagrangians. In such a case, the Legendre map has at a *generic* point of S a stable singularity of the fold type. The Transition Principle forces a phase point reaching S at a generic point to make a jump to another precisely prescribed point of S, in order to continue a "normal" smooth motion up to a possible subsequent jump, etc. In such a jump the system changes not only its momenta, as it occurs in various

collisions, reflection and refraction processes, but also its space-time coordinates. Such a behaviour, it seems, contradicts the common sense, but the naturality of the underlying geometry strongly suggests testing it experimentally. As we show below, the hamiltonian version of the Transition Principle is perfectly confirmed in this sense. In paper [8] we studied in detail the dynamics of a post-galilean harmonic oscillator based on the Transition Principle. The results obtained there show the phase portrait of this system to be intrinsically self-consistent. We also note that independently of the possible physical relevance the dynamical systems constructed on the basis of the Transition Principle generalize naturally billiard systems and as such renew the park of mathematical toys to enjoy.

By studying the geometry of singular lagrangians we observe a simple but, it seems, very important fact: any Lagrangian, however degenerate it is, admits the corresponding Hamiltonian field, which is a *relative* one along the Legendre map. At least to our knowledge, this fact was overlooked previously. We demonstrate its utility by describing by its help the constraint algorithm for constrained lagrangians in few lines.

Finally, it is worth stressing that the analogue of the presented formalism can be developed in field theory as well, which we hope to do in a separate paper.

2. Relative Hamiltonian vector field

2.1. **Relative vector fields.** The notion of relative vector field is a key one for our purposes. Let us recall it briefly. Let M and N be two differentiable manifolds and let $F : M \to N$ be a smooth map. A *relative vector field on N along F* is by definition a derivation of the algebra $C^\infty(N)$ with values in the algebra $C^\infty(M)$, considered as a $C^\infty(N)$-module with respect to the multiplication

$$\phi f \stackrel{def}{=} F^*(\phi) f \qquad\qquad f \in C^\infty(M), \phi \in C^\infty(N)$$

In other words, an $\mathbf{R}$-linear operator

$$X : C^\infty(N) \to C^\infty(M)$$

is a relative vector field along F if it satisfies the Leibniz rule:

$$X(fg) = F^*(f) X(g) + F^*(g) X(f) \qquad\qquad f, g \in C^\infty(N)$$

If $f \in C^\infty(M)$ and X is a relative vector field along F, then fX is also a relative vector field. Therefore, relative vector fields along F form a $C^\infty(M)$-module denoted by $\mathcal{D}(N, M; F)$. Since $C^\infty(M)$ is a $C^\infty(N)$-module, $\mathcal{D}(N, M; F)$ can be considered as a $C^\infty(N)$-module: $(X, g) \mapsto F^*(g) X$, $g \in C^\infty(N)$.

Example 1. $\mathcal{D}(M, M; id_M)$ coincides with the $C^\infty(M)$-module $\mathcal{D}(M)$ of vector fields on M.

Example 2. Let $M \subset N$ be a submanifold, then $\mathcal{D}(M, N; \sigma)$, σ being the inclusion map, is the $C^\infty(M)$-module of smooth fields of vectors tangent to N at points of M.

Example 3. If $X \in \mathcal{D}(M)$ and $F : M \to N$ is a smooth map, then $X \circ F^* \in \mathcal{D}(N, M; F)$.

Example 4. If $Y \in D(N)$, then $F^* \circ Y \in \mathcal{D}(N, M; F)$.

As in the case of usual vector fields, one can associate with each relative vector field $X \in \mathcal{D}(N, M; F)$ the section

$$x \in M \longmapsto X_x \in T_{F(x)}(N)$$

of the pullback $F^*(\pi)$ of the tangent bundle $\pi : T(N) \to N$ given by

$$X_x(g) \stackrel{def}{=} [X(g)](x), \quad \forall g \in C^\infty(N)$$

The same formula, read from the right to the left, associates with each section of $F^*(\pi)$ an element of $\mathcal{D}(N, M; F)$. Thus $\mathcal{D}(N, M; F)$ may be identified with the $C^\infty(M)$-module $\Gamma(F^*(\pi))$ of smooth sections of $F^*(\pi)$. This identification shows that a relative vector field X can be expressed in terms of local coordinates as follows. Let $(x_1, ..., x_m)$ and $(y_1, ..., y_n)$ be local coordinates on M and N, respectively. Then

$$X = \sum_{i=1}^{n} X^i(x) \left.\frac{\partial}{\partial y_i}\right|_{F(x)}$$

or, equivalently

$$X = \sum_{i=1}^{n} X^i(x) \left(F^* \circ \frac{\partial}{\partial y_i}\right)$$

with X^i's being smooth functions on M.

Let $F_t : M \to N, F_0 = F$, be a deformation of F. Then the operator

$$\left.\frac{dF_t^*}{dt}\right|_{t=0} : C^\infty(N) \to C^\infty(M)$$

is, as it is easy to see, a relative (along F) vector field on N. Conversely, any relative vector field can be represented in this form. By this reason, relative vector fields along F are interpreted naturally as infinitesimal deformations of F.

A number of natural operations with differential forms, general covariant tensor fields, etc., involving usual vector fields can be extended to relative vector fields. For instance, if $X \in \mathcal{D}(N, M; F)$ the $C^\infty(N)$-homomorphism

$$i_X : \Lambda^k(N) \to \Lambda^{k-1}(M)$$

between the module of k-forms on N and that of $(k-1)$-forms on M is defined by

$$(i_X(\omega))(X_1, ..., X_{p-1})(x) \stackrel{def}{=} \omega_{F(x)}(X_x, d_xF(X_1|_x), ..., d_xF(X_{p-1}|_x))$$

with $X_i \in \mathcal{D}(M)$, $x \in M$. As in the usual case, we will sometimes write $X \lrcorner \omega$ instead of $i_X(\omega)$.

If $X = Y \circ F^*$, with $Y \in \mathcal{D}(M)$, then, obviously,

$$X \lrcorner \omega = Y \lrcorner F^*(\omega)$$

Similarly, if $X = F^* \circ Z$, with $Z \in \mathcal{D}(N)$, then

$$X \lrcorner \omega = F^*(Z \lrcorner \omega)$$

Evidently, it holds

$$i_X(\omega \wedge \rho) = i_X(\omega) \wedge \rho + (-1)^{\deg \omega} \omega \wedge i_X(\rho) \tag{1}$$

Now the Lie derivative of $\omega \in \Lambda^k(N)$ along X is defined as

$$L_X(\omega) = X(\omega) \stackrel{def}{=} X \lrcorner d\omega + d(X \lrcorner \omega) \in \Lambda^k(M)$$

The Leibniz rule for L_X

$$L_X(\omega \wedge \rho) = L_X(\omega) \wedge \rho + \omega \wedge L_X(\rho)$$

results easily from (1).

In fact, the Lie derivative along a relative field X can be defined for any natural covariant field, for instance for covariant tensors. In the last case the Leibniz rule $L_X(\vartheta \otimes \tau) = L_X(\vartheta) \otimes \tau + \vartheta \otimes L_X(\tau)$ gives the inductive definition of the Lie derivative for general covariant tensor fields.

2.2. **Relative Hamiltonian vector field.** In this section is introduced a relative vector field along the Legendre map , which generalizes the notion of Hamiltonian vector field and is a very useful tool for studying degenerate Lagrangians..

Let M be the configuration space of a dynamical system, described by the Lagrangian $L \in C^\infty(T(M))$. Consider the Legendre map associated with L

$$\mathcal{L} : T(M) \to T^*(M) \quad ,$$

defined by

$$[\mathcal{L}(\xi)](\eta) \stackrel{def}{=} (d_\xi L)(\alpha_\xi(\eta)), \qquad \xi, \eta \in T_q(M), \quad q \in M \ ,$$

where $\alpha_\xi : T_q(M) \to T_\xi(T_q(M))$ is the canonical isomorphism between the vector space $T_q(M)$ and the space tangent to it at point ξ. $\mathcal{L}$ is locally described by equations

$$\begin{aligned} q_i &= q_i \\ p_i &= L_{v_i}(q, v) \end{aligned}$$

where $(q_1, ..., q_n)$ is a local chart of M and (q, v), (q, p) are the corresponding charts on $T(M)$ and $T^*(M)$, respectively. The differential of

$\mathcal{L}$ is expressed with respect to such coordinates by the matrix

$$(d\mathcal{L})_{(q,v)} = \left\| \begin{array}{cc} \mathbf{1} & \mathbf{0} \\ L_{vq} & L_{vv} \end{array} \right\| \quad , \tag{2}$$

where $\mathbf{1}, \mathbf{0}$ are the $n \times n$ identity and null matrices, respectively, and $L_{vq} = \left(L_{v_i q_j}\right)_{i,j=1,...n}$, $L_{vv} = \left(L_{v_i v_j}\right)_{i,j=1,...n}$.

Associate with L the following relative vector field along $\mathcal{L}$:

$$X_L \overset{def}{=} \sum_i v_i \left.\frac{\partial}{\partial q_i}\right|_{\mathcal{L}(q,v)} + \sum_i L_{q_i}(q,v) \left.\frac{\partial}{\partial p_i}\right|_{\mathcal{L}(q,v)} \tag{3}$$

This field is called the *relative Hamiltonian field* associated with L by the reasons below.

Proposition 2.1. . *X_L does not depend on the choice of local coordinates on M and N. Hence, the correspondence $L \mapsto X_L$ is a natural differential operator.*

◀ Namely, one must check that the components (v, L_q) of X_L in the basis $\left(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right)$ of $T_{\mathcal{L}(q,v)}\left(T^*(M)\right)$ change according to the transformation rules induced by a change of coordinates $q \mapsto q' = q'(q)$. It is easy to see that if

$$\xi = \sum_i a_i \frac{\partial}{\partial q_i} + \sum_i b_i \frac{\partial}{\partial p_i} = \sum_i a_i' \frac{\partial}{\partial q_i'} + \sum_i b_i' \frac{\partial}{\partial p_i'}$$

is a vector tangent to $T^*(M)$, then

$$\begin{aligned} a_j' &= \sum_i \frac{\partial q_j'}{\partial q_i} a_i \\ b_j' &= \sum_i \left[\sum_k \frac{\partial}{\partial q_i}\left(\frac{\partial q_k}{\partial q_j'}\right) p_k\right] a_i + \sum_i \frac{\partial q_i}{\partial q_j'} b_i \quad , \end{aligned} \tag{4}$$

and

$$v_j' = \sum_i \frac{\partial q_j'}{\partial q_i} v_i \quad , \tag{5}$$

for $j = 1, ..., n$. So, one has to check that (4) holds for $a_i = v_i$, $b_i = L_{q_i}(q,v), a_i' = v_i'$, $b_i' = L_{q_i'}\left(q', v'\right)$, $(q,p) = \mathcal{L}(q,v)$. Equations $(4)_1$ coincide with (5). As for $(4)_2$ one has:

$$\begin{aligned} L_{q_j'}\left(q', v'\right) &= \frac{\partial}{\partial q_j'}\left(L\left(q\left(q'\right), v\left(q', v'\right)\right)\right) = \\ &= \sum_i L_{q_i} \frac{\partial q_i}{\partial q_j'} + \sum_k L_{v_k} \frac{\partial v_k}{\partial q_j'} = \\ &= \sum_i L_{q_i} \frac{\partial q_i}{\partial q_j'} + \sum_k p_k \frac{\partial v_k}{\partial q_j'} \end{aligned} \tag{6}$$

But in view of (5)

$$\frac{\partial v_k}{\partial q_j^{'}} = \sum_i \frac{\partial^2 q_k}{\partial q_i^{'} \partial q_j^{'}} v_j^{'} = \sum_i \frac{\partial^2 q_k}{\partial q_i^{'} \partial q_j^{'}} \sum_r \frac{\partial q_i^{'}}{\partial q_r} v_r =$$

$$= \sum_r \frac{\partial}{\partial q_r} \left(\frac{\partial q_k}{\partial q_j^{'}} \right) v_r \ , \tag{7}$$

which shows (6) to be identical to $(4)_2$ in the considered situation. ▶

Let

$$E(q,v) \stackrel{def}{=} \sum_{i=1}^{n} v_i L_{v_i}(q,v) - L(q,v) \tag{8}$$

be the energy function associated with Lagrangian L.

Proposition 2.2. *Let L be non-singular, i.e. $\mathcal{L}$ is a diffeomorphism. Then*

$$X_L = \mathcal{L}^* \circ X_H$$

where X_H is the Hamiltonian vector field on $T^*(M)$ corresponding to $H = (\mathcal{L}^{-1})^*(E)$.

◀ A straightforward computation shows that

$$X_L \lrcorner \Omega = -dE \ , \tag{9}$$

where $\Omega = \sum_i dp_i \wedge dq_i$ is the canonical 2-form on $T^*(M)$. By applying now $(\mathcal{L}^{-1})^*$ to (9) one obtains

$$\left((\mathcal{L}^{-1})^* \circ X_L \right) \lrcorner \Omega = -dH$$

Since X_H is by definition the unique solution of

$$X_H \lrcorner \Omega = -dH$$

one can conclude that

$$X_H = (\mathcal{L}^{-1})^* \circ X_L \qquad ▶$$

It is worth stressing that equation (9) defines the vector field X_L uniquely up to a relative field Y such that $Y \lrcorner \Omega = 0$. Since Ω is non-degenerate the $C^\infty(M)$-module $\mathcal{D}_\mathcal{L}$ of relative vector fields along $\mathcal{L}$ that annihilate Ω is generated by relative fields of the form $\mathcal{L}^* \circ Z$, with $Z \in D(T^*(M))$ such that $\mathcal{L}^*(Z \lrcorner \Omega) = 0$. The last condition means that $Z \lrcorner \Omega \in \Lambda^1_\mathcal{L}(T^*(M))$, with

$$\Lambda^1_\mathcal{L}(T^*(M)) = \left\{ \omega \in \Lambda^1(T^*(M)) \middle| \ \mathcal{L}^*(\omega) = 0 \right\}$$

Note that for a generic Lagrangian function L the corresponding Legendre map is a local diffeomorphism except for a nowhere dense subset of $T(M)$. Obviously, $\Lambda^1_\mathcal{L}(T^*(M))|_{Im\,\mathcal{L}} = 0$ for such a Lagrangian. So, the relative Hamiltonian field X_L is uniquely defined by (9) for a generic Lagrangian L. On the contrary, solution of (9) is not unique iff

$\Lambda^1_{\mathcal{L}}(T^*(M))|_{Im\,\mathcal{L}} \neq 0$, which is equivalent to the fact that $\mathcal{L}$ degenerates on an open subset of $T(M)$. This discussion, together with proposition 2.2, shows that (9) admits a unique *natural* solution, namely X_L.

If

$$X = \sum_i a_i(q,v)\frac{\partial}{\partial q_i} + \sum_i b_i(q,v)\frac{\partial}{\partial p_i} \ ,$$

then equation (9) looks in terms of local coordinates as

$$\left\| \begin{array}{cc} L_{qv} & \mathbf{-1} \\ L_{vv} & \mathbf{0} \end{array} \right\| \cdot \left\| \begin{array}{c} a \\ b \end{array} \right\| = \left\| \begin{array}{c} E_q \\ E_v \end{array} \right\| \ ,$$

with $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$.

2.3. **An application: the Constraint Algorithm.** We conclude this section by sketching how the relative Hamiltonian formalism can be used to determine the consistency conditions for constrained Lagrangian systems. The *constraint algorithm* so obtained is, of course, a well-known procedure, going back to the Dirac-Bergmann theory of constrained Hamiltonians, and revisited later on by several authors (see [6] and the references contained in it). However, the geometric interpretation of the algorithm by means of the relative Hamiltonian vector field looks more natural and transparent than the previous ones.

It is useful to notice that the Euler-Lagrange equations

$$\left\| \begin{array}{cc} \mathbf{1} & \mathbf{0} \\ L_{vq} & L_{vv} \end{array} \right\| \cdot \left\| \begin{array}{c} \dot{q} \\ \dot{v} \end{array} \right\| = \left\| \begin{array}{c} v \\ L_q \end{array} \right\| \tag{10}$$

corresponding to the Lagrangian L can be represented in an invariant form as

$$Z_L \circ \mathcal{L}^* = X_L \ , \tag{11}$$

with $Z_L = \dot{q}\,\frac{\partial}{\partial q} + \dot{v}\,\frac{\partial}{\partial v}$ being the vector field on $T(M)$ whose integral curves are the phase trajectories. Notice that, if L is a *constrained* Lagrangian, i.e. $|L_{vv}| = 0$ on the whole phase space, then there is, in general, no vector field $Z_L \in D(T(M))$ satisfying equation (11). However, there may exist a maximal submanifold $W \subset T(M)$ and a vector field $\widetilde{Z}_L \in D(W)$ satisfying (11). The constraint algorithm enables one to find such a submanifold by a finite number of steps. First, notice that in view of equation (11) W can be characterized as the maximal submanifold such that $X_L|_W$ is tangent to $\mathcal{L}(W)$. Consider the sequence $\{W_r\}$of submanifolds of $T(M)$, defined by induction as follows:

1. $W_1 = T(M)$
2. $W_{r+1} = \{x \in W_r |\ X_L|_x \text{ is tangent to } \mathcal{L}(W_r)\}$

Obviously, this sequence stabilizes in a finite number of steps. If the last W_r is not empty, it coincides with W. Notice that the above

formulation of the constraint algorithm is more agreeable, from the computational point of view, than the standard approach (see, for instance, [6]). In fact, if $\mathcal{L}(W_r)$ is given by independent equations:

$$g_i(q,p) = 0\,,\ \ i = 1, ..., s\ ,$$

with s being the codimension of $\mathcal{L}(W_r)$, then the next term $W_{r+1}$of the sequence is given simply by the equations

$$g_i(\mathcal{L}(q,v)) = 0,\ \ X_L(g_i)(q,v) = 0,\ i = 1, ...s$$

It can be shown that the procedure above is equivalent to the standard reduction of a differential equation to the formally integrable form.

That given above is just an example of the usefulness of the relative Hamiltonian formalism in the study of constrained systems and gauge theories. A more detailed exposition of these topics will be given in a separate paper.

3. Singular Lagrangians

In this section we consider a special but important class of non-regular Lagrangians, whose Legendre maps degenerate only along submanifolds of codimension ≥ 1. This class contains all generic Lagrangians. We use the term "singular" for such Lagrangians, reserving the term "degenerate" only for the Lagrangians with everywhere degenerated Legendre map.

3.1. **Submersions with folds.** First, recall briefly some basic facts from the theory of stable mappings, which will be needed below (for further details and proofs see, for instance, [5]).

Let M, N be two manifolds, with $\dim M = m \geq \dim N = n$ and let

$$F : M \to N$$

be a smooth map. Denote by $J^1(M,N)$ the space of 1-jets of maps from M to N. If $(x_1, ..., x_m)$, $(y_1, ..., y_n)$ are two local charts on M and N, respectively, then a system of local coordinates on $J^1(M,N)$ is given by (x, y, p), where $p = (p_{ij})\,,\ i = 1, ..., n\,,\ j = 1, ..., m$. Let us consider the *1-jet graph* of F, i.e. the map

$$j^1F : M \to J^1(M,N)$$

given in local coordinates by

$$j^1F(x) \stackrel{def}{=} \left(x, F(x), \frac{\partial F}{\partial x}(x)\right)$$

Let $S_1 \subset J^1(M,N)$ be the submanifold of 1-jets of corank 1, i.e.

$$S_1 = \left\{(x,y,p) \in J^1(M,N) \middle|\ rank\, p = n-1\right\}$$

Let $S_1(F) = (j^1F)^{-1}(S_1)$. We assume that j^1F is *transversal* to S_1, i.e.

$$Im\, d_x\, j^1F + T_{F(x)}(S_1) = T_{F(x)}J^1(M,N) \qquad \forall x \in S_1(F) \tag{12}$$

In terms of local coordinates, condition (12) is equivalent to the fact that the minors of order n of the Jacobian matrix $\partial F/\partial x$, which vanish on $S_1(F)$, have only simple zeroes on it. Then ([5]) $S_1(F)$ is a submanifold of M having the same codimension, namely $m-n+1$, as S_1 in $J^1(M,N)$.

Definition 3.1. *A point $x \in S_1(F)$ is called a* fold point *if*

$$T_x(S_1(F)) + Ker\, d_xF = T_x(M)$$

Definition 3.2. . *A smooth map $F : M \to N$ is a* submersion with folds *if it satisfies condition (12) and its singularities are fold points only. In such a case the submanifold $S_1(F)$ defined above is called the* fold locus *of F.*

Remark. An obvious consequence of the definition is that the restriction of a submersion with folds to its fold locus is an immersion.

Below we will use the following property of submersions with folds ([5]).

Theorem 3.3. *Let $F : M \to N$ be a submersion with folds and let $a \in S_1(F)$. Then there exist a system of local coordinates $(x_1,...,x_m)$ in a neighbourhood of a and a system of local coordinates $(y_1,...,y_n)$ in a neighbourhood of $F(a)$ such that: 1) $a \equiv (0,...,0), F(a) \equiv (0,...,0)$; 2) the coordinate expression of F is*

$$\begin{cases} y_1 = x_1 \\ ..... \\ y_{n-1} = x_{n-1} \\ y_n = x_n^2 \pm ... \pm x_m^2 \end{cases}$$

◄ ►

3.2. **Legendre maps with fold-type singularities.** Referring to section 2.2 for the notations, consider the case when $d\mathcal{L}$ is regular everywhere except for a hypersurface $S \subset T(M)$. From (2) it follows that S is given by the equation

$$\mathcal{H}(q,v) = 0\ ,$$

where $\mathcal{H}(q,v) = \det L_{vv}$. Assume S to be regular, i.e. that

$$d_x\mathcal{H} \neq 0 \qquad \forall x \in S \tag{13}$$

Assume additionally the following transversality condition:

$$Ker\, d_x\mathcal{L} \cap T_x(S) = \{0\} \qquad \forall x \in S \tag{14}$$

It follows from (14), in particular, that the rank of the matrix L_{vv} is $n-1$, i.e. $\dim Ker\, d\mathcal{L} = 1$ at any point of S, and that $\mathcal{L}|_S$ is a local diffeomorphism between S and $\mathcal{L}(S)$. Hence, $\mathcal{L}$ is a submersion with folds and according to theorem 3.3 it can be represented locally in the normal form

$$\begin{cases} y_1 = x_1 \\ \ldots\ldots\ldots\ldots \\ y_{2n-1} = x_{2n-1} \\ y_{2n} = x_{2n}^2 \end{cases} \tag{15}$$

with respect to suitable local charts $(x_1, ..., x_{2n})$, $(y_1, ..., y_{2n})$ on $T(M)$, $T^*(M)$, respectively. In view of (15) one easily finds that for every point $x \in S$ there exists a neighbourhood U of x in $T(M)$ and a neighbourhood V of $\mathcal{L}(x)$ in $\mathcal{L}(T(M))$ such that: 1) $U \backslash S$ splits into two connected components U_1, U_2 such that $\mathcal{L}(U_1) = \mathcal{L}(U_2) = V \backslash \mathcal{L}(U \cap S)$ and 2) $\overline{\mathcal{L}(U_1)} \cap \overline{\mathcal{L}(U_2)} = \mathcal{L}(U \cap S)$

In other words, if $\mathcal{L}$ is a submersion with folds, then $\mathcal{L}(S)$ locally separates the image of $\mathcal{L}$ from its "local" complement in $T^*(M)$. As we shall see below, this almost obvious property of Legendre maps with folds is essential to carry over the transition principle from the Hamiltonian to the Lagrangian setting.

3.3. **Kernel of the Legendre Map and Characteristics on the Singular Hypersurface.** As it was remarked above, if $\mathcal{L}$ has only fold-type singularities then $Ker\, d_x\mathcal{L}$ is one-dimensional and transversal to S at any point $x \in S$. We will now point out some interesting properties of the kernel and consider its relation with the characteristic distribution on S, which, as we shall see, plays a fundamental role in the Transition Principle.

The following assertion is valid for an arbitrary Lagrangian.

Proposition 3.4. *$Ker\, d\mathcal{L}$ is tangent to the level surfaces of energy at any point of S.*

◀ According to (2) one has

$$Ker\, d_x\mathcal{L} = \left\{ b\frac{\partial}{\partial v} \in T_x(T(M)) \,\middle|\, L_{vv}b = 0 \right\} \tag{16}$$

with the standard matrix notation. On the other hand, it results from (8) that

$$E_v = L_{vv}v \tag{17}$$

which shows that $b^T E_v = 0$ ▶

Recall now the notion of characteristic distribution on S, introduced in [8]. Let $\Omega = \sum_i dp_i \wedge dq_i$ be the canonical 2-form on $T^*(M)$. Then $\mathcal{L}^*(\Omega)$ is a closed 2-form on $T(M)$ degenerate at points of S. More exactly, we have

Proposition 3.5. . *Let $x \in S$ and let $Null\,\mathcal{L}^*(\Omega)_x$ be the null space of $\mathcal{L}^*(\Omega)_x$, i.e.*

$$Null\,\mathcal{L}^*(\Omega)_x = \{\xi \in T_x(T(M)) |\, \xi \lrcorner \mathcal{L}^*(\Omega)_x = 0\}$$

Then $Null\,\mathcal{L}^*(\Omega)_x$ is 2-dimensional and contains $Ker\,d_x\mathcal{L}$.

◀ In special coordinates (q, v) the pullback of Ω takes the form

$$\mathcal{L}^*(\Omega) = \sum_{i<j} \left(L_{v_j q_i} - L_{v_i q_j}\right) dq_i \wedge dq_j - \sum_{i,j} L_{v_i v_j} dq_i \wedge dv_j$$

Hence $Null\,\mathcal{L}^*(\Omega)_x$ consists of the vectors $a\frac{\partial}{\partial q} + b\frac{\partial}{\partial v} \in T_x(T(M))$ such that

$$\left\| \begin{array}{cc} L_{vq} - L_{qv} & L_{vv} \\ L_{vv} & \mathbf{0} \end{array} \right\| \cdot \left\| \begin{array}{c} a \\ b \end{array} \right\| = \left\| \begin{array}{c} 0 \\ 0 \end{array} \right\| \tag{18}$$

Comparing (18) and (16) one sees that $Ker\,d_x\mathcal{L} \subseteq Null\,\mathcal{L}^*(\Omega)_x$. This shows that $\mathcal{L}^*(\Omega)_x$ is degenerate if $x \in S$ and hence the rank of $\mathcal{L}^*(\Omega)_x$ is not greater than $2n-2$. But since $\mathcal{L}|_S : S \to \mathcal{L}(S)$ is a diffeomorphism one has

$$2n - 2 = rank\left(\Omega|_{\mathcal{L}(S)}\right)_{\mathcal{L}(x)} = rank\left(\mathcal{L}^*(\Omega)|_S\right)_x$$

This shows that $rank\,\mathcal{L}^*(\Omega)_x \geq 2n-2$.▶

An immediate consequence of the previous proposition is that $Null\,\mathcal{L}^*_x(\Omega)$ is transversal to $T_x(S)$, so that their intersection is a line l_x tangent to S. This way we get a 1-dimensional distribution $x \mapsto l_x$ on S, called *characteristic distribution*, and its integral curves are called *characteristic curves* or *characteristics* of S.

The characteristics are non-parametrized curves transversal to the fibres of $T(M)$ and hence projecting biunivocally into M. Let $\gamma : I \subset \mathbf{R} \to T(M)$ be a characteristic of S and let $\widetilde{\gamma} = \pi \circ \gamma$ be its projection into M. The problem of determining under what conditions is γ a lifting of $\widetilde{\gamma}$, i.e.

$$\gamma = \lambda\,\dot{\widetilde{\gamma}} \tag{19}$$

for some $\lambda \in C^\infty(I)$, is related with the variational interpretation of the Transition Principle.

Theorem 3.6. *Let $L \in C^\infty(T(M))$ be a singular Lagrangian with folds. The following three conditions are equivalent:*

- **a):** *Any characteristic of S is the lifting of its projection into M.*
- **b):** *Any point $x \equiv (q, v) \in S$, considered as an element of $T_x(T_q(M))$, belongs to $Ker\,d_x\mathcal{L}$.*
- **c):** *The fibres $T_x(M) \subset T(M)$ are tangent to the energy level surfaces at any point of S.*

◀ Equivalence of b) and c) obviously follows from (17). Let us prove the equivalence of a) and b). Assume that b) holds, i.e. that

$$L_{vv}v = 0 \quad \text{on } S \tag{20}$$

Let $\gamma(t) \equiv (q(t), v(t))$ be a characteristic curve of S. We have to show that

$$v(t) = \lambda(t)\,\dot{q}(t)\,, \tag{21}$$

The curve $\overline{\gamma} = \mathcal{L} \circ \gamma$ is a characteristic of the hypersurface $\mathcal{L}(S)$ with respect to Ω, projecting into the same curve $\widetilde{\gamma}$ of M as γ. Let

$$\Phi(q,p) = 0, \qquad \Phi \in C^\infty(T^*(M))$$

be a (local) equation of $\mathcal{L}(S)$. Then the Hamiltonian field $X_\Phi = \Phi_p \frac{\partial}{\partial q} - \Phi_q \frac{\partial}{\partial p}$ is tangent to $\overline{\gamma}$. By reparametrizing γ, if necessary, we can assume that $t \mapsto \overline{\gamma}(t)$ coincides with the parametrization of $\overline{\gamma}$ as an integral curve of X_Φ. As it is immediately seen from the normal form (15) of $\mathcal{L}$, it holds (locally)

$$\mathcal{L}^*(\Phi) = f\mathcal{H}^2 \tag{22}$$

for some $f \in C^\infty(T(M))$. Hence, differentiating (22) with respect to q_i and v_i, one finds that, on $\mathcal{L}(S)$,

$$\left\| \begin{array}{cc} L_{qv} & -\mathbf{1} \\ L_{vv} & \mathbf{0} \end{array} \right\| \cdot \left\| \begin{array}{c} \Phi_p \\ -\Phi_q \end{array} \right\| = \left\| \begin{array}{c} 0 \\ 0 \end{array} \right\| ,$$

with the derivatives of Φ taken at $\mathcal{L}(q,v) = (q, L_v(q,v))$, $(q,v) \in S$. In particular, $L_{vv}\Phi = 0$, so that from condition (20) and the fact that $corank\, L_{vv} = 1$ it follows that on S

$$\Phi_p = \mu v \tag{23}$$

with $\mu \in C^\infty(S)$. But, according to Hamilton equations associated with Φ, $\Phi_p = \dot{q}$, so that one gets (21), with $\lambda = 1/\mu$. ▶

Corollary (of the proof). Let $\gamma(t)$ be a characteristic of S. Then $\alpha_{\gamma(t)}\left(\overset{\bullet}{q}(t)\right) \in Ker\, d_{\gamma(t)}\mathcal{L}$, with $\alpha_{\gamma(t)}$ being the canonical identification of $T_{q(t)}(M)$, $q(t) = \pi(\gamma(t))$, and $T_{\gamma(t)}\left(T_{q(t)}(M)\right)$.

◀ It is sufficient to notice that $\alpha_{\gamma(t)}\left(\overset{\bullet}{q}(t)\right) = \Phi_p(\mathcal{L}(\gamma(t)))$.▶

This corollary suggests a natural way to parametrize characteristics of S. Namely, let γ be such a characteristic and $\widetilde{\gamma}$ be its projection. Any point $(q,v) \in \gamma$ corresponds to a vector $v\,\frac{\partial}{\partial q}\Big|_q$, tangent to M (but not to $\widetilde{\gamma}$, in general) at the point $q \in \widetilde{\gamma}$. But this vector is canonically identified with $v\,\frac{\partial}{\partial v}\big|_{(q,v)} \in T_{(q,v)}(T_q(M))$, which can be univocally represented as the sum of a vector $w_q \in Ker\,(d\mathcal{L})_{(q,v)}$ and another vector tangent to S. In this way one associates with any point $q \in \widetilde{\gamma}$ a vector $w_q \in T_{(q,v)}(T_q(M)) \equiv T_q(M)$ which, according to the previous

theorem, is tangent to $\widetilde{\gamma}$. Hence we obtain a parametrization of $\widetilde{\gamma}$, and consequently of γ.

The above theorem is illustrated with a special class of Lagrangians which appears frequently in post-galilean models of interacting particles ([3], [12]). One of them, a post-galilean oscillator, was considered in [8] with the purpose to exhibit the "jumping dynamics" resulting from the Transition Principle.

Example. Consider Lagrangians of the form

$$L = L\left(q, Q\left(q, v\right)\right) \quad , \tag{24}$$

where

$$Q\left(q, v\right) = \frac{1}{2} v^T G\left(q\right) v$$

is a pseudo-Riemannian metric on M, whose matrix is $1/2G\left(q\right)$. So, $G\left(q\right)$ is symmetric and non degenerate, for any point $q \in M$. The singular set S of the corresponding Legendre map coincides with the set of points at which the operator represented by the matrix $L_{vv} = \left\| L_{v_i v_j} \right\|$ has a non-trivial kernel. Note that

$$L_{vv} = L^{'} G + L^{''} A \ , \tag{25}$$

with the $n \times n$ matrix $A = A\left(q, v\right) = \left(Gv\right)\left(Gv\right)^T$, and the prime standing for $\partial / \partial Q$. If $v \neq 0$, then A is an operator of rank 1 whose image is generated by vector Gv while its kernel, $Ker\, A$, is composed of vectors w such that $v^T G w = 0$. There are two qualitatively different cases: 1) $v \notin Ker\, A\left(q, v\right) \Longleftrightarrow Q\left(q, v\right) \neq 0$, and 2) $v \in Ker\, A\left(q, v\right) \Longleftrightarrow Q\left(q, v\right) = 0$. Let us consider them separately.

1) In this case by identifying $T_q\left(M\right)$ and $T_{(q,v)}\left(T_q\left(M\right)\right) \subset T\left(T\left(M\right)\right)$ we see that the latter space is decomposed into the direct sum

$$T_{(q,v)}\left(T_q\left(M\right)\right) = Ker\, A\left(q, v\right) \oplus \left\{v\right\}$$

with $\left\{v\right\} = \left\{w \middle| \ w = \lambda v, \ \lambda \in \mathbf{R}\right\}$. So, if $z = w + u$, with $w \in Ker\, A\left(q, v\right)$, $u \in \left\{v\right\}$, then according to (25)

$$L_{vv}\left(w + u\right) = G\left(L' w + \left(L' + v^T G v L''\right) u\right)$$

Hence

$$w + u \in Ker\, L_{vv} \Longleftrightarrow L' w + \left(L' + v^T G v L''\right) u = 0 \tag{26}$$

So, $Ker\, d_{(q,v)} \mathcal{L} = Ker\, L_{vv}\left(q, v\right) \neq 0$ in one of the following cases:

i): $L^{'} + 2L'' Q = 0$, $L' \neq 0 \iff Ker\, d_{(q,v)} \mathcal{L} = \left\{v\right\}$

ii): $L' = 0$, $L' + 2L'' Q \neq 0 \iff Ker\, d_{(q,v)} \mathcal{L} = Ker\, A\left(q, v\right)$

iii): $L' = L' + 2L'' Q = 0 \iff Ker\, d_{(q,v)} \mathcal{L} = T_{(q,v)}\left(T_q\left(M\right)\right)$

As it is easy to see, in the case i) $Ker\, d\mathcal{L}$ is 1-dimensional and is transversal to the hypersurface $S_1 = \left\{L' + 2L'' Q = 0\right\}$ except for its submanifold $S_1^0 = \left\{3L'' + 2L''' Q = 0\right\}$. So, $S_1 \backslash S_0$ is

composed of fold points. On the contrary, $Ker\, d\mathcal{L}$ is tangent to $S_2 = \{L' = 0\}$.

2) It is convenient in this case to consider the direct decomposition

$$T_{(q,v)}\left(T_q\left(M\right)\right) = Ker\, A\left(q,v\right) \oplus \{Gv\}\,, \quad v \neq 0$$

where $Im\, A\left(q,v\right) = \{Gv\} = \{z \,|\, z = \lambda Gv, \lambda \in \mathbf{R}\}$. Note that $A\left(Gv\right) = \lambda\left(Gv\right)$, with $\lambda \in \mathbf{R}$. So, if $z = w+u$, $w \in Ker\, A\left(q,v\right)$, $u \in Im\, A\left(q,v\right)$, then in view of (25) one has

$$L_{vv}\left(z\right) = G\left(L'w + L'u\right) + \lambda L''u = G\left(L'w + L'u\right) + \lambda\mu L''Gv$$

with $u = \mu Gv$. This shows that

$$L_{vv}\left(z\right) = 0 \Longleftrightarrow \left(L'w + \lambda\mu L''v\right) + L'u = 0$$

Since $L'w + \lambda\mu L''v \in Ker\, A$ we see that $L'Au = 0$. This implies either $L'\left(q, Q\left(v\right)\right) = 0$ or $A\left(q,v\right)u = 0$, i.e. $u = 0$. In the last case $\mu = 0$ and $L'w = 0$ $(w \neq 0)$, so we find again that $L' = 0$. Hence $S \cap \{Q\left(q,v\right) = 0\} \subset S_2$.

Thus we see that $S = S_1 \cup S_2$ and $S_1 \backslash S_1^0$ consists of fold points, while S_2 is not a generic singularity of $\mathcal{L}$.

3.4. **Characteristic curves and symmetries.** In this subsection it is shown that symmetries of a Lagrangian respect the characteristic curves of its singular hypersurface S. Namely, the first integral of Euler-Lagrange equations corresponding to an infinitesimal symmetry of the Lagrangian is constant along characteristic curves.

First, recall some definitions. Let $F : M \to M$ be a smooth map. Its natural liftings $\widetilde{F} : T\left(M\right) \to T\left(M\right)$ and $\widehat{F} : T^*\left(M\right) \to T^*\left(M\right)$ are defined by formulas

$$\widetilde{F}\Big|_{T_q(M)} = d_q F\,, \quad q \in M$$

and

$$\widehat{F}\Big|_{T_q^*(M)} = \left(d_q F^*\right)^{-1}\,, \quad q \in M$$

respectively. Obviously, if F is a diffeomorphism, then such are also $\widetilde{F}$ and $\widehat{F}$. F is a *symmetry* of a Lagrangian L if $\widetilde{F}^*\left(L\right) = L$. Further, it is easy to see that if F is a symmetry of L, then

$$\mathcal{L} \circ \widetilde{F} = \widehat{F} \circ \mathcal{L} \tag{27}$$

Another relation we need is (see, for instance, [10])

$$\widehat{F}^*\left(\rho\right) = \rho\,, \tag{28}$$

where $\rho = \sum p_i dq_i$ is the Liouville form on $T^*\left(M\right)$.

Similarly, any vector field $X \in \mathcal{D}\left(M\right)$ can be lifted canonically to both $T\left(M\right)$ and $T^*\left(M\right)$. Denote these liftings by $\widetilde{X} \in D\left(T\left(M\right)\right)$ and $\widehat{X} \in D\left(T^*\left(M\right)\right)$, respectively. If $\{F_t\}$ is the flow generated by X, then

$\widetilde{X}$ (resp., $\widehat{X}$) is the vector field generating the flow $\left\{\widetilde{F}_t\right\}$ (resp., $\left\{\widehat{F}_t\right\}$). The infinitesimal analogue of (28) is

$$\widehat{X}(\rho) = 0 \tag{29}$$

Recall that $X \in \mathcal{D}(M)$ is called an infinitesimal (non-hidden) *symmetry* of L if

$$\widetilde{X}(L) = 0$$

If $X \in \mathcal{D}(M)$ is an infinitesimal symmetry of L, then

$$\widetilde{X} \circ \mathcal{L}^* = \mathcal{L}^* \circ \widehat{X}\,, \tag{30}$$

which is the infinitesimal analogue of (27).

According to Noether's theorem, the function

$$I_X = \mathcal{L}^*(\rho)\left(\widetilde{X}\right) \tag{31}$$

is a first integral of the Euler-Lagrange equations.

Theorem 3.7. *I_X is constant along the characteristics of S.*

◀ Since $\widehat{X}(\rho) = X \lrcorner\, d\rho + d\left(\widehat{X} \lrcorner\, \rho\right)$, from (30), (29) and (31) we get

$$dI_X = -\widetilde{X} \lrcorner\, \mathcal{L}^*(\Omega)$$

Hence, if $x \in S$ and $\xi \in T_x(S)$ is a characteristic vector, then

$$\xi(I_X) = -\mathcal{L}^*(\Omega)_x\left(\widetilde{X}_x, \xi\right) = 0 \qquad \blacktriangleright$$

4. Transition Principle.

The Transition Principle discussed in this section is a prescription describing the behaviour of a dynamical system in some *irregular* situations when the standard "smooth" principles are no longer applicable. This principle was introduced by one of the authors (see [11]) in the context of Hamiltonian mechanics when the Hamiltonian function is discontinuous. The examples given below and in [11], [8] show it to be "experimentally" confirmed. On the other hand, the concept of relative Hamiltonian vector field associated with a Lagrangian allows one to observe that the geometrical background of the Hamiltonian Transition Principle is essentially the same as that for fold-singular Lagrangians. This motivates the "Lagrangian" Transition Principle as it is presented below.

4.1. **The Hamiltonian formulation.** Let us recall the Transition Principle for discontinuous Hamiltonians.

Let (Φ, Ω) be a symplectic manifold. Suppose that Φ is divided by a hypersurface Γ into two closed domains Φ_+, Φ_- and that the hamiltonian function of the system is bi-valued on Γ. In other words, the restrictions $H_\pm = H|_{\Phi_\pm}$ are C^∞-smooth in their respective domains, but they do not necessarily coincide on the common boundary Γ.

Recall ([10]) that on Γ is defined the one-dimensional distribution of *characteristic directions*. Such distribution associates with each point $x \in \Gamma$ the characteristic line

$$l_x \stackrel{def}{=} \left\{ \xi \in T_x(\Phi) \mid (\xi \lrcorner \Omega_x)|_{T_x(\Gamma)} = 0 \right\}$$

The integral curves of this distribution are called *characteristics* of Γ.

Definition 4.1. *Let $x \in \Gamma$. We say that x is an* in-point *(resp. an* out-point*) for H_+ if $X_{H_+}\big|_x$ is directed toward Φ_+ (resp. toward Φ_-). Similarly, in- and out-points for H_- are defined.*

Definition 4.2. *Let $x \in \Gamma$ and $E = H_+(x)$. Denote by γ_x the characteristic of Γ passing through x. A point $y \in \gamma_x$ is called* decisive *for (x, H_+) if it is an in-point for H_+ (resp. for H_-) and $H_+(y) = E$ (resp. $H_-(y) = E$). Analogously, one can define decisive points for (x, H_-).*

Transition Principle (Hamiltonian formulation). *When the moving phase point reaches from Φ_+ (resp., Φ_-) the hypersurface Γ at a point x, its trajectory is to be prolonged, starting from any decisive point for (x, H_+) (resp., (x, H_-)), as the trajectory of the corresponding Hamiltonian.*

Remark. The transition principle is applied as well to the situation when the Hamiltonian H is smooth but the phase space has a non-empty boundary $\partial\Phi$. In such a case one has just to put formally $\Gamma = \partial\Phi, \Phi_+ = \Phi, \Phi_- \backslash \Gamma = \emptyset, H_+ = H$ and $H_- = \infty$. In other words, only in- and out-points for $H_+ = H$ are to be taken into consideration. Elastic collisions of rigid bodies (see the example below) are described by the transition principle in this form.

It is worth stressing that, according to the transition principle, the phase trajectory generally splits into several parts after having reached the hypersurface Γ. Such behaviour is not, however, abnormal and happens, for instance, in geometrical optics (see [8]) when a light ray splits into reflected and refracted rays, in perfect accordance with the transition principle.

We remark that from the above principle it follows that the energy of the system does not change after an impact with Γ. In the example below we will see that in some special but important cases also other first integrals of the equations of motion are preserved.

The transition principle can be modified naturally to take into account further peculiarities of the studied physical system. For instance, the energy can be replaced by an alternative constant of motion in the formulation of the principle, or some further rules describing the energy balance in the process of impact may be added (non-elastic collisions, etc.).

Example 1. (The transition principle for natural Hamiltonians). Let us consider the case when $\Phi = \Phi_+ \cup \Phi_-$ is the cotangent bundle $T^*(M)$ of the configuration space $M = M_+ \cup M_-$ and $\Gamma = \pi^{-1}(S)$, with $S = \partial M_+ \cap \partial M_-$ being a regular hypersurface in M and $\pi : T^*(M) \to M$ being the bundle projection. Further, let each of the Hamiltonians H_+, H_- be of the form

$$H_\pm(q,p) = p^T G_\pm(q)\, p + V_\pm(q) \ , \tag{32}$$

with $V_\pm \in C^\infty(M_\pm)$ potential energy and $G_\pm$ Riemannian metric on $M_\pm$ (kinetic energy). Let $F(q) = 0$ be the equation of S and hence of Γ too (more exactly, $\pi^*(F) = 0$ is the equation of Γ). Then the characteristics of Γ are described by the Hamilton equations:

$$\begin{cases} \dot{q}_i = F_{p_i} = 0 \\ \dot{p}_i = -F_{q_i}(q) \end{cases}$$

together with $F(q) = 0$. Hence, the characteristic passing through the point $\overline{x} \equiv (\overline{q}, \overline{p}) \in \Gamma$ is the line $\gamma_{\overline{x}} \subset T^*_{\overline{q}}(M)$ given in parametric form by

$$\begin{cases} q_i(t) = \overline{q}_i \\ p_i(t) = \overline{p}_i - F_{q_i}(\overline{q})\, t \end{cases}$$

Suppose that the phase point, coming from Φ_+, reaches Γ at the point $\overline{x}$. According to the transition principle the part of the trajectory reflected from Γ starts from the in-points for H_+ at which the characteristic $\gamma_{\overline{x}}$ intersects the hyperquadric $H_+(q,p) = H_+(\overline{q}, \overline{p})$. This intersection consists of the two points $\overline{x} = x(0)$ and $x^* = x(t^*)$, with $t^* = 2\left(F_q^T G_+ p\right) / \left(F_q^T G_+ F_q\right)$ (all the functions are evaluated at $\overline{x}$). Note that $X_{H_+}(F) = -X_F(H_+) = 2F_q^T G_+ p$. Therefore:

$$\overline{x} = x^* \Leftrightarrow X_{H_+} \text{ tangent to } \Gamma \text{ at } \overline{x} \Leftrightarrow \gamma_{\overline{x}} \text{tangent to } \left\{H_+ = \overline{H}_+\right\} \text{ at } \overline{x}$$

If $\overline{x} \neq x^*$, then

$$X_{H_+}(F)\big|_{x^*} = -2\, X_{H_+}(F)\big|_{\overline{x}}$$

so that if $\overline{x}$ is an out-point for H_+, x^* is an in-point for it (and viceversa). Therefore, if the phase point reaches Γ at $\overline{x}$ transversally, then according to the transition principle there is one and only one "reflected" trajectory issuing from x^*. As for "refracted" trajectories, they can be two, one or none, depending on how the characteristic straight line intersects the quadric $\{H_-(q,p) = H_+(\overline{x})\}$.

The following fact concerning the relation between symmetries of $H_\pm$ and the characteristics on Γ is analogous to theorem 3.7. So, we omit the proof. For the notation, see section 3.4.

Theorem 4.3. . *Let $X \in D(M)$ be a non-hidden symmetry of the Hamiltonian H, i.e. $\widehat{X}(H) = 0$. Then the corresponding first integral of Hamilton equations, $\rho\left(\widehat{X}\right)$ (see [10]), is constant along the characteristics of Γ iff X is tangent to $S = \pi^{-1}(\Gamma)$.*

◄ ►

Example 2. (Elastic collisions between rigid bodies). Let C_1, C_2 be two rigid bodies with perfectly smooth outer surfaces, and assume that in case of an impact between them the total mechanical energy does not change (elastic collision). Then one can think of the 12-dimensional configuration space M as the union of two domains: M_+, corresponding to the admissible configurations, in which the motion is described by a Hamiltonian of the form (32); and a domain M_-, formed by the configurations such that $(C_1 \cap C_2)^\circ \neq \emptyset$, with which one associates a potential energy "identically equal to $+\infty$". In other words, the rigidity constraints do not allow for refracted trajectories.

The value of the momenta p after a collision can obviously be determined in an elementary way by taking into account the integrals of motion (total mechanical energy, projections of linear and angular momenta). Apart from mechanical energy, all the other integrals are associated with 1-parameter groups of space symmetries of the system (invariance with respect to translations and rotations), and it is evident that such isometry groups leave fixed the hypersurface $S = \partial M_+ \cap \partial M_-$ of tangency configurations. Hence, according to the previous theorem, the corresponding first integrals keep constant along the characteristics of $\Gamma = \pi^{-1}(S)$. Therefore, by applying the transition principle we automatically get the conservation of all the above integrals of motion. This also shows that the transition principle leads to the same result as the classical theory in describing elastic collisions. Moreover, it continues to work as well when the lack of symmetries does not allow symmetry considerations.

4.2. **The Lagrangian formulation.** The transition principle, originally considered only in the Hamiltonian case described above, can be extended to the Lagrangian situation, as it has been shown for the first time in [8]. Namely, suppose that a dynamical system is described by a singular Lagrangian with folds as in section 3.2. Then the motion of the system is uniquely determined outside the singular hypersurface S by Euler-Lagrange equations (10), or, equivalently, (11), where $Z_L = \overset{\bullet}{q}\frac{\partial}{\partial q} + \overset{\bullet}{v}\frac{\partial}{\partial v} \in D(T(M)\setminus S)$ is the Lagrangian vector field associated with L. On the other hand, equations (10) cannot be solved

uniquely with respect to $\dot{v}$ on S. So accelerations are undetermined on S, in general, and there are discontinuities in the motion. The Lagrangian formulation of the transition principle is a natural way to determine these discontinuities.

First, let us extend to the Lagrangian case the notion of in- and out-points.

Definition 4.4. *Let $x \in S$ and let the neighbourhood V of $\mathcal{L}(x)$ in $\mathcal{L}(T(M))$ be given as at the end of section 3.2. A point $x \in S$ is called an* in-point *for L if $X_L|_x$ is directed towards V, and* out-point *otherwise.*

X_L is, generally, transversal to $\mathcal{L}(S)$ everywhere on S, except for a (possibly non regular) $(2n-2)$-dimensional submanifold S_1. This, in turn, contains a submanifold S_2 of dimension $(2n-2)$ on which X_L is tangent at S_1, etc., just as in the case of the constraint algorithm (see section 2.3).

Definition 4.5. *Let $x \in S$. A point $y \in S$ is called* decisive *for x if it is an in-point such that $E(y) = E(x)$.*

Given the above definitions, the extension of the Transition Principle to the Lagrangian case is obvious.

Transition Principle (Lagrangian formulation). *When the phase point reaches the hypersurface S at a point x, its trajectory can be prolonged, starting from any point decisive for x, as a trajectory of the Lagrangian vector field Z_L.*

Recall that the characteristics of a singular Lagrangian L are naturally parametrized curves, according to the construction of section 3.3. The physical meaning of this parameter is at the moment not very clear. It seems natural to interpret it as the time during which the phase point remains captured by the singular surface S.

Obviously, physical applications of the Lagrangian Transition Principle depend on whether there exist physically meaningful Lagrangians with fold singularities. For instance, are the singularities appearing in various post-galilean models (see [3], [12]) just the artificial by-products of the adopted approximation procedure? In other words, the dilemma is whether the presence of singularities indicates only that the Lagrangian model is inadequate or, on the contrary, a new phenomenon. An experimental test of it would be very interesting. From the mathematical point of view, singular lagrangian dynamical systems generalize naturally the billiard systems and as such are worth being studied. In [8] such a dynamics (post-galilean oscillator) was studied in detail, showing its intrinsic self-consistency.

4.3. **The Transition Principle and the Weierstrass-Erdmann conditions.** A natural question whether the singular Lagrangian dynamics based on the transition principle corresponds to a variational

problem is briefly discussed in this subsection. In fact, functionals corresponding to singular Lagrange densities appear in various mathematical problems and have been studied by various authors. The following classical theorem is one of the main results in this direction.

Theorem 4.6. *(Weierstrass-Erdmann conditions). Consider the action functional*

$$\mathcal{F}(q) \overset{def}{=} \int_a^b L\left(q(t), \dot{q}(t)\right) dt \tag{33}$$

defined on the space of piecewise smooth functions $q : [a, b] \to M$ with fixed end-points $q', q'' \in M$. Let $\overline{q}(t)$ be an extremal for $\mathcal{F}$, and suppose its derivative to have a jump discontinuity at $t^* \in]a, b[$. Then the following two conditions must be satisfied:

1. $E(t^*-) = E(t^*+)$
2. $L_{v_i}(t^*-) = L_{v_i}(t^*+), \qquad i = 1, ..., n$

where by $f(t^*+)$, $f(t^*-)$ we denote the right and left limit of f at t^*, respectively, and the energy E and momenta L_{v_i} are evaluated on $\overline{q}(t)$.

◀See, for instance, [4]▶

According to Hamilton's principle, the variational problem associated with the functional (33) is equivalent to Euler-Lagrange equations outside the singular hypersurface S. On the other hand, it is easy to see that the derivative of a broken extremal $\overline{q}(t)$ of $\mathcal{F}$ has a jump at t^* only if $\left(\overline{q}(t^*), \dot{\overline{q}}(t^*)\right) \in S$. At a first glance, it would seem natural to interpret the extremals of (33) on the class of piecewise smooth curves. However, such an assumption presents serious drawbacks from the dynamical point of view. First, the behaviour of the system *on* S is not taken into any account. The natural geometrical structure described above is simply ignored and the singular hypersurface itself is just considered as a "black box". Furthermore, the second Weierstrass-Erdmann condition is a consequence of an ad hoc restriction on the class of possible trajectories. Namely, it is assumed that the extremal curve $q(t)$ has discontinuities only in the derivative, i.e. jumps of position are not allowed. That this need not necessarily be the case is shown by the following example.

Example. Consider the Lagrangian function

$$L(q, v) = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - kq^2\left(1 + \frac{v^2}{2c^2}\right) \tag{34}$$

describing the motion of a one-dimensional relativistic oscillator in the post-galilean approximation ([7], [8]). The constants m, q, k, c are the mass and the charge of the oscillating particle, the elastic constant and

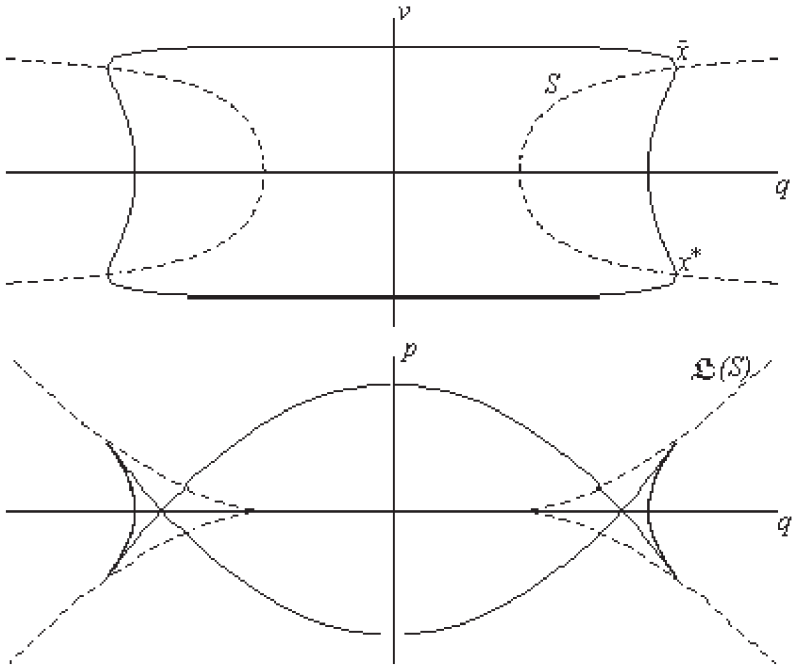

FIGURE 1.

the light velocity, respectively. The energy function corresponding to (34) is

$$E(q,v) = \frac{mc^2}{\sqrt{1-v^2/c^2}} + kq^2\left(1 - \frac{v^2}{2c^2}\right) ,$$

while the equation of S is

$$q^2 = \frac{mc^2}{k\left(1-v^2/c^2\right)^{3/2}}$$

As it is shown in figure 1, for a sufficiently high value of the energy the phase point reaches S at the point $\overline{x} \equiv (\overline{q}, \overline{v})$. Then, according to the Transition Principle, the motion continues after the jump at the point $x^* \equiv (\overline{q}, -\overline{v})$. On the other hand, Weierstrass-Erdmann conditions tell that after the system reaches $\overline{x}$ its trajectory can be prolonged no more, because $L_v(x^*) = -L_v(\overline{x}) \neq 0$. In other words, time ceases to exist for this system. So, the Transition Principle seems to be more reasonable in the dynamical context.

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