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Coverings and Integrability of the Gauss–Mainardi–Codazzi Equations

Joseph Krasil′shchik, Michal Marvan

Abstract. Using covering theory approach (zero-curvature representations with the gauge group SL_2), we insert the spectral parameter into the Gauss–Mainardi–Codazzi equations in Tchebycheff and geodesic coordinates. For each choice, four integrable systems are obtained.

Introduction

When immersed in the Euclidean space E^3, surfaces with metric $g = g_{11}\,dx^2 + 2g_{12}\,dx\,dy + g_{22}\,dy^2$ and the second fundamental form $b = b_{11}\,dx^2 + 2b_{12}\,dx\,dy + b_{22}\,dy^2$ satisfy the Gauss–Mainardi–Codazzi equations (GMC) $R^k_{ijl} = b_{ij}b^k_l - b_{il}b^k_j$ and $b_{ij,k} = b_{ik,j}$, where R^k_{ijl} are components of the curvature tensor. By imposing an algebraic constraint of the form $L(g_{ij}, b_{ij}, x, y) = 0$ we obtain a special instance of what is called *reduced GMC equations*. Many reduced GMC systems have been found integrable in the sense of soliton theory, e.g., in works [1, 2, 3, 4, 5, 10, 13], thus leading to *integrable* classes of surfaces. An example is provided by the so-called *linear Weingarten surfaces* determined by a linear relation $\alpha K + \beta H = \gamma$ ($\alpha, \beta, \gamma = \mathrm{const}$) between their Gauss and main curvatures, see the recent book [15] and references therein.

In this paper we apply the methods of [11] to a related problem. Namely, we check GMC equations, written here in Tchebycheff and geodesic coordinates, for existence of a zero-curvature representation (ZCR) with coefficients in the complex Lie algebra sl_2. For each choice of the coordinates we have found four cases possessing a non-removable parameter.

An sl_2*-valued ZCR* is determined by an sl_2-valued form $A\,dx + B\,dy$ satisfying $A_{,y} - B_{,x} + [A, B] = 0$. A *gauge-equivalent* ZCR is given by $A'\,dx + B'\,dy$ with $A' = S_{,x}S^{-1} + SAS^{-1}$, $B' = S_{,y}S^{-1} + SBS^{-1}$ for a

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suitable SL_2-matrix S. Any ZCR gauge-equivalent to the zero form is called *trivial.* A non-removable parameter is a parameter that cannot be removed by a gauge transformation.

As it happens, the theory of zero-curvature representations is naturally formulated in terms of coverings [9]. To obtain a well-defined construction, one needs to consider a linear covering [16] endowed with an action of a Lie group G. Then a ZCR related to this covering is determined by a closed $\mathfrak{g}$-valued horizontal 1-form. To make exposition self-contained, we expose here a geometrical theory of ZCR based on the covering theory in the category of differential equations.

1. Linear coverings and zero-curvature representations

Let $\mathcal{O}$ be a smooth manifold (possibly, infinite-dimensional) with an integrable finite-dimensional distribution $\mathcal{C}$ of dimension n. Integrability is understood in the formal sense here: a distribution is said to be integrable, if the module of vector fields lying in this distribution is closed with respect to commutator. Recall (see [9]) that a *covering* over the pair $(\mathcal{O},\mathcal{C})$ is a locally trivial fiber bundle $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$ such that

1. The manifold $\tilde{\mathcal{O}}$ is endowed with an n-dimensional integrable distribution $\tilde{\mathcal{C}}$.
2. The mapping $d\tau\big|_{\tilde{\mathcal{C}}_{\tilde\theta}}\colon \tilde{\mathcal{C}}_{\tilde\theta} \to T_\theta\mathcal{O}$ is an isomorphism onto the plane $\mathcal{C}_{\tau(\tilde\theta)} \subset T_\theta\mathcal{O}$ for any $\tilde\theta \in \tilde{\mathcal{O}}$.

Our main concern will be with manifolds $\mathcal{O}$ of the form $\mathcal{E}^\infty$, where $\mathcal{E}^\infty$ is the infinite prolongation of a differential equation $\mathcal{E} \subset J^k(\pi)$, $J^k(\pi)$ being the manifold of k-jets for some locally trivial bundle $\pi\colon E \to M$. The corresponding distribution $\mathcal{C}$ is the *Cartan distribution* (see details in [8]).

Denote by $\mathrm{D}(N)$ the Lie algebra of vector fields on a smooth manifold N and by $\mathcal{C}\mathrm{D}(\mathcal{O}) \subset \mathrm{D}(\mathcal{O})$ the subalgebra of vector fields lying in the distribution $\mathcal{C}$. Then a covering structure in the bundle $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$ is identified with a flat $\mathcal{C}$*-connection* in τ, i.e., with a mapping $\nabla = \nabla^\tau\colon \mathcal{C}\mathrm{D}(\mathcal{O}) \to \mathrm{D}(\tilde{\mathcal{O}})$ such that

1. ∇ is an $\mathcal{F}(\mathcal{O})$-linear mapping, where $\mathcal{F}(\mathcal{O})$ is the algebra of smooth functions on $\mathcal{O}$.
2. For any $X \in \mathcal{C}\mathrm{D}(\mathcal{O})$, the field $\nabla_X \in \mathrm{D}(\tilde{\mathcal{O}})$ projects to X by $d\tau$.
3. For any X, $Y \in \mathcal{C}\mathrm{D}(\mathcal{O})$ one has

$$\nabla_{[X,Y]} = [\nabla_X, \nabla_Y]. \tag{1}$$

We shall also say that ∇ determines a *covering structure* in τ.

Two coverings $\tau_1\colon \tilde{\mathcal{O}}_1 \to \mathcal{O}$, $\tau_2\colon \tilde{\mathcal{O}}_2 \to \mathcal{O}$ with the connections ∇^1 and ∇^2 respectively are said to be *equivalent*, if there exists an equivalence φ of the bundles τ_1, τ_2, i.e., a diffeomorphism $\varphi\colon \tilde{\mathcal{O}}_1 \to \tilde{\mathcal{O}}_2$ satisfying $\tau_1 = \tau_2 \circ \varphi$, such that $d\varphi \circ \nabla^1 = \nabla^2$.

If $\mathcal{U} \subset \mathcal{O}$ is a trivialization of the bundle τ, i.e., a domain such that the bundle $\tau\big|_{\tau^{-1}(\mathcal{U})} : \tau^{-1}(\mathcal{U}) \to \mathcal{U}$ is trivial, then the covering structure over $\mathcal{U}$ is given by the splitting

$$\nabla_X = X + V_X, \quad V_X \in \mathrm{D}(\tau^{-1}(\mathcal{U})), \tag{2}$$

where V_X is a τ-vertical vector field, while the flatness condition is expressed in the form

$$V_{[X,Y]} = [X, V_Y] + [V_X, Y] + [V_X, V_Y]. \tag{3}$$

Let now τ be a vector bundle and $\mathcal{LF}(\tilde{\mathcal{O}})$ denote the $\mathcal{F}(\mathcal{O})$-module of fiber-wise linear smooth functions on $\tilde{\mathcal{O}}$.

Definition 1.1. Let $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$ be a vector bundle.

1. A covering $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$ is called *linear*, if the field ∇_X preserves the module $\mathcal{LF}(\tilde{\mathcal{O}})$ for any $X \in \mathcal{C}\mathrm{D}(\mathcal{O})$ (cf. [16], where the same concept is introduced, though without using the term "covering").
2. Two linear coverings are said to be *equivalent*, if there exists their equivalence φ, which is a morphism of vector bundles (i.e., is fiber-wise linear).

Let $\mathcal{U} \subset \mathcal{O}$ be a trivialization of the bundle τ and $w^1, \dots, w^r, \dots$ be coordinates along the fiber. Then a covering τ is linear if and only if the field V_X represents in the form

$$V_X = \sum_r \Big(\sum_s V_X^{rs} w^s \Big) \frac{\partial}{\partial w^r} \tag{4}$$

for any $X \in \mathcal{C}\mathrm{D}(\mathcal{O})$, where V_X^{rs} are smooth functions on $\mathcal{O}$ and all internal sums are finite. Thus, any linear covering is locally determined by the system of matrices $\left\| V_i^{rs} \right\|$, where $V_i^{rs} = V_{X_i}^{rs}$, X_i, $i = 1, \dots, n$, being a local basis of the distribution $\mathcal{C}$.

If an automorphism of the vector bundle τ is locally given by a matrix S, then the matrices V_X are transformed by the formula

$$V_X \mapsto X(S)S^{-1} + SV_X S^{-1}, \tag{5}$$

where $X(S)$ denotes the component-wise action.

Example 1.1. Let $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$ be a covering. Denote by $\mathcal{C}\Lambda^1(\mathcal{O}) \subset \Lambda^1(\mathcal{O})$ the $\mathcal{F}(\mathcal{O})$-submodule consisting of forms such that $\mathrm{i}_X\omega = 0$ for any $X \in \mathcal{C}\mathrm{D}(\mathcal{O})$. In a similar way, we obtain the $\mathcal{F}(\tilde{\mathcal{O}})$-submodule $\tilde{\mathcal{C}}\Lambda^1(\tilde{\mathcal{O}})$ in $\Lambda^1(\tilde{\mathcal{O}})$. Then, since $\tilde{\mathcal{C}}\mathrm{D}(\tilde{\mathcal{O}})$ projects to $\mathcal{C}\mathrm{D}(\mathcal{O})$ by τ_*, one has $\tau^*(\mathcal{C}\Lambda^1(\mathcal{O})) \subset \tilde{\mathcal{C}}\Lambda^1(\tilde{\mathcal{O}})$. Denote by $\mathcal{C}\Lambda^1(\tilde{\mathcal{O}}) \subset \tilde{\mathcal{C}}\Lambda^1(\tilde{\mathcal{O}})$ the $\mathcal{F}(\tilde{\mathcal{O}})$-submodule generated by $\tau^*(\mathcal{C}\Lambda^1(\mathcal{O}))$. Then $\mathcal{C}\Lambda^1(\tilde{\mathcal{O}})$ is stable with respect to the Lie action of vector fields lying in $\mathcal{C}\mathrm{D}(\tilde{\mathcal{O}})$ and we can extend this action to the quotient module $\tilde{\mathcal{C}}\Lambda^1(\tilde{\mathcal{O}})/\mathcal{C}\Lambda^1(\tilde{\mathcal{O}})$. One can easily see that the corresponding vector bundle $\tau_\ell\colon T^*_{\mathcal{C}}(\tilde{\mathcal{O}}) \to \tilde{\mathcal{O}}$ is endowed in this way with a linear covering structure.

Example 1.2. Consider a construction dual to that of Example 1.1. Let again $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$ be an arbitrary covering. In the tangent bundle $T\tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$, take the subbundle $\tau^\ell\colon T^{\mathrm{v}}\tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$ of *τ-vertical* tangent vectors. Obviously, the bundle τ^ℓ is dual to τ_ℓ. If a vector field $Y \in \mathrm{D}(\tilde{\mathcal{O}})$ is such that $d\tau Y_{\tilde{\theta}}$ lies in the plane $\mathcal{C}_\theta$, $\theta = \tau(\tilde{\theta})$, we can define its *vertical component* Y^{v} by setting

$$Y^{\mathrm{v}}_{\tilde{\theta}} := Y_{\tilde{\theta}} - \nabla(d\tau(Y_{\tilde{\theta}})),$$

at any point $\tilde{\theta} \in \tilde{\mathcal{O}}$. Take a τ-vertical vector field Z, i.e., a section of the bundle τ^ℓ, and a field $X \in \tilde{\mathcal{C}}\mathrm{D}(\tilde{\mathcal{O}})$. Then the field $[X, Z]$ possesses the above formulated property and the relation

$$\tilde{\nabla}_X(Z) := [X, Z]^{\mathrm{v}}$$

determines a covering structure in the projection $\tau^\ell\colon T^{\mathrm{v}}\tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$. This covering structure in the bundle τ^ℓ is linear and is called the *linearization* of the covering τ (cf. [7, 12]).

Example 1.3. Let A be a commutative $\Bbbk$-algebra, $\Bbbk$ being a commutative ring, and P be an A-module. Recall that a *derivation* of the module P is a linear mapping $\bar{X}\colon P \to P$ satisfying $\bar{X}(ap) = X(a)p + a\bar{X}(p)$ for all $a \in A$, $p \in P$, and some derivation $X\colon A \to A$ of the algebra A. Denote the A-module of all such derivations by $\mathrm{Der}(P)$. Obviously, $\mathrm{Der}(P)$ is a Lie $\Bbbk$-algebra with respect to the commutator.

In particular, let $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$ be a linear covering. Then, by definition, the covering structure in τ determines the homomorphism of Lie algebras $\nabla\colon \mathcal{C}\mathrm{D}(\mathcal{O}) \to \mathrm{Der}(\mathcal{LF}(\tilde{\mathcal{O}}))$, which splits the natural projection $\mathrm{Der}(\mathcal{LF}(\tilde{\mathcal{O}})) \to \mathrm{D}(\mathcal{O})$, $\bar{X} \mapsto X$. The kernel of this projection coincides with the module $\mathrm{End}(\mathcal{LF}(\tilde{\mathcal{O}})) = \Gamma(\mathrm{End}(\tau))$, where $\Gamma(\cdot)$ denotes the module of sections. This kernel is an ideal of the Lie algebra $\mathrm{Der}(\mathcal{LF}(\tilde{\mathcal{O}}))$. Thus, we obtain a linear covering structure in the bundle $\mathrm{End}(\tau)$.

In local coordinates, an endomorphism φ is represented by a matrix A_φ and the action of a field $X \in \mathcal{C}\mathrm{D}(\mathcal{O})$ on A_φ is expressed by the formula

$$X(A_\varphi) + [V_X, A_\varphi],$$

where $X(A_\varphi)$ is understood as the component-wise action.

Assume now that a Lie group G acts in the bundle τ by linear automorphisms of this bundle, i.e., a representation $\rho\colon G \to \mathrm{Aut}(\tau)$ is given. Let $\mathfrak{g}$ be the corresponding Lie algebra. Then the representation

ρ yields the following commutative diagram:

$$\begin{tikzcd} \mathcal{O} \times \mathfrak{g} \arrow[r, "\bar{\rho}"] \arrow[d, "\mathrm{pr}_{\mathcal{O}}"'] & T^{\mathrm{v}}\tilde{\mathcal{O}} \arrow[d, "\tau^{\ell}"] \\ \mathcal{O} & \tilde{\mathcal{O}} \arrow[l, "\tau"] \end{tikzcd}$$

Definition 1.2. Let ∇^0, ∇^1 be two linear covering structures in the vector bundle $\tau\colon \tilde{\mathcal{O}} \to \mathcal{O}$.

1. We say that the covering structures ∇^0 and ∇^1 differ by a *zero-curvature representation* (with the *gauge group* G, or with the *gauge algebra* $\mathfrak{g}$), if for any $X \in \mathcal{C}\mathrm{D}(\mathcal{O})$ there exists a $\mathfrak{g}$-valued function $g_X \in \Gamma(\mathrm{pr}_{\mathcal{O}})$ linear in X and such that
$$\nabla^1_X - \nabla^0_X = \bar{\rho} \circ g_X \circ \tau.$$
In other words, the vertical vectors ∇^1_X and ∇^0_X differ by an element of the Lie algebra $\mathfrak{g}$ at any point $\theta \in \mathcal{O}$.
2. Let ∇^0, ∇^1 and $\square^0$, $\square^1$ be two pairs satisfying the above definition. They are said to be *equivalent*, if there exists an element $g \in G$ such that $\rho(g)$ is an equivalence of ∇^0 to $\square^0$ and of ∇^1 to $\square^1$.

Thus, equivalence in the sense of the previous definition means that the matrices S in (5) belong to the group G.

Remark 1.1. In coordinate computations for particular differential equations $\mathcal{E}^\infty = \mathcal{O}$, the bundle τ is usually trivial, $\tau = \mathrm{pr}_{\mathcal{E}^\infty}\colon \mathcal{O} = \mathcal{E}^\infty \times \mathbb{R}^r \to \mathcal{E}^\infty$, and the structure ∇^0 is also trivial (i.e., is such that the matrices V_X vanish for all $X \in \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$, see eq. (4)). This should be the reason why it is never taken into consideration explicitly and is accepted "by default": only the covering ∇^1 is included into definition.

From now on we take $\mathcal{O} = \mathcal{E}^\infty$, fix a vector bundle $\tau\colon \tilde{\mathcal{E}}^\infty \to \mathcal{E}^\infty$, where $\mathcal{E} \subset J^k(\pi)$ is a differential equation, and assume that a Lie group G acts in τ in the above described way. Consider a covering structure ∇^0 in the bundle τ. Then ∇^0 determines a flat connection in the bundle $\tilde{\mathcal{E}}^\infty \xrightarrow{\tau} \mathcal{E}^\infty \xrightarrow{\pi_\infty} M$ extending the Cartan connection in $\pi_\infty\colon \mathcal{E}^\infty \to M$. This connection is uniquely determined by the corresponding connection form U^0 and the condition ∇^0 to be a covering structure is equivalent to the identity $[\![U^0, U^0]\!] = 0$, where $[\![\cdot\,,\cdot]\!] = 0$ is the *Frölicher–Nijenhuis bracket* (see [6] for more details).

If ∇^1 is another covering structure in τ, then the connection forms U^0 and U^1 differ by a horizontal vector-valued form $\omega_{01} = U^1 - U^0$. From [6] we immediately obtain the following

Proposition 1.1. *Two linear covering structures in the vector bundle τ satisfy Definition* 1.2 (1) *if and only if*

1. *The form ω_{01} is $\mathfrak{g}$-valued.*

2. The identity

$$\partial^0 = \omega_{01} + \tfrac{1}{2}[\![\omega_{01}, \omega_{01}]\!] = 0 \tag{6}$$

holds, where $\partial^0 = [\![\omega_{01}, \cdot]\!]$ *is the differential associated to* ∇^0 *by the Frölicher–Nijenhuis bracket.*

Remark 1.2. In the situation discussed in Remark 1.1, the action of the differential ∂^0 on the form ω_{01} coincides with the component-wise action of $-d_h$, where d_h is the *de Rham horizontal differential* on $\mathcal{E}^\infty$. Thus, eq. (6) can be rewritten as

$$d_h\omega_{01} = \tfrac{1}{2}[\![\omega_{01}, \omega_{01}]\!]$$

in this case.

2. Tchebycheff coordinates

This section contains results of computation of sl_2-valued ZCR's for reduced GMC equations in Tchebycheff coordinates. The method used is taken from [11] and shortly explained at the end of this section.

Fix an arbitrary system of *Tchebycheff coordinates* x, y. Then we have

$$\begin{aligned} g &= dx^2 + 2\cos f\, dx\, dy + dy^2, \\ b &= b_{11}\, dx^2 + 2b_{12}\, dx\, dy + b_{22}\, dy^2. \end{aligned}$$

The Gauss–Mainardi–Codazzi equations [14], eq. (74b,c), are

$$\begin{aligned} f_{,xy} &= \frac{b_{12}^2 - b_{11}b_{22}}{\sin f}, \\ b_{11,y} &= b_{12,x} + \frac{b_{22} - b_{12}\cos f}{\sin f} f_{,x}, \\ b_{12,y} &= b_{22,x} - \frac{b_{11} - b_{12}\cos f}{\sin f} f_{,y}. \end{aligned} \tag{7}$$

There always exists a nonparametric sl_2-valued zero-curvature representation (see [14], eq. (84)), derived from the Gauss–Weingarten equations. In Tchebycheff coordinates we have

$$\begin{aligned} A &= \frac{\mathrm{i}}{2}\begin{pmatrix} f_x & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{11} - b_{12}}{\sin f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{11} - b_{12}}{\sin f} & -f_x \end{pmatrix}, \\ B &= \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{12} - b_{22}}{\sin f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{12} - b_{22}}{\sin f} & 0 \end{pmatrix}. \end{aligned} \tag{8}$$

It is well known that this zero-curvature representation does not belong to any 1-parametric family. However, imposing one additional relation between the unknowns $x, y, f, b_{11}, b_{12}, b_{22}$ we obtain four distinct classes of surfaces admitting a 1-parametric ZCR in the above sense. The following proposition may be proved by straightforward computation. We obtained it by methods of [11].

Proposition 2.1. *Let $X_1(x), X_2(x), Y_1(y), Y_2(y)$ be arbitrary functions and Z be a constant. Denote*

$$
\begin{aligned}
L := {} & (b_{11}b_{22} - b_{12}^2)Z + Y_1 b_{11} + X_1 b_{22} \\
& + [(X_2 - Y_2)\sin f - (X_1 + Y_1)\cos f]b_{12} \\
& - (X_1Y_1 + X_2Y_2)\sin^2 f + (X_1Y_2 - X_2Y_1)\sin f \cos f.
\end{aligned}
\tag{9}
$$

If $L = 0$, then the matrices

$$
A = \frac{1}{2}\begin{pmatrix} \mathrm{i}f_x & \mathrm{i}\dfrac{\mathrm{e}^{\mathrm{i}f}b_{11} - b_{12}}{\sin f} + X_1 - \mathrm{i}X_2 \\ \mathrm{i}\dfrac{\mathrm{e}^{-\mathrm{i}f}b_{11} - b_{12}}{(Z+1)\sin f} - \dfrac{X_1 + \mathrm{i}X_2}{Z+1} & -\mathrm{i}f_x \end{pmatrix},
$$

$$
B = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i}\dfrac{\mathrm{e}^{\mathrm{i}f}b_{12} - b_{22}}{\sin f} + (Y_1 - \mathrm{i}Y_2)\mathrm{e}^{\mathrm{i}f} \\ \mathrm{i}\dfrac{\mathrm{e}^{-\mathrm{i}f}b_{12} - b_{22}}{(Z+1)\sin f} - \dfrac{Y_1 + \mathrm{i}Y_2}{Z+1}\mathrm{e}^{-\mathrm{i}f} & 0 \end{pmatrix}
$$

give a zero-curvature representation for equation (7).

Assume now that the functions X_1, X_2, Y_1, Y_2, Z depend on an auxiliary variable t. To be a possible parameter of the ZCR, t must not be a parameter of the equation (7) itself. In other words, the condition $L = 0$ must not depend on t.

If $\partial L/\partial t = 0$, then one easily derives from (9) that $X_1 = X_2 = Y_1 = Y_2 = Z = 0$ and the ZCR is the same as (8), without any parameter.

If $\partial L/\partial t - L/t = 0$, then $L(t) = tL(1)$, and the condition $L(t) = 0$ does not depend on t either. Combining with (9), we arrive at the classification given below. We use the notation

$$
K = \frac{b_{11}b_{22} - b_{12}^2}{\sin^2 f}
$$

for the Gauss curvature and

$$
H = \frac{b_{11} - 2b_{12}\cos f + b_{22}}{2\sin^2 f}
$$

for the mean curvature. Functions X_i, Y_i are as in the proposition above.

Case 1. $X_1 = Y_1 \neq 0$. This class corresponds to *linear Weingarten surfaces* determined by a linear relation

$$
K + 2\kappa H + \lambda = 0
$$

between the two curvatures. Here κ, λ are arbitrary constants.

Our t-parametrized zero-curvature representation is

$$A = \frac{1}{2}\begin{pmatrix} \mathrm{i} f_x & \mathrm{i}\,\dfrac{\mathrm{e}^{\mathrm{i}f} b_{11} - b_{12}}{\sin f} + \Delta^-(\kappa, \lambda, t) \\ \mathrm{i}\,\dfrac{\mathrm{e}^{-\mathrm{i}f} b_{11} - b_{12}}{(t+1)\sin f} - \dfrac{\Delta^+(\kappa, \lambda, t)}{t+1} & -\mathrm{i} f_x \end{pmatrix},$$

$$B = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i}\,\dfrac{\mathrm{e}^{\mathrm{i}f} b_{12} - b_{22}}{\sin f} + \Delta^-(\kappa, \lambda, t)\mathrm{e}^{\mathrm{i}f} \\ \mathrm{i}\,\dfrac{\mathrm{e}^{-\mathrm{i}f} b_{12} - b_{22}}{(t+1)\sin f} - \dfrac{\Delta^+(\kappa, \lambda, t)}{t+1}\mathrm{e}^{-\mathrm{i}f} & 0 \end{pmatrix},$$

where $\Delta^\pm(\kappa, \lambda, t) = \kappa t \pm \sqrt{\kappa^2 t^2 + \lambda t}$.

Case 2. $X_1 = Y_1 = 0$. If

$$K + 2\kappa \frac{b_{12}}{\sin f} - \lambda = 0,$$

where κ, λ are constants, then there is a t-parametrized zero-curvature representation

$$A = \frac{\mathrm{i}}{2}\begin{pmatrix} f_x & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{11} - b_{12}}{\sin f} - \Delta^+(\kappa, \lambda, t) \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{11} - b_{12}}{(t+1)\sin f} - \dfrac{\Delta^+(\kappa, \lambda, t)}{t+1} & -f_x \end{pmatrix},$$

$$B = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{12} - b_{22}}{\sin f} + \Delta^-(\kappa, \lambda, t)\mathrm{e}^{\mathrm{i}f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{12} - b_{22}}{(t+1)\sin f} + \dfrac{\Delta^-(\kappa, \lambda, t)}{t+1}\mathrm{e}^{-\mathrm{i}f} & 0 \end{pmatrix}.$$

Case 3a. $X_1 = 0$, $Y_1 \neq 0$. If

$$K \sin^2 f - Y_1 b_{11} + (Y_1 \cos f + Y_2 \sin f) b_{12} = 0,$$

then

$$A = \frac{\mathrm{i}}{2}\begin{pmatrix} f_x & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{11} - b_{12}}{\sin f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{11} - b_{12}}{(t+1)\sin f} & -f_x \end{pmatrix},$$

$$B = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{12} - b_{22}}{\sin f} + (\mathrm{i}Y_1 + Y_2) t \mathrm{e}^{\mathrm{i}f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{12} - b_{22}}{(t+1)\sin f} - \dfrac{\mathrm{i}Y_1 - Y_2}{t+1} t \mathrm{e}^{-\mathrm{i}f} & 0 \end{pmatrix}.$$

is a t-parametrized zero-curvature representation.

Case 3b. $X_1 \neq 0$, $Y_1 = 0$: The relation is

$$K \sin^2 f + (X_1 \cos f + X_2 \sin f) b_{12} - X_1 b_{22} = 0$$

and the t-parametrized zero-curvature representation is

$$A = \frac{\mathrm{i}}{2} \begin{pmatrix} f_x & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{11} - b_{12}}{\sin f} + (\mathrm{i}X_1 - X_2)t \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{11} - b_{12}}{(t+1)\sin f} - \dfrac{\mathrm{i}X_1 + X_2}{t+1} t & -f_x \end{pmatrix},$$

$$B = \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{12} - b_{22}}{\sin f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{12} - b_{22}}{(t+1)\sin f} & 0 \end{pmatrix}.$$

Cases 3a and 3b transform one to another under the $x \leftrightarrow y$ symmetry.

Case 4. $0 \neq X_1 \neq Y_1 \neq 0$: If

$$K \sin^2 f - Y_1 b_{11} + (X_1 \mathrm{e}^{\mathrm{i}f} + Y_1 \mathrm{e}^{-\mathrm{i}f}) b_{12} - X_1 b_{22} = 0,$$

then

$$A = \frac{\mathrm{i}}{2} \begin{pmatrix} f_x & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{11} - b_{12}}{\sin f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{11} - b_{12}}{(t+1)\sin f} - \dfrac{2\mathrm{i}X_1 t}{t+1} & -f_x \end{pmatrix},$$

$$B = \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & \dfrac{\mathrm{e}^{\mathrm{i}f} b_{12} - b_{22}}{\sin f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f} b_{12} - b_{22}}{(t+1)\sin f} - \dfrac{2\mathrm{i}Y_1 t}{t+1} \mathrm{e}^{-\mathrm{i}f} & 0 \end{pmatrix}.$$

is a t-parametrized zero-curvature representation.

Case 4 is incompatible with the real geometry of E^3. A formally real class is obtained after transformation $f \mapsto \mathrm{i}f$, $x \mapsto \mathrm{i}x$, and $y \mapsto \mathrm{i}y$.

Two natural questions arise: whether the parameter t cannot be removed by gauge transformation and whether there might be integrable cases outside this classification. To give our answers, we first recall some facts from [11].

In the simplest style, still applicable to the GMC equations, for a system of PDE $\{F^l = 0\}_{l=1}^N$ the *characteristic N-tuple* of a ZCR $A\,dx + B\,dy$ consists of sl_2 matrices C_l, $l = 1, \ldots, N$, satisfying $A_{,y} - B_{,x} + [A, B] = C_l F^l$. Then the gauge-equivalent ZCR with respect to a gauge matrix S has $\{SC_lS^{-1}\}_{l=1}^N$ as its characteristic N-tuple. Incidentally, in all four cases the characteristic triple is one and the same:

$$\frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{\mathrm{i}}{2\sin f}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}f} \\ \dfrac{\mathrm{e}^{-\mathrm{i}f}}{t+1} & 0 \end{pmatrix}, \quad \frac{-\mathrm{i}}{2\sin f}\begin{pmatrix} 0 & 1 \\ \dfrac{1}{t+1} & 0 \end{pmatrix}.$$

As one obviously cannot remove t by conjugation, t is a true parameter.

To give at least partial answer to our second question, we explain the assumptions we used in derivation of Proposition 2.1. By [11], the characteristic triple $C = \{C_1, C_2, C_3\}$ also satisfies

$$\widehat{R^*}(C) = 0, \tag{10}$$

where R^* is the formal adjoint to the operator of universal linearization, and $\widehat{R^*}$ is obtained from R^* by replacement of total derivatives D_x, D_y with "covariant total derivatives" $D_x - \mathrm{ad}_A$, $D_y - \mathrm{ad}_B$ (or, equivalently, by lifting the operator R^* to the corresponding covering space by means of the connection defining the covering). The procedure then consists in solving the equation $\widehat{R^*}(\bar{C}) = 0$ on unknowns $A, B, \bar{C}$, where $\bar{C}$ is a normal form of C with respect to conjugation.

For the above ZCR (8), the characteristic triple $C = \{C_1, C_2, C_3\}$ is

$$\frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{\mathrm{i}}{2\sin f}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}f} \\ \mathrm{e}^{-\mathrm{i}f} & 0 \end{pmatrix}, \quad -\frac{\mathrm{i}}{2\sin f}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

In (11), C_1 is in Jordan normal form and is diagonal. Then, if the zero-curvature representation (A, B) belongs to a 1-parameter family, then the Jordan form for C_1 will be diagonal for adjacent members of the family as well. Similarly, since C_2, C_3 are both non-diagonal, they will be so for the adjacent members. By conjugation with an appropriate diagonal matrix leaving C_1 unchanged, we may set one of the non-diagonal terms of C_2 or C_3 to 1. Consequently, during our computation we may assume the characteristic element to be of the form

$$C_1 = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}, \quad C_2 = \begin{pmatrix} p_1 & p_2 \\ p_3 & -p_1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} q_1 & 1 \\ q_3 & -q_1 \end{pmatrix}.$$

The functions $r, p_1, p_2, p_3, q_1, q_3$ together with the entries of the sl_2-matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}$$

are twelve unknowns that are to be determined along with the yet unknown dependence of b_{22} on x, y, f, b_{11}, b_{12}.

The determining system (10) is rather complex (typeset here, it would fill approximately four pages) and manageable only with the aid of computer algebra. To solve it, we demanded that the expressions a_1, b_1 depend on x, y, f, f_x, f_y, the expressions a_2, a_3, b_2, b_3 depend on

x, y, f, b_{11}, b_{12} (cf. eq. (8)), and the expressions $r, p_1, p_2, p_3, q_1, q_3$ depend on x, y, f (cf. (11)). Finally, we demanded that the function L in the constraint $L = 0$ actually depends on b_{11}, b_{12}, b_{22} and is nonlinear in these variables. Under assumptions stated, the result in Proposition 2.1 is exhaustive.

3. Geodesic coordinates

Assume now that the immersed surface is endowed with the geodesic coordinates, so that the metric is $g = dx^2 + f\,dy^2$, $f > 0$. As before, the second fundamental form is $b = b_{11}\,dx^2 + 2b_{12}\,dx\,dy + b_{22}\,dy^2$. Then the Gauss–Mainardi–Codazzi equations are

$$\begin{aligned} f_{,xx} &= \frac{f_{,x}^2}{2f} + 2(b_{12}^2 - b_{11}b_{22}), \\ b_{11,y} &= b_{12,x} + \frac{f_{,x}}{2f} b_{12}, \\ b_{12,y} &= b_{22,x} + \frac{f_{,y}}{2f} b_{12} - \frac{f_{,x}}{2f}(b_{22} + f b_{11}). \end{aligned} \tag{12}$$

The nonparametric zero-curvature representation is $A\,dx + B\,dy$ with

$$\begin{aligned} A &= -\frac{1}{2}\begin{pmatrix} 0 & \dfrac{\mathrm{i}b_{12}}{\sqrt{f}} + b_{11} \\ \dfrac{\mathrm{i}b_{12}}{\sqrt{f}} - b_{11} & 0 \end{pmatrix}, \\ B &= -\frac{1}{2}\begin{pmatrix} \dfrac{\mathrm{i}f_x}{2\sqrt{f}} & \dfrac{\mathrm{i}b_{22}}{\sqrt{f}} + b_{12} \\ \dfrac{\mathrm{i}b_{22}}{\sqrt{f}} - b_{12} & -\dfrac{\mathrm{i}f_x}{2\sqrt{f}} \end{pmatrix}. \end{aligned} \tag{13}$$

Proposition 3.1. *Let $Y_1(y), Y_2(y), Y_3(y), Y_4(y)$ be arbitrary functions and Z be a constant. Use the notation*

$$\begin{aligned} L :=\ & \frac{1 - Z}{Z\sqrt{f}}(b_{11}b_{22} - b_{12}^2) - \left(Y_1\sqrt{f} + x\frac{\partial Y_2}{\partial y} + Y_4\right) b_{11} \\ & + \left(x\frac{\partial Y_1}{\partial y} + Y_3\right) \frac{1}{\sqrt{f}}\, b_{12} - \frac{Y_1}{\sqrt{f}}\, b_{22} \\ & - Y_1Y_4 + Y_2Y_3 - xY_1\frac{\partial Y_2}{\partial y} + x\frac{\partial Y_1}{\partial y}Y_2 - ({Y_1}^2 + {Y_2}^2)\sqrt{f}. \end{aligned} \tag{14}$$

If $L = 0$, then the matrices

$$A = -\frac{1}{2}\begin{pmatrix} 0 & \left(\frac{\mathrm{i}b_{12}}{\sqrt{f}} + b_{11} + \Delta^-\right) Z \\ \frac{\mathrm{i}b_{12}}{\sqrt{f}} - b_{11} - \Delta^+ & 0 \end{pmatrix},$$

$$B = -\frac{1}{2}$$

$$\times \begin{pmatrix} \frac{\mathrm{i}f_x}{2\sqrt{f}} & \left(\frac{\mathrm{i}b_{22}}{\sqrt{f}} + b_{12} + x\frac{\partial\Delta^+}{\partial y} + \mathrm{i}\Delta^+\sqrt{f} + Y_3 + \mathrm{i}Y_4\right)Z \\ \frac{\mathrm{i}b_{22}}{\sqrt{f}} - b_{12} - x\frac{\partial\Delta^-}{\partial y} + \mathrm{i}\Delta^-\sqrt{f} - Y_3 + \mathrm{i}Y_4 & -\frac{\mathrm{i}f_x}{2\sqrt{f}} \end{pmatrix},$$

where $\Delta^\pm = Y_1 \pm \mathrm{i}Y_2$, form a zero-curvature representation for equation (12).

Denoting

$$K = \frac{b_{11}b_{22} - b_{12}^2}{f}, \qquad H = \frac{1}{2}\left(b_{11} + \frac{b_{22}}{f}\right)$$

the Gauss and mean curvature, respectively, we get the following five integrable classes.

Case 1. Linear Weingarten surfaces

$$K + \alpha H + \beta = 0,$$

where α, β are arbitrary constants. Then

$$A = -\frac{1}{2}\begin{pmatrix} 0 & (t+1)\left(\frac{\mathrm{i}b_{12}}{\sqrt{f}} + b_{11}\right) + \square^- \\ \frac{\mathrm{i}b_{12}}{\sqrt{f}} - b_{11} - \frac{\square^+}{t+1} & 0 \end{pmatrix},$$

$$B = -\frac{1}{2}\begin{pmatrix} \frac{\mathrm{i}f_x}{2\sqrt{f}} & (t+1)\left(\frac{\mathrm{i}b_{22}}{\sqrt{f}} + b_{12}\right) + \square^-\,\mathrm{i}\sqrt{f} \\ \frac{\mathrm{i}b_{22}}{\sqrt{f}} - b_{12} + \frac{\square^+}{t+1}\mathrm{i}\sqrt{f} & -\frac{\mathrm{i}f_x}{2\sqrt{f}} \end{pmatrix},$$

where $\square^\pm = \square^\pm(\alpha, \beta, t) = \alpha t \pm \sqrt{\alpha^2 t^2 - \beta t - \beta t^2}$.

Case 2. The class

$$K + Y\frac{b_{11}}{\sqrt{f}} + \gamma = 0,$$

where Y is an arbitrary function of y and γ is a constant. In this case

$$A = -\frac{1}{2}\begin{pmatrix} 0 & (t+1)\left(\dfrac{\mathrm{i}b_{12}}{\sqrt{f}} + b_{11} + \mathrm{i}\sqrt{\dfrac{\gamma t}{t+1}}\right) \\ \dfrac{\mathrm{i}b_{12}}{\sqrt{f}} - b_{11} + \mathrm{i}\sqrt{\dfrac{\gamma t}{t+1}} & 0 \end{pmatrix},$$

$$B = -\frac{1}{2}\begin{pmatrix} \dfrac{\mathrm{i}f_x}{2\sqrt{f}} & (t+1)\left(\dfrac{\mathrm{i}b_{22}}{\sqrt{f}} + b_{12} + \bar{Y} - \sqrt{\dfrac{\gamma f t}{t+1}}\right) \\ \dfrac{\mathrm{i}b_{22}}{\sqrt{f}} - b_{12} + \bar{Y} + \sqrt{\dfrac{\gamma f t}{t+1}} & -\dfrac{\mathrm{i}f_x}{2\sqrt{f}} \end{pmatrix},$$

where $\bar{Y} = \mathrm{i}\,tY/(t+1)$.

Case 3. The class

$$K + Y_1\frac{b_{11}}{\sqrt{f}} + Y_2\frac{b_{12}}{f} = 0,$$

where Y_1, Y_2 are arbitrary functions of y. Then

$$A = -\frac{1}{2}\begin{pmatrix} 0 & (t+1)\left(\dfrac{\mathrm{i}b_{12}}{\sqrt{f}} + b_{11}\right) \\ \dfrac{\mathrm{i}b_{12}}{\sqrt{f}} - b_{11} & 0 \end{pmatrix},$$

$$B = -\frac{1}{2}\begin{pmatrix} \dfrac{\mathrm{i}f_x}{2\sqrt{f}} & (t+1)\left(\dfrac{\mathrm{i}b_{22}}{\sqrt{f}} + b_{12}\right) + (\mathrm{i}Y_1 - Y_2)t \\ \dfrac{\mathrm{i}b_{22}}{\sqrt{f}} - b_{12} + \dfrac{(\mathrm{i}Y_1 + Y_2)t}{t+1} & -\dfrac{\mathrm{i}f_x}{2\sqrt{f}} \end{pmatrix}.$$

Case 4. The class

$$K + Y_1 H - \frac{1}{2}Y\left(\frac{b_{12}}{f} + \mathrm{i}\frac{b_{11}}{\sqrt{f}}\right) = 0, \qquad Y = x\frac{\partial Y_1}{\partial y} + Y_2,$$

where Y_1, Y_2 are arbitrary functions of y. Then

$$A = -\frac{1}{2}\begin{pmatrix} 0 & (t+1)\left(\dfrac{\mathrm{i}b_{12}}{\sqrt{f}} + b_{11}\right) + Y_1 t \\ \dfrac{\mathrm{i}b_{12}}{\sqrt{f}} - b_{11} & 0 \end{pmatrix},$$

$$B = -\frac{1}{2}\begin{pmatrix} \dfrac{\mathrm{i}f_x}{2\sqrt{f}} & (t+1)\left(\dfrac{\mathrm{i}b_{22}}{\sqrt{f}} + b_{12}\right) + tY + \mathrm{i}Y_1 t\sqrt{f} \\ \dfrac{\mathrm{i}b_{22}}{\sqrt{f}} - b_{12} & -\dfrac{\mathrm{i}f_x}{2\sqrt{f}} \end{pmatrix}.$$

Case 4 is incompatible with the real geometry of E^3, but nevertheless we may turn it into a formally real class by allowing f to be negative.

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