The Diffiety Institute Preprint Series

Preprint DIPS–10/98 **June 19, 1995**

The "three-line" theorem for the Vinogradov $\mathcal{C}$-spectral sequence of the Yang–Mills equations

by

Dmitri GESSLER

Available via INTERNET:
http://ecfor.rssi.ru/~diffiety/
http://www.botik.ru/~diffiety/
anonymous FTP:
ftp://ecfor.rssi.ru/general/pub/diffiety
ftp://www.botik.ru/~diffiety/preprints

The Diffiety Institute
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

Preprint SISSA 71/95/FM

The "three-line" theorem for the Vinogradov $\mathcal{C}$-spectral sequence of the Yang–Mills equations

Dmitri GESSLER

ABSTRACT. The Vinogradov $\mathcal{C}$-spectral sequence for the Yang–Mills equations is considered and the "three-line" theorem for the term E_1 of the $\mathcal{C}$-spectral sequence is proved: $E_1^{p,q} = 0$ if $p > 0$ and $q < n-2$, where n is the dimension of spacetime.

1. Introduction

Homological methods play an important role in the study of systems of differential equations. The $\mathcal{C}$-spectral sequence (variational bicomplex) introduced by A. M. Vinogradov [11] contains important invariants of differential equations such as conservation laws, characteristic classes of families of solutions, etc. It provides a means for studying various aspects of Lagrangian formalism, the inverse problem of calculus of variations. The term E_1 of the $\mathcal{C}$-spectral sequence is an analog of the de Rham complex in the category of nonlinear partial differential equations (for a very enlightening discussion see [15]).

General methods for calculating this important spectral sequence are based on the Spencer type cohomology techniques. In [13], the "two-line" theorem estimating the term E_1 for determined systems of differential equations is proved, a concrete method of calculating for the term $E_1^{1,n-1}$, which is related to the theory of conservation laws, is given, where n is the number of independent variables. Due to this method, a complete description of the set of conservation laws is possible for determined differential equations (see, for example, [14]). Further development of Vinogradov's results is done in [10, 9], where the Janet sequence for involutive differential equations is used and a

1991 *Mathematics Subject Classification.* Primary 35A30; Secondary 58A20, 58G05.

Key words and phrases. nonlinear differential equations, spectral sequences, Spencer cohomology, Yang–Mills equations.

general approach to calculation of the horizontal cohomology is proposed. In [5], this approach is applied to the $\mathcal{C}$-spectral sequence for overdetermined equations and a method of calculating the $\mathcal{C}$-spectral sequence, generalizing Vinogradov's one, is given. This method enables one to reduce the problem of computing the term $E_1^{p,n-1}$, $p > 0$, to the problem of finding the kernel of some differential operator. The terms $\overline{H}^q = E_1^{0,q}$ also known as the horizontal de Rham cohomology, or the characteristic cohomology, are studied for exterior differential systems from a different point of view in [3].

Now it has been realized that many fundamental questions of the local field theory can be reformulated as cohomological ones. In [2, 6], the analysis of the antifield-BRST method of quantization from the homological point of view is done, in particular, the characteristic (horizontal) cohomology for the Yang–Mills equations is considered and it is proved that $\overline{H}^q = E_1^{0,q} = 0$ if $q < n-2$, where n is the dimension of the spacetime. In this paper, we extend the results of [2] concerning the characteristic cohomology for the Yang–Mills equations and prove the "three-line" theorem, which is the first step in studying the $\mathcal{C}$-spectral sequence. The method used in this paper is the same as in [5, 4].

This paper is organized as follows. Section 2 is a summary of the geometrical theory of nonlinear differential equations and the $\mathcal{C}$-spectral sequence. In Section 3, the formally exact resolution of the universal linearization operator for the Yang–Mills equation is considered and the "three-line" theorem is proved.

2. Jet manifolds and infinitely prolonged differential equations

In this section, we define the basic concepts of the geometrical theory of differential equations and the theory of the $\mathcal{C}$-spectral sequence ([1, 5, 8, 12, 13]).

2.1. **Jets.** Let M be a smooth manifold and $\pi\colon E \to M$ be a smooth fiber bundle over M, $\dim M = n$, $\dim E = m + n$. Denote by $\Gamma(\pi)$ the set of all (local) sections of π.

Let $\pi_k\colon J^k \to M$ be the *bundle of k-jets* for π,

$$J^k(\pi) = \{\, [f]_x^k \mid f \in \Gamma(\pi), x \in M \,\},$$

where $[f]_x^k$ denotes the k-jet of a local section f at x. Denote by $J^\infty(\pi)$ the manifold of infinite jets for π. This manifold is the inverse limit with respect to the following system of mappings

$$\pi_{k,l}\colon J^k(\pi) \to J^l(\pi), \quad \pi_{k,l}\left([f]_x^k\right) = [f]_x^l,\ k \geq l,$$
$$\pi_k\colon J^k(\pi) \to M, \quad \pi_k\left([f]_x^k\right) = x.$$

By definition, one has the natural projections

$$\pi_{\infty,k}\colon J^\infty(\pi) \to J^k(\pi), \quad \pi_\infty\colon J^\infty(\pi) \to M.$$

Let us choose a coordinate neighborhood U in M such that the restriction of the bundle E to U is trivial. Let $x_1, \dots, x_n$ be local coordinates in the neighborhood U and $u^1, \dots, u^m$ be coordinates along the fiber of π in $\pi^{-1}(U)$. Each local section $f \in \Gamma(\pi)$ is of the form $f = (u^1(x_1, \dots, x_n), \dots, u^m(x_1, \dots, x_n))$. Define functions p^j_σ by

$$p^j_\sigma\left([f]^k_{x_0}\right) = \left.\frac{\partial^{|\sigma|} u^j}{\partial x_1^{\sigma_1} \dots \partial x_n^{\sigma_n}}\right|_{x=x_0},$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$, $|\sigma| = \sigma_1 + \dots + \sigma_n$. Then the smooth functions (x^i, p^j_σ), $1 \le i \le n$, $1 \le j \le m$, $0 \le |\sigma| \le k$, form local coordinates in $J^k(\pi)$, $0 \le k \le \infty$.

Let $\mathcal{F}_k(\pi)$ denote the algebra $C^\infty\left(J^k(\pi)\right)$. Then one has the system of embeddings

$$\pi^*_{k,l} \colon \mathcal{F}_l(\pi) \to \mathcal{F}_k(\pi), l \le k, \quad \pi^*_k \colon C^\infty(M) \to \mathcal{F}_k(\pi).$$

The direct limit $\mathcal{F}(\pi)$ with respect to the system $\{\pi^*_{k,l}\}$ is called the *algebra of smooth functions* on $J^\infty(\pi)$. We identify $\mathcal{F}_k(\pi)$ and $C^\infty(M)$ with their images in $\mathcal{F}(\pi)$,

$$\mathcal{F}(\pi) = \bigcup_{k \ge 0} \mathcal{F}_k(\pi).$$

In the same manner, one can define the module of *i-forms* $\Lambda^i(\pi)$, $i \ge 0$, on $J^\infty(\pi)$

$$\Lambda^i(\pi) = \bigcup_{k \ge 0} \Lambda^i\left(J^k(\pi)\right)$$

and consider the graded algebra $\Lambda^*(\pi) = \sum_{i=0}^\infty \Lambda^i(\pi)$ of differential forms on $J^\infty(\pi)$.

A *vector field* X on $J^\infty(\pi)$ is an $\mathbb{R}$-linear mapping $X \colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ satisfying the Leibniz rule $X(\varphi\psi) = \varphi X(\psi) + \psi X(\varphi)$, for any φ, $\psi \in \mathcal{F}(\pi)$, and preserving the filtration in the algebra $\mathcal{F}(\pi)$, i.e., fulfilling the condition that there exists $r \ge 0$ such that

$$X\left(\mathcal{F}_k(\pi)\right) \subset \mathcal{F}_{k+r}(\pi)$$

for each $k \ge 0$. Denote by $D(\pi)$ the set of all vector fields on $J^\infty(\pi)$. Obviously, $D(\pi)$ is an $\mathcal{F}(\pi)$-module and a Lie algebra over $\mathbb{R}$. Locally, each $X \in D(\pi)$ can be represented as

$$X = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_{j,\sigma} b^j_\sigma \frac{\partial}{\partial p^j_\sigma},$$

where $b^j_\sigma \in \mathcal{F}_{|\sigma|+r}(\pi)$ for some r.

Let $\xi_i \colon F_i \to J^\infty(\pi)$, $i = 1, 2$, be vector bundles over $J^\infty(\pi)$, $P_i = \Gamma(\xi_i)$ be $\mathcal{F}(\pi)$-modules of sections. An $\mathbb{R}$-linear differential operator

$\Delta\colon P_1 \to P_2$ is called $\mathcal{C}$*-differential* if it can be restricted to the manifolds of the form $[f]^\infty$, $f \in \Gamma(\pi)$. That is

$$\Delta(\varphi)|_{[f]^\infty} = 0 \text{ when } \varphi|_{[f]^\infty} = 0,\ \varphi \in P_1.$$

If vector bundles ξ_1, ξ_2 are finite dimensional, $m_i = \dim \xi_i$, then in local coordinates each $\mathcal{C}$-differential operator Δ can be represented as an $m_2 \times m_1$ matrix

$$\Delta = \begin{pmatrix} \sum_\sigma a^\sigma_{11} D_\sigma & \dots & \sum_\sigma a^\sigma_{1m_1} D_\sigma \\ \multicolumn{3}{c}{\dotfill} \\ \sum_\sigma a^\sigma_{m_21} D_\sigma & \dots & \sum_\sigma a^\sigma_{m_2m_1} D_\sigma \end{pmatrix},$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$, $D_\sigma = (D_1)^{\sigma_1} \circ \dots \circ (D_n)^{\sigma_n}$, and

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} p^j_{\sigma+1_i} \frac{\partial}{\partial p^j_\sigma}, \quad i = 1, \dots, n,$$

is the *i-th total derivative*. The set of $\mathcal{C}$-differential operators from P_1 to P_2 is clearly an $\mathcal{F}(\pi)$-module and is denoted by $\mathcal{C}\mathrm{Diff}(P_1, P_2)$.

In any module $\mathcal{C}\mathrm{Diff}(Q, R)$, there exists a filtration by the modules $\mathcal{C}\mathrm{Diff}^{(k)}(Q, R)$ consisting of $\mathcal{C}$-differential operators of order $\leq k$. Consider the *module of $\mathcal{C}$-symbols*

$$\begin{aligned} \mathcal{C}\mathrm{smbl}(Q, R) &= \sum_{k=0}^{\infty} \mathcal{C}\mathrm{smbl}^{(k)}(Q, R), \\ \mathcal{C}\mathrm{smbl}^{(k)}(Q, R) &= \mathcal{C}\mathrm{Diff}^{(k)}(Q, R)/\mathcal{C}\mathrm{Diff}^{(k-1)}(Q, R) \\ &= S^k(\overline{\Lambda}^1)^* \otimes Q^* \otimes R. \end{aligned}$$

By definition, one has the projections

$$\mathcal{C}\mathrm{smbl}^{(k)}\colon\ \mathcal{C}\mathrm{Diff}^{(k)}(Q, R) \to \mathcal{C}\mathrm{smbl}^{(k)}(Q, R).$$

For each $\Delta \in \mathcal{C}\mathrm{Diff}(Q, R)$, $\Delta \neq 0$, define the *order* of Δ by

$$\operatorname{ord} \Delta = \min\{k \mid \Delta \in \mathcal{C}\mathrm{Diff}^{(k)}(Q, R)\},$$

and the $\mathcal{C}$*-symbol* $s_0(\Delta) = \mathcal{C}\mathrm{smbl}^{(\operatorname{ord}\Delta)}(\Delta)$.

The representative object for the functor $\mathcal{C}\mathrm{Diff}^{(k)}(P, \cdot)$, $0 \leq k \leq \infty$, is called the *module of horizontal k-jets* and is denoted by $\overline{\mathcal{J}}^k(P)$. By definition, we have

$$\mathcal{C}\mathrm{Diff}^{(k)}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(\overline{\mathcal{J}}^k(P), Q).$$

For any $\mathcal{C}$-differential operator $\Delta\colon P \to Q$ denote by $\overline{\mathcal{J}}^k(\Delta)\colon \overline{\mathcal{J}}^k(P) \to Q$ the corresponding homomorphism.

The filtration in the module $\mathcal{C}\mathrm{Diff}(P, Q)$ yields projections

$$\overline{\pi}_{k,k-1}\colon \overline{\mathcal{J}}^k(P) \to \overline{\mathcal{J}}^{k-1}(P), k > 0,$$

and the following sequence is exact

$$0 \to S^k\overline{\Lambda}^1 \otimes P \to \overline{\mathcal{J}}^k(P) \to \overline{\mathcal{J}}^{k-1}(P) \to 0.$$

If $\Delta\colon P \to Q$ is a $\mathcal{C}$-differential operator of order k, then $s_0(\Delta) \in S^k(\overline{\Lambda}^1)^* \otimes Q^* \otimes P$ is an $\mathcal{F}$-homomorphism

$$s_0(\Delta)\colon S^k\overline{\Lambda}^1 \otimes P \to Q.$$

The *l-th prolongation* $s_l(\Delta)$ of the $\mathcal{C}$-symbol $s_0(\Delta)$ is, by definition, the composition

$$s_l(\Delta)\colon S^{k+l}\overline{\Lambda}^1 \otimes P \xrightarrow{\sigma\otimes 1} S^l\overline{\Lambda}^1 \otimes S^k\overline{\Lambda}^1 \otimes P \xrightarrow{1\otimes s_0(\Delta)} S^l\overline{\Lambda}^1 \otimes Q,$$

where $\sigma\colon S^{k+l}\overline{\Lambda}^1 \to S^l\overline{\Lambda}^1 \otimes S^k\overline{\Lambda}^1$ is the natural inclusion.

Let $\xi_i\colon F_i \to M$, $i = 1, 2$, be vector bundles, $P_i = \Gamma(\xi_i)$, and $\Delta\colon P_1 \to P_2$ be an $\mathbb{R}$-linear differential operator, $\Delta \in \mathrm{Diff}(P_1, P_2)$. By definition, put $\overline{P_i} = \mathcal{F}(\pi) \otimes_{C^\infty(M)} P_i = \Gamma\left(\pi_\infty^*(\xi_i)\right)$, where $\pi_\infty^*(\xi_i)$ is the induced vector bundle over $J^\infty(\pi)$. A unique $\mathcal{C}$-differential operator $\overline{\Delta}\colon \overline{P_1} \to \overline{P_2}$ such that $\overline{\Delta}(1 \otimes p) = 1 \otimes \Delta(p)$ is called the *lifting of* Δ.

Consider a vector field $X \in D(M)$ on M. It is a first-order differential operator acting from $C^\infty(M)$ to $C^\infty(M)$. Then the lifting $\overline{X} \in \mathcal{C}\mathrm{Diff}\left(\mathcal{F}(\pi), \mathcal{F}(\pi)\right)$ is a vector field on $J^\infty(\pi)$. Consider the submodule $\mathcal{C}D(\pi) \subset D(\pi)$ generated by vector fields of the form $\overline{X}$. Thus we have a distribution on $J^\infty(\pi)$ which is called the *Cartan distribution.* The Cartan distribution is completely integrable in the sense that it satisfies the Frobenius integrability condition

$$[\mathcal{C}D(\pi), \mathcal{C}D(\pi)] \subset \mathcal{C}D(\pi). \tag{1}$$

In local coordinates, if $X = \sum_i a_i\partial/\partial x_i$, $a_i \in C^\infty(M)$, then $\overline{X} = \sum_i a_i D_i$, and

$$\mathcal{C}D(\pi) = \Big\{ \sum_i \varphi_i D_i \mid \varphi_i \in \mathcal{F}(\pi)\Big\}.$$

A vector field $X \in D(\pi)$ is called vertical if $X(\varphi) = 0$ for each $\varphi \in C^\infty(M) \subset \mathcal{F}(\pi)$. Denote by $D^V(\pi) \subset D(\pi)$ the $\mathcal{F}(\pi)$-module of vertical vector fields. The module D^V is also a subalgebra of the Lie algebra $D(\pi)$,

$$[D^V(\pi), D^V(\pi)] \subset D^V(\pi), \tag{2}$$

and $D(\pi)$ splits into the direct sum

$$D(\pi) = \mathcal{C}D(\pi) \oplus D^V(\pi).$$

Dually, the module of 1-forms on $J^\infty(\pi)$ splits into the direct sum

$$\Lambda^1(\pi) = \mathcal{C}^1\Lambda^1(\pi) \oplus \overline{\Lambda}^1(\pi), \tag{3}$$

where

$$\overline{\Lambda}^1(\pi) = \big\{\omega \in \Lambda(\pi) \mid \omega(X) = 0 \text{ for any } X \in D^V(\pi)\big\}$$

is the module of *horizontal* 1*-forms*, and

$$\mathcal{C}^1\Lambda^1(\pi) = \{\omega \in \Lambda(\pi) \mid \omega(X) = 0 \quad \text{for any} \quad X \in \mathcal{C}D(\pi)\}$$

is the module of *Cartan forms*.

Locally,

$$\overline{\Lambda}^1(\pi) = \Big\{ \sum_i \varphi_i dx_i \mid \varphi_i \in \mathcal{F}(\pi) \Big\},$$

$$\mathcal{C}^1\Lambda^1(\pi) = \Big\{ \sum_{j,\sigma} \varphi_j^\sigma \omega_\sigma^j \mid \varphi_j^\sigma \in \mathcal{F}(pi) \Big\},$$

where $\omega_\sigma^j = dp_\sigma^j - \sum_i p_{\sigma+1_i}^j dx_i$.

From (3) it follows that each $\Lambda^i(\pi)$, $i > 0$, splits into

$$\Lambda^i(\pi) = \sum_{\alpha+\beta=i} \overline{\Lambda}^\alpha(\pi) \otimes \mathcal{C}^\beta\Lambda^1(\pi),$$

where

$$\overline{\Lambda}^\alpha(\pi) = \underbrace{\overline{\Lambda}^1(\pi) \wedge \cdots \wedge \overline{\Lambda}^1(\pi)}_{\alpha \text{ times}}, \quad \mathcal{C}^\beta\Lambda^1(\pi) = \underbrace{\mathcal{C}^1\Lambda^1(\pi) \wedge \cdots \wedge \mathcal{C}^1\Lambda^1(\pi)}_{\beta \text{ times}}.$$

The graded algebra $\overline{\Lambda}^*(\pi)$ is the lifting of the graded algebra $\Lambda^*(M)$ of differential forms on M. The lifting of the de Rham differential $d\colon \Lambda^i(M) \to \Lambda^{i+1}(M)$ is called the *horizontal differential* and is denoted by $\overline{d}\colon \overline{\Lambda}^i(\pi) \to \overline{\Lambda}^{i+1}(\pi)$. The complex

$$0 \to \mathcal{F}(\pi) \xrightarrow{\overline{d}} \overline{\Lambda}^1(\pi) \xrightarrow{\overline{d}} \cdots \xrightarrow{\overline{d}} \overline{\Lambda}^n(\pi) \to 0$$

is called the *horizontal de Rham complex* and its cohomology at the term $\overline{\Lambda}^i(\pi)$ is denoted by $\overline{H}^i(\pi)$. In local coordinates

$$\overline{d}(dx_i) = 0, \quad \overline{d}(\varphi) = \sum_i D_i(\varphi)\, dx_i, \ \varphi \in \mathcal{F}(\pi).$$

Let $V(\pi)\colon V(E) \to E$ be the vector bundle of vertical vector fields on E, $V(E) = \{v \in T(E) \mid \pi_* v = 0\}$. By definition, put $\varkappa = \overline{\Gamma(V(\pi))} = \Gamma\left(\pi^*_{\infty,0}(V(\pi))\right)$. Then there exists a map

$$\mathcal{Э}\colon \varkappa \to D^V(\pi), \quad \varphi \mapsto \mathcal{Э}_\varphi,$$

where $\mathcal{Э}_\varphi$ is called an *evolutionary derivation* and is defined by the formula

$$\mathcal{Э}_\varphi(\psi)|_{[f]^\infty} = \left.\frac{d}{dt}\right|_{t=0} \left(\psi|_{[f_t]^\infty} \right),$$

where $f \in \Gamma(\pi)$, $\varphi \in \varkappa$, $\psi \in \mathcal{F}(\pi)$, and f_t is a 1-parameter family of sections of π such that $d/dt|_{t=0} f_t = \varphi|_{[f]^\infty}$, $f_0 = f$.

In local coordinates, if $\varphi = (\varphi^1, \ldots, \varphi^m)$, then

$$\mathcal{Э}_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial p_\sigma^j}.$$

2.2. **Differential operators and equations.** A *system of nonlinear differential equations* of order k imposed on sections of $\pi\colon E \to M$ is a submanifold $\mathcal{Y}^k \subset J^k(\pi)$. Denote by $\mathcal{Y}^{k+s} \subset J^{k+s}(\pi)$ the s-th prolongation of $\mathcal{Y}^k$. We shall always assume that $\mathcal{Y}^k$ is formally integrable. Then $\mathcal{Y}^s$, $s \geq k$, is a smooth manifold, and $\pi_{s,t}$, $s \geq t \geq k$, maps $\mathcal{Y}^s$ onto $\mathcal{Y}^t$ surjectively. The inverse limit of the system of maps

$$\pi_{s,t}\colon \mathcal{Y}^s \to \mathcal{Y}^t, \quad s \geq t \geq k,$$

is called an *infinitely prolonged differential equation*, or simply a *differential equation*, and is denoted by $\mathcal{Y} = \mathcal{Y}^\infty$. In a trivial way, the infinite jet manifold $J^\infty(\pi)$ is a differential equation of zero order with $\mathcal{Y}^k = J^k(\pi)$, $k \geq 0$.

One can construct a calculus on a differential equation $\mathcal{Y} \subset J^\infty(\pi)$ in the same way as for the jet manifold $J^\infty(\pi)$. Let $\mathcal{F}, \Lambda^*, D$ denote the algebra of smooth functions, the graded algebra of differential forms and the module of vector fields on $\mathcal{Y}$ respectively.

As for the jet manifold $J^\infty(\pi)$, there exists the splitting of the modules of vector fields D and 1-form Λ^1

$$D = D^V \oplus \mathcal{C}D, \quad \Lambda^1 = \overline{\Lambda}^1 \oplus \mathcal{C}^1\Lambda^1. \tag{4}$$

Let $\xi\colon E' \to M$ be a vector bundle, $F\colon \Gamma(\pi) \to \Gamma(\xi)$ be a nonlinear differential operator. The operator F can be considered as a section of the induced vector bundle $\pi_\infty^*(\xi)$ over $J^\infty(\pi)$. Define a smooth map

$$J(F)\colon J^\infty(\pi) \to J^\infty(\xi), \qquad [f]_x^\infty \mapsto [F(f)]_x^\infty.$$

Let $\mathcal{Y} = \mathcal{Y}(F)$ be the differential equation defined by F,

$$\mathcal{Y}(F) = \ker J(F) = \{\theta \in J^\infty(\pi) \mid J(F)(\theta) = 0\}.$$

In local coordinates, if $F = (F^1, \dots, F^s)$, $F^l = F^l(x_i, p_\sigma^j)$, then $\mathcal{Y}(F)$ is the infinite prolongation of the system

$$\begin{cases} F^1 = 0, \\ \dots\dots\dots \\ F^s = 0. \end{cases}$$

Denote by $P = \Gamma\left(\pi_\infty^*(\xi)\right)$ the $\mathcal{F}(\pi)$-module of sections of the induced vector bundle over $J^\infty(\pi)$. The *universal linearization of* F is a $\mathcal{C}$-differential operator $\ell_F \in \mathcal{C}\mathrm{Diff}(\varkappa, P)$ such that

$$\ell_F(\varphi)|_{[f]^\infty} = \left.\frac{d}{dt}\right|_{t=0} F(f_t), \quad \varphi \in \varkappa, f \in \Gamma(\pi),$$

where f_t is a 1-parameter family of sections of π such that $d/dt|_{t=0} f_t = \varphi|_{[f]^\infty}$, $f_0 = f$. In local coordinates,

$$\ell_F = \begin{pmatrix} \sum_\sigma \dfrac{\partial F^1}{\partial p^1_\sigma} D_\sigma & \dots & \sum_\sigma \dfrac{\partial F^1}{\partial p^m_\sigma} D_\sigma \\ \dots & \dots & \dots \\ \sum_\sigma \dfrac{\partial F^s}{\partial p^1_\sigma} D_\sigma & \dots & \sum_\sigma \dfrac{\partial F^s}{\partial p^m_\sigma} D_\sigma \end{pmatrix}.$$

Denote by $I_{\mathcal{Y}} \subset \mathcal{F}(\pi)$ the ideal of functions vanishing on $\mathcal{Y}$. For each $\mathcal{F}(\pi)$-module R denote by $[R]$ the *restriction of* R *to* $\mathcal{Y}$, $[R] = R/(I_{\mathcal{Y}} \cdot R)$. If R is a projective $\mathcal{F}(\pi)$-module, then $[R]$ is a projective $\mathcal{F}$-module as well.

A differential equation $\mathcal{Y} = \mathcal{Y}(F)$ is called *regular* with respect to the nonlinear differential operator $F = (F^1, \dots, F^s)$ if

1. The ideal $I_{\mathcal{Y}}$ is generated by the functions $D_\sigma(F^l)$, $1 \le l \le s$, $0 \le |\sigma| < \infty$;
2. The module of 1-forms Λ^1 is projective.

This definition does not depend on the choice of coordinates.

If the $\mathcal{F}$-module Λ^1 is projective, then from (4) it follows that modules $\overline{\Lambda}^1$ and $\mathcal{C}^1\Lambda^1$ are also projective. From now on we assume that all modules under consideration are projective.

Let $i\colon \mathcal{Y} \to J^\infty(\pi)$ be the natural inclusion, $i^*\colon [\mathcal{C}^1\Lambda^1(\pi)] \to \mathcal{C}^1\Lambda^1$ be the induced homomorphism, $[\ell_F]\colon [\varkappa] \to [P]$ be the restriction of the universal linearization to $\mathcal{Y}$.

2.3. **Adjoint operator.** Let $\mathcal{Y}$ be a differential equation. For any $\mathcal{F}$-module Q consider the following complex $(\mathcal{S}(Q), \overline{d}_1)$:

$$0 \to \mathcal{C}\mathrm{Diff}(Q, \mathcal{F}) \xrightarrow{\overline{d}_1} \mathcal{C}\mathrm{Diff}(Q, \overline{\Lambda}^1) \xrightarrow{\overline{d}_1} \cdots \xrightarrow{\overline{d}_1} \mathcal{C}\mathrm{Diff}(Q, \overline{\Lambda}^n) \to 0, \quad (5)$$

where $\overline{d}_1(\Delta) = -\overline{d} \circ \Delta$. The cohomology of complex (5) is described by

Proposition 1 ([13])**.** *1.* $H^i(\mathcal{S}(Q)) = 0$ *if* $i \neq n$;
2. $H^n(\mathcal{S}(Q)) = \mathrm{Hom}_{\mathcal{F}}(Q, \overline{\Lambda}^n)$.

By definition, put $\widehat{Q} = \mathrm{Hom}_{\mathcal{F}}(Q, \overline{\Lambda}^n)$ for any module Q.

Each $\mathcal{C}$-differential operator $\nabla\colon Q_1 \to Q_2$ induces a homomorphism of complexes

$$\mathcal{S}(\nabla)\colon \mathcal{S}(Q_2) \to \mathcal{S}(Q_1),$$

$\mathcal{S}(\nabla)(\Delta) = \Delta \circ \nabla$, $\Delta \in \mathcal{C}\mathrm{Diff}(Q_2, \overline{\Lambda}^i)$, and an $\mathbb{R}$-linear map of the cohomology

$$\nabla^*\colon \widehat{Q_2} \to \widehat{Q_1}.$$

The operator ∇^* is called the *adjoint operator* for ∇.

Proposition 2 ([13])**.** *1.* $\nabla^* \in \mathcal{C}\mathrm{Diff}(\widehat{Q_2}, \widehat{Q_1})$.

2. *For all* $\nabla_1 \in \mathcal{C}\mathrm{Diff}(Q_1, Q_2)$, $\nabla_2 \in \mathcal{C}\mathrm{Diff}(Q_2, Q_3)$,
$$(\nabla_2 \circ \nabla_1)^* = \nabla_1^* \circ \nabla_2^*.$$
3. *If in local coordinates* ∇ *is an* $m_1 \times m_2$*-matrix*
$$\nabla = \left\| \sum_\sigma a_{ij}^\sigma D_\sigma \right\|,$$
then ∇^* *is the* $m_2 \times m_1$*-matrix*
$$\nabla^* = \left\| \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_{ji}^\sigma \right\|.$$

2.4. **The $\mathcal{C}$-spectral sequence.** Consider the $\mathcal{C}$*-filtration* in the de Rham complex on an equation $\mathcal{Y}$
$$\Lambda = \mathcal{C}^0\Lambda \supset \mathcal{C}^1\Lambda \supset \mathcal{C}^2\Lambda \supset \cdots,$$
where
$$\mathcal{C}^p\Lambda = \sum_{\substack{\alpha \geq p \\ \beta \geq 0}} \mathcal{C}^\alpha\Lambda^1 \otimes \overline{\Lambda}^\beta.$$
The spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ determined by this filtration is called the $\mathcal{C}$*-spectral sequence* for the differential equation $\mathcal{Y}$. As usual, p denotes the filtration and $p+q$ denotes the degree.

Proposition 3 ([13])**.** *1.* $E_0^{p,q} = \mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^q$;
2. $E_0^{0,q} = \overline{\Lambda}^q$, $d_0^{0,q} = \overline{d}$;
3. *The* $\mathcal{C}$*-spectral sequence converges and the term* $E_\infty = \{E_\infty^{p,q}\}$ *is attached to* $H^*_{\text{de Rham}}(\mathcal{Y})$.

Keeping in mind Statement 2 of Proposition 3, denote
$$d_0^{p,q} = \overline{d}\colon \mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^q \to \mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^{q+1}.$$
If $\mathcal{Y} = J^\infty(\pi)$, then in local coordinates $\overline{d}$ is uniquely defined by the following equalities
$$\overline{d}(f) = \sum_i D_i(f)\, dx_i, \; f \in \mathcal{F};$$
$$\overline{d}(dx_i) = 0, \; dx_i \in \overline{\Lambda}^1;$$
$$\overline{d}(\omega_\sigma^j) = d\omega_\sigma^j = \sum_i dx_i \wedge \omega_{\sigma+1_i}^j, \; \omega_\sigma^j \in \mathcal{C}^1\Lambda^1.$$

2.5. **The term E_1 of the Vinogradov $\mathcal{C}$-spectral sequence.** Consider a sequence of $\mathcal{C}$-differential operators
$$P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3, \quad \operatorname{ord}\Delta_1 = k_1, \operatorname{ord}\Delta_2 = k_2. \tag{6}$$
Sequence (6) is called *formally exact* if for any $s \geq 0$ the sequence of homomorphisms
$$\overline{\mathcal{J}}^{k_1+k_2+s}(P_1) \xrightarrow{\overline{\mathcal{J}}(\Delta_1)} \overline{\mathcal{J}}^{k_2+s}(P_2) \xrightarrow{\overline{\mathcal{J}}(\Delta_2)} \overline{\mathcal{J}}^s(P_3)$$

is exact.

The following lemma is a fact from the formal theory of differential equations.

Lemma 1. *Consider a sequence of differential operators*

$$P_0 \xrightarrow{\Delta_0} P_1 \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_{l-1}} P_l \to P_{l+1} = 0, \quad \operatorname{ord} \Delta_i = k_i. \tag{7}$$

Suppose that for any $m = 1, \dots, l$ and $i \geq 0$ the sequence of the prolongations of the $\mathcal{C}$-symbols

$$S^{k_{m-1}+i}\overline{\Lambda}^1 \otimes P_{m-1} \xrightarrow{s_i(\Delta_{m-1})} S^i\overline{\Lambda}^1 \otimes P_m \xrightarrow{s_{i-k_m}(\Delta_m)} S^{i-k_m}\overline{\Lambda}^1 \otimes P_{m+1}$$

is exact, where $s_i(\Delta_m) = 0$ and $S^i\overline{\Lambda}^1 = 0$ if $i < 0$. Then sequence (7) is formally exact.

The following proposition estimates the term E_1 of the $\mathcal{C}$-spectral sequence.

Proposition 4 ([5])**.** *Let $\mathcal{Y} = \mathcal{Y}(F)$ be a regular differential equation. Assume that there exists a formally exact sequence of $\mathcal{C}$-differential operators*

$$P_0 = [\varkappa] \xrightarrow{[\ell_F]} P_1 \xrightarrow{\Delta_1} \cdots \xrightarrow{\Delta_{k-1}} P_k \to 0.$$

Then

1. *$E_1^{p,q} = 0$ if $p > 0$ and $q < n - k$;*
2. *For any $p > 0$ there exists a complex of $\mathcal{C}$-differential operators*

$$0 \to \mathcal{C}^{p-1}\overline{\Lambda}^1 \otimes \widehat{P_k} \xrightarrow{(\Delta^*_{k-1})_{p-1}} \mathcal{C}^{p-1}\overline{\Lambda}^1 \otimes \widehat{P_{k-1}} \xrightarrow{(\Delta^*_{k-2})_{p-1}} \cdots \xrightarrow{[\ell^*_F]_{p-1}} \mathcal{C}^{p-1}\overline{\Lambda}^1 \otimes [\widehat{\varkappa}] \to 0 \tag{8}$$

and the term $E_1^{p,q}$ is a direct summand in the cohomology group of complex (8) at term $\mathcal{C}^{p-1}\overline{\Lambda}^1 \otimes \widehat{P_{n-q}}$.

3. The "three-line" theorem for the $\mathcal{C}$-spectral sequence of the Yang–Mills equations

3.1. Let V be an oriented $\mathbb{R}$-linear space, $\dim V = n$, and $\eta\colon V \times V \to \mathbb{R}$ be a nondegenerate metric. Denote by $H\colon \Lambda^k \to \Lambda^{n-k}$, $k = 0, \dots, n$, the Hodge isomorphism. If $e_1, \dots, e_n$ is a positively oriented orthonormal basis, $\eta(e_i, e_j) = 0$ if $i \neq j$ and $\eta(e_i, e_i) = \eta_{ii} = \pm 1$, then

$$H(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \eta_{i_1 i_1} \dots \eta_{i_k i_k} \epsilon_{i_1 \dots i_k j_1 \dots j_{n-k}} e_{j_1} \wedge \cdots \wedge e_{j_{n-k}},$$

where $\{j_1, \dots, j_{n-k}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$.

Consider the following d-complex

$$0 \to S^l V \xrightarrow{d} S^{l-1}V \otimes V \xrightarrow{d} S^{l-2}V \otimes \Lambda^2 V \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^l V \to 0,$$

$$d(f \otimes \theta) = \sum_{i=1}^{n} \frac{\partial f}{\partial e_i} \otimes e_i \wedge \theta, \quad f \in S^k V, \theta \in \Lambda^{l-k} V, 0 \leq k \leq n,$$

which is the de Rham complex with polynomial coefficients.

By definition, put

$$\delta = (1 \otimes H) \circ d \circ (1 \otimes H) \colon S^k V \otimes \Lambda^i V \to S^{k-1} V \otimes \Lambda^{i-1} V$$

and consider the δ-complex

$$0 \xrightarrow{S} {}^{l} V \otimes \Lambda^n V \xrightarrow{\delta} S^{l-1} V \otimes \Lambda^{n-1} V \xrightarrow{\delta} \cdots \xrightarrow{\delta} S^{l-n} V \to 0,$$

which is clearly isomorphic to the d-complex. It is easy to see that

$$\delta(f \otimes \theta) = \eta_{11} \cdots \eta_{nn} \sum_{i=1}^{n} \eta_{ii} \frac{\partial f}{\partial e_i} \otimes \frac{\partial \theta}{\partial e_i}.$$

The following lemma is a fact from linear algebra.

Lemma 2. *The d-complex (resp. δ-complex) is acyclic if $l > 0$ (resp. $l > n$).*

Lemma 3. *The linear map*

$$d + \delta \colon S^l V \otimes \Lambda^{k-1} V \oplus S^l V \otimes \Lambda^{k+1} V \to S^{l-1} V \otimes \Lambda^k V$$

is an epimorphism.

Proof. Choose a positively oriented orthonormal basis $e_1, \ldots, e_n$ of V. Denote $\epsilon = \eta_{11} \cdots \eta_{nn}$. Consider an element $e_1^l \otimes \theta \in S^l V \otimes \Lambda^k V$. Obviously, $\theta = e_1 \wedge \theta' + \theta''$, where $\partial\theta'/\partial e_1 = \partial\theta''/\partial e_1 = 0$. Then we have

$$\begin{aligned} e_1^l \otimes \theta &= e_1^l \otimes e_1 \wedge \theta' + e_1^l \otimes \theta'' \\ &= d\Big(\frac{e_1^{l+1}}{l+1} \otimes \theta'\Big) + \delta\Big(\epsilon\eta_{11} \frac{e_1^{l+1}}{l+1} \otimes e_1 \wedge \theta''\Big), \end{aligned}$$

that is $e_1^l \otimes \theta \in \operatorname{im}(d + \delta)$.

By induction, suppose that $e_1^i f \otimes \theta \in \operatorname{im}(d + \delta)$, where $i = m, m + 1, \ldots, l$ and $\partial f/\partial e_1 = 0$. Consider $e_1^{m-1} f \otimes \theta$, $\partial f/\partial e_1 = 0$. As above

$$\begin{aligned} e_1^{m-1} f \otimes \theta &= e_1^{m-1} f \otimes e_1 \wedge \theta' + e_1^{m-1} f \otimes \theta'' \\ &= d\Big(\frac{e_1^m}{m} f \otimes \theta'\Big) - \frac{e_1^m}{m}\Big(\sum_{i=2}^{n} \frac{\partial f}{\partial e_i} \otimes e_i \wedge \theta'\Big) \\ &+ \delta\Big(\epsilon\eta_{11} \frac{e_1^m}{m} f \otimes e_1 \wedge \theta''\Big) - \frac{e_1^m}{m}\Big(\epsilon\eta_{11} \sum_{i=2}^{n} \eta_{ii} \frac{\partial f}{\partial e_i} \otimes e_1 \wedge \frac{\partial \theta''}{\partial e_i}\Big). \end{aligned}$$

Hence, by inductive hypothesis, $e_1^{m-1} f \otimes \theta \in \operatorname{im}(d+\delta)$. Thus $\operatorname{im}(d+\delta) = S^l V \otimes \Lambda^k V$ and the lemma is proved. $\square$

Corollary 1. *The following complex*

$$S^lV\otimes\Lambda^kV \xrightarrow{\Box} S^{l-2}V\otimes Lambda^{n-k}V \xrightarrow{d} S^{l-3}V\otimes\Lambda^{n-k+1}V \xrightarrow{d}\cdots$$
$$\xrightarrow{d} S^{l-2-k}V\otimes\Lambda^nV\to 0,$$

where $\Box = d\circ H\circ d$ *and* $0\le k\le n$, *is exact.*

3.2. **Yang–Mills equations.** We choose spacetime to be the manifold $M=\mathbb{R}^n$ with local coordinates x^i, $i=1,\dots,n$, equipped with a flat metric η. By definition, put $\partial_i=\partial/\partial x^i$. To define the Yang–Mills field consider a trivial fiber bundle over the spacetime with the structure group G, $\dim G=m$, $M\times G\to M$. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and τ^α, $\alpha=1,\dots,m$, be a basis of $\mathfrak{g}$ with the structure constants $c^\alpha_{\beta\gamma}$. Consider a trivial vector bundle $\xi\colon\mathfrak{g}\times M\to M$. Sections of this bundle can be identified with infinitesimal gauge transformation. Denote the $C^\infty(M)$-module of sections by $\Gamma(\xi)=\mathcal{G}$. In local coordinates, $\epsilon\in\mathcal{G}$ is written in the form $\epsilon=\epsilon^\alpha\tau_\alpha$, $\epsilon^\alpha\in C^\infty(M)$.

We can represent a connection on the principal fiber bundle by a 1-form A taking values in $\mathfrak{g}$. This 1-form is called the Yang–Mills field and is a section of a vector bundle $\pi\colon\xi\otimes T^*M\to M$ over M with the fiber $\mathfrak{g}\otimes T^*_{x_0}M$ at point $x_0\in M$. In local coordinates one has

$$A=A^\alpha_i dx^i\otimes\tau_\alpha.$$

Consider the infinite jets bundle $J^\infty(\pi)$. The module $\varkappa$ can be identified with

$$\varkappa=\overline{\Lambda}^1(\pi)\otimes\overline{\mathcal{G}},$$

where the $\mathcal{F}$-module $\overline{\mathcal{G}}$ is the lifting of the $C^\infty(M)$-module $\mathcal{G}$ up to $J^\infty(\pi)$. Then

$$\widehat{\varkappa}=\widehat{\overline{\Lambda}^1}(\pi)\otimes\overline{\mathcal{G}}^*=\overline{\Lambda}^{n-1}(\pi)\otimes\overline{\mathcal{G}}^*.$$

Using the Hodge isomorphism $H\colon\overline{\Lambda}^{n-1}(\pi)\to\overline{\Lambda}^1$ and the chosen basis τ_α of $\mathfrak{g}$, we identify

$$\varkappa=\widehat{\varkappa}=\overline{\Lambda}^1(\pi)\otimes\overline{\mathcal{G}}.$$

The curvature of a connection is a 2-form F on M with values in $\mathfrak{g}$, which is called the Yang–Mills field strength. In local coordinates,

$$F=F^\alpha_{ij}dx^i\wedge dx^j\otimes\tau_\alpha,\quad F^\alpha_{ij}=\partial_iA^\alpha_j-\partial_jA^\alpha_i+c^\alpha_{\beta\gamma}A^\beta_iA^\gamma_j.$$

The Yang–Mills field equations are given by a differential operator of the 2-nd order $\Phi\colon\xi\to\xi$ given in local coordinates as follows

$$\Phi(A)^\alpha_j=\partial^iF^\alpha_{ij}+c^\alpha_{\beta\gamma}A^{\beta i}F^\gamma_{ij},\quad \alpha=1,\dots,m, j=1,\dots,n. \qquad (9)$$

Equations (9) define an infinitely prolonged equation $\mathcal{Y}=\mathcal{Y}(\Phi)\subset J^\infty(\pi)$. Let $\ell_\Phi\colon\varkappa\to\varkappa$ be the universal linearization of Φ. Since equations (9) are Euler–Lagrange equations, the $\mathcal{C}$-differential operator $\ell^*_\Phi\colon\widehat{\varkappa}\to\widehat{\varkappa}$ in our identification $\varkappa=\widehat{\varkappa}$ coincides with ℓ_Φ.

An infinitesimal gauge transformation is given by a differential operator $R\colon \mathcal{G} \to \varkappa$. Let $\overline{R}\colon \overline{\mathcal{G}} \to \varkappa$ be the lifting of the operator R up to $J^\infty(\pi)$. In local coordinates the $\mathcal{C}$-differential operator $\overline{R}$ is written as follows

$$\overline{R}(\epsilon)_i^\alpha = D_i(\epsilon^\alpha) + c^\alpha_{\beta\gamma} A_i^\beta \epsilon^\gamma,$$

where $\epsilon = (\epsilon^\beta) \in \overline{\mathcal{G}}$, $\beta = 1, \dots, m$, $i = 1, \dots, n$, $D_i = \overline{\partial_i}$ is a total derivative. Since the Yang–Mills equations are invariant under infinitesimal gauge transformations, we have

$$[\ell_\Phi] \circ [\overline{R}] = 0, \tag{10}$$

where $[\cdot]$ denotes the restriction of a $\mathcal{C}$-differential operator to the equation $\mathcal{Y}$. From (10) one immediately obtain

$$[\overline{R}]^* \circ [\ell_\Phi]^* = [\overline{R}]^* \circ [\ell_\Phi] = 0. \tag{11}$$

Equality (11) is a corollary of the second Noether theorem. Thus one has the complex

$$\overline{\mathcal{G}} \otimes \overline{\Lambda}^1 \xrightarrow{[\ell_\Phi]} \overline{\mathcal{G}} \otimes \overline{\Lambda}^{n-1} \xrightarrow{[\overline{R}]^*} \overline{\mathcal{G}} \otimes \overline{\Lambda}^n \to 0. \tag{12}$$

Theorem 1. *Complex* (12) *is formally exact.*

Proof. Put $c^\alpha_{\beta\gamma} = 0$, that is consider the Abelian case. Clearly, the symbols of the $\mathcal{C}$-differential operators ℓ_Φ and $\overline{R}^*$ do not change. Then one easily obtains

$$\Phi(A)^\alpha_j = \partial^i \partial_i A^\alpha_j - \partial^i \partial_j A^\alpha_i, \quad \overline{R}(\epsilon)^\alpha_i = D_i(\epsilon^\alpha).$$

It is not difficult to see that the $\mathcal{C}$-symbols of the $\mathcal{C}$-differential operators ℓ_Φ and $\overline{R}^*$ are written as follows

$$s_0(\ell_\Phi) = 1 \otimes (d \circ H \circ d)\colon \overline{\mathcal{G}} \otimes S^2\overline{\Lambda}^1 \otimes \overline{\Lambda}^1 \to \overline{\mathcal{G}} \otimes \overline{\Lambda}^{n-1},$$
$$s_0(\overline{R}^*) = 1 \otimes d\colon \overline{\mathcal{G}} \otimes \overline{\Lambda}^1 \otimes \overline{\Lambda}^{n-1} \to \overline{\mathcal{G}} \otimes \overline{\Lambda}^n.$$

The theorem follows now from Corollary 1 and Lemma 1. □

Combining Theorem 1 and Proposition 4, one obtains the following theorem.

Theorem 2 (the "three-line" theorem)**.** $E_1^{p,q} = 0$ *if* $p > 0$ *and* $q < n - 2$.

ACKNOWLEDGEMENTS

The author is grateful to Professor A. M. Vinogradov for attention to this work and to A. M. Verbovetsky for useful discussions.

He also wishes to thank the Scuola Internatzionale Superiore di Studi Avanzati (SISSA) in Trieste, where this work was written, for warm hospitality and the Italian Ministero degli Affari Esteri for a fellowship.

References

[1] Anderson, I. M.: Introduction to the variational bicomplex, *Contemporary Mathematics, Mathematical Aspects of Classical Field Theory*, **132**, Amer. Math. Soc., Providence, R.I., 1992, 51–73.

[2] Barnich G., Brandt F., and Henneaux M.: Local BRST cohomology in the antifield formalism: I. General theorems, *Comm. Math. Phys.* **174** (1995) 57–92, hep-th/9405109.

[3] Bryant R. L. and Griffiths P. A.: Characteristic cohomology of differential systems (I): General theory, *J. Amer. Math. Soc.* **8** (1995) 507–596.

[4] Gessler D.: On the Vinogradov $\mathcal{C}$-spectral sequence for determined systems of differential equations, *Differential Geom. Appl.* **7** (1997) 303–324.

[5] Verbovetsky A.: Notes on the horizontal cohomology, In: M. Henneaux, J. Krasil'shchik, and A. Vinogradov (eds.) *Secondary Calculus and Cohomological Physics, Contemporary Mathematics*, **219**, Amer. Math. Soc., Providence, R.I., 1998, 211–231.

[6] Henneaux M. and Teitelboim C.: *Quantization of Gauge Systems*, Princeton University Press, 1992.

[7] Khorkova N. G.: On the $\mathcal{C}$-spectral sequence of differential equations, *Differential Geom. Appl.* **3** (1993) 219–243.

[8] Krasil'shchik I. S., Lychagin V. V., and Vinogradov, A. M.: *Geometry of Jet Spaces and Nonlinear Differential Equations*, Gordon and Breach, New York, 1986.

[9] Marvan M.: On the $\mathcal{C}$-spectral sequence with "general" coefficients, In: *Differential Geometry and Its Applications*, Proc. Conf., Aug. 27 – Sept. 2, 1989, Brno, Czechoslovakia, World Scientific, Singapore, 1990, 361–371.

[10] Tsujishita, T.: Homological method of computing invariants of systems of differential equations, *Differential Geom. Appl.* **1** (1991) 3–34.

[11] Vinogradov A. M.: A spectral sequence associated with a nonlinear differential equation, and algebro-geometric foundations of Lagrangian field theory with constraints, *Soviet Math. Dokl.* **19**, no. 1 (1978) 144–148.

[12] Vinogradov A. M.: Geometry of nonlinear differential equations, *Itogi nauki i tekniki, Problemy geometrii*, **11** (1980) 89–134 (Russian); English transl. in *J. Soviet Math.* **17** (1981) 1624–1649.

[13] Vinogradov A. M.: The $\mathcal{C}$-spectral sequence, Lagrangian formalism and conservation laws, *J. Math. Anal. Appl.* **100** (1984) 1–129.

[14] Vinogradov A.M., ed.: *Symmetries of Partial Differential Equations: Conservation Laws, Applications, Algorithms*, Kluwer Acad. Publ., the Netherlands, 1989.

[15] Vinogradov A. M.: From symmetries of partial differential equations towards secondary ("quantized") calculus, *J. Geom. Phys.* **14** (1994) 146–194.