The Diffiety Institute Preprint Series

Preprint DIPS–5/98 **March 26, 1998**

Introduction to Secondary Calculus

by

Alexandre VINOGRADOV

Available via INTERNET:
http://ecfor.rssi.ru/~diffiety/
http://www.botik.ru/~diffiety/
anonymous FTP:
ftp://ecfor.rssi.ru/general/pub/diffiety
ftp://www.botik.ru/~diffiety/preprints

The Diffiety Institute
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

Introduction to Secondary Calculus

Alexandre VINOGRADOV

Abstract. First we exhibit some basic notions and constructions of modern geometry of partial differential equations that lead to the concept of diffiety, an analogue of affine algebraic varieties for partial differential equations. Then it is shown how the differential calculus on diffieties which respects the underlying infinite order contact structure is self-organized into Secondary Calculus in such a way that higher symmetries of PDE's become secondary vector fields and the first term of the $\mathcal{C}$-spectral sequence becomes the algebra of secondary differential forms. Then the general secondarization problem is formulated. Its solution for modules and multi-vector-valued differential forms is proposed and the relevant homological algebra is discussed. Eventually, relations with gauge theories are briefly outlined at the end.

Secondary Calculus sprang up at the fall of seventies as a result of a natural evolution of ideas in the geometric theory of nonlinear partial differential equations. Since basic facts and constructions of that theory are not yet of common knowledge, the first goal of these notes is to exhibit a minimal set of them in order to give to a nonexpert a first feeling of what Secondary Calculus is. This is done in Sec. 1 − 7. Formulation of Secondarization Problem culminates this part.

After that, in Sec. 8 − 10 we present and discuss for the first time the key notion of a *secondary module* over a "secondary smooth function algebra". This subject turned out to be a rather delicate matter which very successively slipped out an exact formalization for a long time. Indeed, it appears very surprising that secondary modules are suitable homotopy classes for some kind of differential complexes over diffieties. Not less remarkable is that the simplest class of secondary modules is formed by homotopy classes of complexes naturally associated with flat connections. The concept of a secondary module appears to be a natural junction point of Secondary Calculus and QFT if understood as "cohomological physics".

In Sec. 11 we solve the secondarization problem for multi-vector-valued differential forms (or form-valued multi-vectors). The corresponding cohomologies are studied. These cohomologies are, in a sense, dual to those that appear in the first term $\mathcal{C}E_1^{*,*}$ of the $\mathcal{C}$-spectral sequence. Secondary multi-vector-valued differential forms act naturally on secondary differential forms, i.e., on $\mathcal{C}E_1^{*,*}$. It looks remarkable that the k-lines theorem for secondary multi-vector-valued differential forms holds under the same conditions as for $\mathcal{C}E_1^{*,*}$. Moreover, bigradings (p, q) of nontrivial secondary multi-vector-valued differential forms are perfectly complementary to those of nontrivial secondary differential forms by guaranteeing a nontrivial action of the formers on the latters (see Fig. 5 and Fig. 6 below).

All natural operations with secondary multi-vector-valued differential forms are proved secondarizable. To illustrate these topics, we show how the secondary Frölicher − Nijenhuis bracket and some other secondary operations look like. A systematic exposition of this subject will appear in [66].

Date: August 20, 1997.

1991 *Mathematics Subject Classification.* Primary 81Q30; Secondary 81R99,18G55, 58E99, 55U99, 16E40, 16W30.

Except for some general remarks in Sec. 11, we do not go to discuss here physical aspects of Secondary Calculus in more details. Instead, the reader is strongly suggested to compare notions and constructions of this paper with other contributions to these proceedings.

About fifteen years ago the author presented in [62] his expectations concerning an eventual role of Secondary Calculus in QFT (see [63] for more details and the updated version). These proceedings seems to confirm them.

Contents

1. Infinite jets

Let $E = E^{n+m}$ be an $(n+m)$-dimensional smooth manifold, $m, n \geqslant 0$. Fix non-negative integers n and k. Then n-dimensional submanifolds of E passing through a point $a \in E$ can be subdivided into equivalence classes with respect to the relation: "to be tangent one another with the k-th order". These equivalence classes are called *k-jets* of n-dimensional submanifolds of E at a. If $L \subset E$ is such a submanifold and $a \in L$, then $[L]^k_a$ denotes its k-jet at a, i.e., the above equivalence class to which L belongs. For our purposes it is important to note that this definition is also valid for $k = \infty$.

The totality of all k-jets of n-dimensional submanifolds of E is in a natural way a smooth manifold denoted by $J^k = J^k(E, n)$ and called *k-th jet manifold* (of n-dimensional submanifolds of E). For instance, as it is easy to see, $J^0(E, n) = E$ while $J^1(E, n)$ is the Grassmannian fibre bundle of n-dimensional tangent subspaces to E.

If $L \subset E$, $\dim L = n$, then

$$j_k(L) : L \to J^k(E, n) \quad \text{with} \quad j_k(L)(a) = [L]^k_a$$

is the natural lift of L to $J^k(E, n)$.

For $k \geqslant l$ there exists a natural projection $J^k \to J^l$, $[L]^k_a \mapsto [L]^l_a$. The corresponding chain of maps

$$E = J^0 \leftarrow J^1 \leftarrow \cdots \leftarrow J^k \leftarrow \cdots \leftarrow J^\infty \tag{1}$$

shows J^∞ to be its inverse limit.

A standard coordinate-wise description of jet manifolds starts with a choice of a local chart $(y_1, \ldots, y_{m+n})$ in E. Divide it then into two parts $(y_{\alpha_1}, \ldots, y_{\alpha_n})$ and

$(y_{\beta_1}, \ldots, y_{\beta_m})$ and put $x_i = y_{\alpha_i}$, $i = 1, \ldots, n$, $u^j = y_{\beta_j}$, $j = 1, \ldots, m$. Conditionally, the variables x_i's can be thought as *independent* while u^j's as *dependent* ones. Such a *divided chart* in E induces a local chart in J^k as follows.

Consider n-dimensional submanifolds of E that can be represented locally in the form $L = \{u^i = f_i(x_1, \ldots, x_n),\ i = 1, \ldots, m\}$. With a multi-index $\sigma = (\sigma_1, \ldots, \sigma_n)$, $|\sigma| = \sigma_1 + \cdots + \sigma_n \leqslant k$, one can associate a local function u^i_σ on J^k such that

$$u^i_\sigma \circ j_k(L) = \frac{\partial^{|\sigma|} f_i}{\partial x_\sigma},$$

where $\partial^{|\sigma|}/\partial x_\sigma$ stands for the derivation corresponding to σ. Then

$$(x_1, \ldots, x_n, u^1, \ldots, u^m, \ldots, u^i_\sigma, \ldots)$$

with $|\sigma| \leqslant k$ is a local chart on J^k. Obviously, for $k = \infty$ there is no limitations on σ.

The sequence of inclusions of smooth functions algebras is associated with sequence (1):

$$C^\infty(J^o) \hookrightarrow C^\infty(J^1) \hookrightarrow \cdots \hookrightarrow C^\infty(J^k) \hookrightarrow \ldots \tag{2}$$

The direct limit of (2) is called the *smooth functions algebra* on J^∞. We denote it by $\mathcal{F} = \mathcal{F}(J^\infty)$ though $C^\infty(J^\infty)$ would be more expressive. Denote also by $\mathcal{F}_k = \mathcal{F}_k(J^\infty)$ the image of $C^\infty(J^k)$ in $\mathcal{F}$. This way one gets a filtration of $\mathcal{F}$:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset \mathcal{F}. \tag{3}$$

In terms of coordinates a smooth function f on J^∞ is a function depending on a finite number of coordinates $(x, u, \ldots, u^i_\sigma, \ldots)$.

The same procedure is applied to define any kind of "covariant" objects on J^∞. For instance, a differential form on J^∞ is a differential form on one of jet spaces J^k's, $k < \infty$, lifted to J^∞. Its local expression is standard with respect to a finite number of aforementioned jet coordinates.

A pithy differential calculus on J^∞ can be developed if working with filtered $\mathcal{F}$-modules $P = \{P_i\}$. This means that P_i is an $\mathcal{F}_i$-module and P is the direct limit of the sequence of inclusions $P_0 \hookrightarrow P_1 \hookrightarrow \ldots$. Coordinate-wisely, any $p \in P$ can be described by means of its components $p_\alpha \in \mathcal{F}$. A differential operator

$$\Delta : P = \{P_i\} \to Q = \{Q_i\}$$

of order $\leqslant k$ is an $\mathbb{R}$-linear map such that

$$[f_o, [f_1, \ldots [f_k, \Delta] \ldots]] = 0, \quad \forall f_o, \ldots, f_k \in \mathcal{F}$$

and $\Delta(P_i) \subset Q_{j(i)}$ for all i.

The above construction can be specified to the case when E is supplied with a fiber structure $\pi : E \to M$, $\dim M = n$. In this case considering only submanifolds of the form $L = s(M)$ with $s : M \to E$ being a (local) section of π one gets the jet space $J^k(\pi)$, $k = 0, 1, \ldots, \infty$. A local chart on E respecting the fibre structure is composed of a local chart $\{x_j\}$ on M completed by "fibre coordinates" $\{u^i\}$. Such a situation is typical in field theory where M is the space-time manifold and u^i are "fields". Note that $J^k(\pi) \subset J^k(E, n)$ is an open domain whose closure is the whole $J^k(E, n)$.

Denote by $L^{(k)} = \operatorname{im} j_k(L) \subset J^k$, a natural lift of $L \subset E$ to J^k. The following assertion is almost evident.

Proposition 1.1. *Let $\theta \in J^k$ and let L_1 and L_2 be n-dimensional submanifolds of E such that $\theta = [L_1]^\infty_a = [L_2]^\infty_a$. Then $T_\theta(L_1^{(\infty)}) = T_\theta(L_2^{(\infty)})$, i.e., tangent spaces at θ to infinite lifts of L_1 and L_2 at θ coincides.*

This fact shows correctness of the following definition.

Definition 1.1. The *Cartan subspace* $C_\theta \subset T_\theta(J^\infty)$ is defined to be $C_\theta = T_\theta(L^{(\infty)})$ if $\theta = [L]_a^\infty$.

Obviously, $\dim C_\theta = n$. So, the correspondence $\theta \stackrel{\mathcal{C}}{\longmapsto} C_\theta$ is an n-dimensional distribution on J^∞, called the *Cartan distribution*, or *infinite order contact structure.*

The well-known total derivative operators $D_k = \partial/\partial x_k + \sum_{i,\sigma} u^i_{\sigma+1_k} \partial/\partial u^i_\sigma$, $1 \leqslant k \leqslant n$, can be seen as vector fields on J^∞. They annihilate Cartan forms $\omega^i_\sigma = du^i_\sigma - \sum_k u^i_{\sigma+1_k}\, dx_k$ for any i, σ.

Proposition 1.2. *The Cartan distribution $\mathcal{C}$ can be given either as the span of vector fields $D_1, \dots, D_n$, or as the solution of the Pfaff system $\omega^i_\sigma = 0, \forall\, i, \sigma$. The latter the is a Frobenius one, i.e., satisfies the hypothesis of Frobenius theorem.*

This result means that the linear space C_θ is generated by vectors $D_{1,\theta}, \dots, D_{n,\theta}$ and a vector $\xi \in T_\theta(T^\infty)$ belongs to C_θ iff $\xi \rfloor \omega^i_\sigma = 0$ for all i, σ.

A submanifold $N \subset J^\infty$ is called *integral* (with respect to $\mathcal{C}$), if $T_\theta N = C_\theta$ for any $\theta \in N$.

Proposition 1.3. *A manifold N is integral iff it is locally of the form $L^{(\infty)}$. Therefore, the set of integral submanifolds is identified with the set of immersed n-dimensional submanifolds of E, or with multi-valued sections of π, if $J^\infty = J^\infty(\pi)$.*

2. Infinite prolongations of PDE's and diffieties

A system of k-th order partial differential equations $\mathcal{E}$ imposed on n-dimensional submanifolds of E (sections of π) may be viewed geometrically as a submanifold $\mathcal{E} \subset J^k$. Indeed, such a submanifold can be given locally as $F = 0$, $F = (F_1, \dots, F_l)$ with $F_i = F_i(x, u, \dots, u^i_\sigma, \dots)$, $|\sigma| \leqslant k$. So, F_i's are functions on J^k and the system $F = 0$ defines locally a submanifold $\mathcal{E}$ in J^k. As it is easy to see, a submanifold $L \subset E$ is a solution of $\mathcal{E}$ iff $L^{(k)} \subset E$.

The infinite prolongation of $\mathcal{E}$ is defined locally as a submanifold of J^∞ given by the following infinite system of equations

$$\mathcal{E}_\infty = \{F_i = 0,\ D_k(F_i) = 0,\ D_k D_l(F_i) = 0, \dots\} \subset J^\infty. \tag{4}$$

In terms of jet coordinates, a point $\theta \in \mathcal{E}_\infty$ may be viewed as a formal solution of the system $\mathcal{E}$ at a point $x^0 = (x^0_1, \dots, x^0_u)$ of the space of independent variables. In other words, jet coordinates of θ are coefficients of a formal series at x^0 which is a formal solution of $\mathcal{E}$. The following rather elementary fact is, however, of fundamental importance.

Proposition 2.1. *If $\mathcal{E}$ is a formally integrable system and $\theta \in \mathcal{E}_\infty$, then C_θ is tangent to $\mathcal{E}_\infty$, i.e., $C_\theta \subset T_\theta(\mathcal{E}_\infty)$. Therefore, the Cartan distribution $\mathcal{C}$ on J^∞ can be restricted to $\mathcal{E}_\infty$.*

Put $\mathcal{C}(\mathcal{E}_\infty) = \mathcal{C}\,|_{\mathcal{E}_\infty}$ and call this n-dimensional distribution on $\mathcal{E}_\infty$ the *Cartan distribution* on $\mathcal{E}_\infty$. Its fundamental role in geometric theory of PDE's is due to the assertion:

Proposition 2.2. *An n-dimensional submanifold $L \subset E$ is a solution of $\mathcal{E}$ iff $L^{(\infty)} \subset \mathcal{E}_\infty$ and as such is an integral submanifold of $\mathcal{C}(\mathcal{E}_\infty)$. Conversely, any integral submanifold of $\mathcal{C}(\mathcal{E}_\infty)$ is locally of the form $L^{(\infty)}$ for a solution L of $\mathcal{E}$.*

So, we see that integral submanifolds of $\mathcal{C}(\mathcal{E}_\infty)$ may be treated as multi-valued solutions of $\mathcal{E}$. Hence, the pair $\mathcal{O} = (\mathcal{E}_\infty, \mathcal{C}(\mathcal{E}_\infty))$ contains all necessary informations concerning solutions of $\mathcal{E}$ and, speaking informally, may be viewed as a store, where all solutions of $\mathcal{E}$ are stored.

Definition 2.1. A pair of the form $(\mathcal{E}_\infty, \mathcal{C}(\mathcal{E}_\infty))$ is called an *elementary diffiety.*

Diffieties are geometric objects which play the same role in the theory of partial differential equations as affine algebraic varieties in the theory of algebraic equations.

The smooth function algebra on a diffiety $\mathcal{O} = (\mathcal{E}_\infty, \mathcal{C}(\mathcal{E}_\infty))$ is composed of functions g which are locally of the form $g = f\,|_{\mathcal{E}_\infty}$ with $f \in \mathcal{F}(J^\infty)$. Denote it by $\mathcal{F}(\mathcal{O})$ or $\mathcal{F}(\mathcal{E}_\infty)$. This is a filtered algebra with a filtration inherited from $\mathcal{F}(J^\infty)$ and what was said above with respect to $\mathcal{F}(J^\infty)$ remains also valid for $\mathcal{F}(\mathcal{O})$.

It is convenient to introduce the *diffiety dimension* $\operatorname{Dim}\mathcal{O}$ of an elementary diffiety $\mathcal{O} = (\mathcal{E}_\infty, \mathcal{C}(\mathcal{E}_\infty))$ as the dimension of the structure distribution $\mathcal{C}(\mathcal{E}_\infty)$, i.e., the number of independent variables.

Also, it is natural to introduce *diffiety morphisms* which will be called *smaps.* If $\mathcal{O} = (\mathcal{E}_\infty, \mathcal{C}(\mathcal{E}_\infty))$ and $\mathcal{O}' = (\mathcal{E}'_\infty, \mathcal{C}(\mathcal{E}'_\infty))$, then a smap of $\mathcal{O}$ to $\mathcal{O}'$ is a smooth map $h : \mathcal{E}_\infty \to \mathcal{E}'_\infty$ respecting the Cartan distribution. More exactly, "smooth" means that $h \circ f \in \mathcal{F}(\mathcal{E}_\infty)$ if $f \in \mathcal{F}(\mathcal{E}'_\infty)$. "Respects" means that the image of the distribution $\mathcal{C}(\mathcal{E}_\infty)$ under the differential of h belongs to $\mathcal{C}(\mathcal{E}'_\infty)$, i.e., $d_\theta h : C_\theta \to C_{h(\theta)}$, for all $\theta \in \mathcal{E}_\infty$, where $d_\theta h$ stands for the differential of h at θ.

Coordinate-wisely, smaps from $\mathcal{O}$ to $\mathcal{O}'$ may be seen as infinite prolongations of differential operators sending solutions of $\mathcal{E}$ to solutions of $\mathcal{E}'$. More exactly, if Δ is such an operator and $u = f(x)$ is the local description of a submanifold $L \subset E$ such that $\theta = [L]^\infty_{x^o} \in \mathcal{E}_\infty$, then the derivatives of $\Delta(f)$ evaluated at the corresponding point y are jet coordinates of $h(\theta) \in \mathcal{E}'_\infty$.

The following class of smaps is to be singled out. A smap h is called a *covering* if h is surjective and $d_\theta h : C_\theta \to C_{h(\theta)}$ is an isomorphism for any $\theta \in \mathcal{E}_\infty$. Various differential substitutions, quotiening of PDE's, Wahlquist – Estabrook prolongation structures, etc., are, in fact, coverings. This reveals the importance of this concept. A *Bäcklund transformation* connecting solutions of equations $\mathcal{E}'$ and $\mathcal{E}''$ can be seen as a diagram of coverings

$$\mathcal{O}' \xleftarrow{h'} \mathcal{O} \xrightarrow{h''} \mathcal{O}''$$

with $\mathcal{O}' = (\mathcal{E}'_\infty, \mathcal{C}(\mathcal{E}'_\infty))$, $\mathcal{O}'' = (\mathcal{E}''_\infty, \mathcal{C}(\mathcal{E}''_\infty))$ and $\mathcal{O} = (\mathcal{E}, \mathcal{C}(\mathcal{E}_\infty))$ for a suitable $\mathcal{E}$.

For our current purposes the concept of a covering is important because it allows us to introduce the general concept of a diffiety. These are inverse limits of, generally, infinite sequences of coverings:

$$\mathcal{O}_1 \xleftarrow{h_1} \mathcal{O}_2 \xleftarrow{h_2} \cdots \longleftarrow \mathcal{O}_i \xleftarrow{h_i} \ldots$$

Diffieties are objects on which Secondary Calculus grows naturally. The study of diffieties of the most general form is now at its origin and many things can be only suspected. One of them is that a more sophisticated definition of the diffiety dimension could take values in $\mathbb{R}$.

An n-dimension manifold can be viewed as a diffiety in two different but natural ways depending on how the structure distribution is chosen. If we supply M with the 0-dimensional distribution $M \ni a \mapsto \{0\} = C_a \subset T_a M$, then M becomes a 0-Dimensional diffiety. The only integral submanifolds of it are points of M. Another possibility is to put $C_a = T_a M$. This way one gets an n-dimensional diffiety the single integral submanifold of which is M itself. So, n-dimensional manifolds are smallest possible n-Dimensional diffieties and as such they appear to be solutions of PDE's.

So, an n-dimensional manifold M viewed as a 0-Dimensional diffiety is a "society" whose "members", i.e., points, have no internal structure. On the other hand, M viewed as n-Dimensional diffiety, is a social singleton having, however, a rich internal structure. The general case is a mixture of these extremal options.

Now we are ready to give a first vague idea of Secondary Calculus: this is a kind of Calculus on diffieties which respects the Cartan distribution (infinite order contact structure) on them.

It is important to stress that in view of what was said before the category of finite-dimensional manifolds is identified with the 0-dimensional part of the category of diffieties. By this reason, the classical mathematics (Calculus, differential geometry, etc.) appears to be a very degenerated particular case of the diffiety theory. So, we are led to face the *secondarization problem*: to extend the classical Calculus to the whole category of diffieties.

Note the analogy with the quantization problem. While the latter requires to respect Bohr's correspondence principles to reach the classical limit when $h \to 0$, the former requires the same when $\operatorname{Dim} \mathcal{O} \to 0$. This analogy is, in fact, more instructive than superficial as it could appear at first glance.

Now we pass to describe some facts of geometry of PDE's that really gave birth to Secondary Calculus.

3. Higher symmetries of PDE's

It is convenient to interpret vector fields on a diffiety $\mathcal{O}$ as derivations of the smooth function algebra $\mathcal{F}(\mathcal{O})$. For instance, any vector field on $\mathcal{O} = J^\infty$ as it is easy to see is an operator of the form

$$X = \sum_{i,\sigma} a^i_\sigma \frac{\partial}{\partial x_k} + \sum_k b_k \frac{\partial}{\partial x_k}$$

This is an infinite series but no convergence problem arises because smooth functions on J^∞ depend only on a finite number of jet variables. Similar situations appear in many other circumstances regarding Calculus on diffieties. But the general algebraic approach [53], being coordinate-free, guarantees us of any difficulties of that kind. A vector field X "respecting" a distribution is, obviously, an infinitesimal contact transformations of it. This means, for instance, that the vector field $[X, Y]$ belongs to the distribution if Y does. So, denote by $D_\mathcal{C}(\mathcal{O})$ the totality of all contact fields on a diffiety $\mathcal{O}$. This is a Lie algebra with respect to the standard commutation operation of vector fields. By starting from these observations it is rather simple to describe all contact fields on J^∞. Below D_σ for $\sigma = (i_1, \dots i_k,)$ stands for $D_1^{i_1} \circ \cdots \circ D_n^{i_n}$.

Proposition 3.1. *Any vector field $X \in D_\mathcal{C}(J^\infty)$ can be uniquely presented in the form*

$$X = \ni_\varphi + \sum_k a_k D_k \tag{5}$$

with $\varphi = (\varphi_1, \dots, \varphi_m)$, $\varphi_i, a_k \in \mathcal{F}(J^\infty)$ and

$$\ni_\varphi = \sum_{\sigma,i} D_\sigma(\varphi_i) \frac{\partial}{\partial u^i_\sigma}.$$

The vector field $\ni_\varphi$ is an *evolutionary derivation* corresponding to the *generating function* φ.

Two components of the splitting (5) are of rather different kinds. Observe that in view of Proposition 1.2 the vector field Y belongs to $\mathcal{C}(J^\infty)$ iff it is of the form $Y = \sum_k a_k D_k$. Denote the totality of such fields by $\mathcal{C}D(J^\infty)$. By the construction of the Cartan distribution $\mathcal{C}(J^\infty)$ any, vector field $Y \in \mathcal{C}D(J^\infty)$ is tangent to any integral submanifold in J^∞. By this reason, the virtual flow generated by such a field makes to slide any integral submanifold along itself. So, such a flow generates the trivial one in the "space" of all integral submanifolds. On the contrary, an evolutionary derivation $\ni_\varphi$ generates a nontrivial virtual flow in this space if $\varphi \neq 0$.

Observing now that $\mathcal{C}D(J^\infty)$ is an ideal of the Lie algebra $D_\mathcal{C}(J^\infty)$ it is natural to introduce the quotient Lie algebra

$$\varkappa = D_\mathcal{C}(J^\infty)/\mathcal{C}D(J^\infty)$$

in order to get the Lie algebra of vector fields on the "space" of all integral submanifolds of J^∞. The Lie algebra $\varkappa$ is identified in view of Proposition 3.1 with the $\mathcal{F}$-module of generating functions. In terms of generating functions the Lie bracket operations in $\varkappa$ looks as

$$\{\varphi, \psi\} = \ni_\varphi(\psi) - \ni_\psi(\varphi), \tag{6}$$

where evolutionary derivations act on generating functions component-wisely.

The above general arguments are, evidently, valid for a general diffiety $\mathcal{O}$ and we can define the quotient Lie algebra

$$\operatorname{Sym}\mathcal{O} = D_\mathcal{C}(\mathcal{O})/\mathcal{C}D(\mathcal{O})$$

with $\mathcal{C}D(\mathcal{O})$ being the totality of all vector fields on $\mathcal{O}$ belonging to the structure distribution $\mathcal{C}(\mathcal{O})$. If $\mathcal{O} = \mathcal{E}_\infty$ it is natural denote $\operatorname{Sym}\mathcal{O}$ by $\operatorname{Sym}\mathcal{E}$ and to call elements of $\operatorname{Sym}\mathcal{E}$ higher (infinitesimal) symmetries of the equation $\mathcal{E}$. It is not difficult to verify that hierarchies associated with well-known completely integrable systems like the KdV equation are composed of higher symmetries of the original equation.

The basic technique used to find higher symmetries is as follows. If the equation $\mathcal{E}$ is given by $F = 0$ with $F = (F_1, \ldots, F_l)$, $F_i \in \mathcal{F}(J^\infty)$, then the matrix differential operator

$$l_F = \left\| \sum_\sigma \frac{\partial F_i}{\partial u^j_\sigma} D_\sigma \right\|$$

is called the *universal linearization operator* (for $\mathcal{E}$). The operator l_F is a $\mathcal{C}$-differential operator (see below) and as such admits restrictions to infinitely prolonged equations.

Theorem 3.1. *If* $\mathcal{E} = \{F = 0\}$, *then* $\operatorname{Sym}\mathcal{E} = \ker l_{[F]}$ *with* $l_{[F]} = l_F \,|_{\mathcal{E}_\infty}$.

This result gives origin to some rather efficient methods of computing algebra $\operatorname{Sym}\mathcal{E}$ for a concrete equation $\mathcal{E}$.

The following notion will play an important role in what follows. A differential operator $\Delta : P \to Q$ between two $\mathcal{F}(\mathcal{O})$-modules is called $\mathcal{C}$-*differential*, if it can be restricted to any integral submanifold W of $\mathcal{O}$. This means that the restriction of $\Delta(p)$ to W is determined uniquely by the restriction of p to W for any $p \in P$. Proposition 1.2 shows that vector fields from $\mathcal{C}D(J^\infty)$ are tangent to integral submanifolds of J^∞, i.e., are restrictable. Therefore, any matrix differential operator whose entries are of the form $\sum_\sigma a_\sigma D_\sigma$ with $a_\sigma \in \mathcal{F}$ is a $\mathcal{C}$-differential one. Moreover, Proposition 2.1 shows that such operators can be restricted to $\mathcal{E}_\infty \subset J^\infty$ and, therefore, are $\mathcal{C}$-differential on $\mathcal{O} = \mathcal{E}_\infty$.

Proposition 3.2. *The operators described above exhaust the class of* $\mathcal{C}$*-differential operators on* $\mathcal{E}_\infty$.

4. The $\mathcal{C}$-spectral sequence

Denote by $\Lambda^i(M)$ the $C^\infty(M)$-module of i-th order differential forms on the manifold M. The sequence of projections (1) generates the sequence of inclusions

$$\Lambda^i(J^o) \hookrightarrow \Lambda^i(J^1) \hookrightarrow \cdots \hookrightarrow \Lambda^i(J^k) \hookrightarrow \ldots$$

Its direct limit denoted by $\Lambda^i(J^\infty)$ is a $\mathcal{F}(J^\infty)$-module filtered by the images of $\Lambda^i(J^k)$. By definition, differential forms on J^∞ are elements of $\Lambda^i(J^\infty)$. Local expressions of such forms in terms of jet coordinates look as finite sums

$$\sum a^{j_1\dots j_l}_{i_1\dots i_k\sigma_1\dots\sigma_l} dx_{i_1}\wedge\dots\wedge dx_{i_k}\wedge du^{j_1}_{\sigma_1}\wedge\dots\wedge du^{j_l}_{\sigma_l}. \qquad (7)$$

Denote by $\mathcal{C}\Lambda^i(J^\infty)\subset\Lambda^i(J^\infty)$ the $\mathcal{F}$-submodule of $\Lambda^i(J^\infty)$ composed of differential forms vanishing on the Cartan distribution $\mathcal{C}(J^\infty)$ or, equivalently, when being restricted to any integral submanifold of J^∞. The following proposition describe the structure of $\mathcal{C}\Lambda^i$.

Proposition 4.1. *It holds*

(i) $\mathcal{C}\Lambda^1=\left\{\rho=\sum a^i_\sigma\omega^i_\sigma \mid a^i_\sigma\in\mathcal{F}\right\}$ (*the summation is finite*),
(ii) $\mathcal{C}\Lambda^i=\mathcal{C}\Lambda^1\wedge\Lambda^{i-1}$,
(iii) $\mathcal{C}\Lambda^*=\sum\mathcal{C}\Lambda^i$ *is a differentially closed ideal in* $\Lambda^*=\sum_i\Lambda^i(J^\infty)$.

The above definitions remains valid for an arbitrary diffiety $\mathcal{O}$ together with assertions (ii) and (iii) of Proposition 4.1. We will denote by $\mathcal{C}\Lambda^i(\mathcal{O})$ the $\mathcal{F}(\mathcal{O})$-module of differential forms on $\mathcal{O}$ vanishing on the distribution $\mathcal{C}(\mathcal{O})$. Below we write sometimes $\mathcal{C}\Lambda^i,\Lambda^i$, etc. without referring to the base diffiety $\mathcal{O}$.

The k-th power of the ideal $\mathcal{C}\Lambda^*$ of Λ^* is defined as

$$\mathcal{C}^k\Lambda^*=\mathcal{C}\Lambda^1\wedge\dots\wedge\mathcal{C}\Lambda^1\wedge\Lambda^* \quad (k\text{ times}).$$

Evidently, all ideals $\mathcal{C}^k\Lambda^*$ are differentially closed and so we get the *$\mathcal{C}$-filtration* in the de Rham complex Λ^* on a diffiety $\mathcal{O}$:

$$\Lambda^*\supset\mathcal{C}\Lambda^1\supset\dots\supset\mathcal{C}\Lambda^k\supset\dots$$

The spectral sequence associated with the $\mathcal{C}$-filtration is called the *$\mathcal{C}$-spectral sequence* (on $\mathcal{O}$).

Below we use the standard notation for terms of the $\mathcal{C}$-spectral sequence (see, for instance, ([36]) with added specifying $\mathcal{C}$:

$$\mathcal{C}E^{p,q}_r=\mathcal{C}E^{p,q}_r(\mathcal{O}),\quad d_r=d^{p,q}_r:\mathcal{C}E^{p,q}_r\to\mathcal{C}E^{p-r,q+r-1}_r$$
$$\mathcal{C}E_r=\textstyle\sum_{p,q}\mathcal{C}E^{p,q}_r,\quad \mathcal{C}E^{p,*}_0=\sum_q\mathcal{C}E^{p,q}_0.$$

Recall that $\mathcal{C}E_{r+1}$ is the cohomology of $\mathcal{C}E_r$ with respect to $d_r=\{d^{p,q}_r\}$.

By definition,

$$\mathcal{C}E^{p,q}_0=\mathcal{C}^p\Lambda^{p+q}\,/\mathcal{C}^{p+1}\Lambda^{p+q}$$

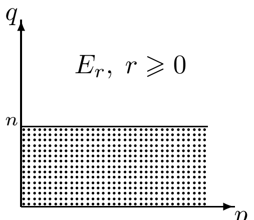

Fig. 1

and $d_0:\mathcal{C}E^{p,q}_0\to\mathcal{C}E^{p,q+1}_0$. So, the term $\mathcal{C}E_0$ is splitting into subcomplexes $\mathcal{C}E^{p,*}_0$, $p=0,1,2,\dots$, and $\mathcal{C}E^{p,q}_0=0$ for $p<0$. Also, the fact that $\mathcal{C}\Lambda^i=\Lambda^i$, if $i>n=\operatorname{Dim}\mathcal{O}$, and Proposition 4.1 shows that $E^{p,q}_0=0$, if $q>n$. Therefore, all nontrivial terms of the $\mathcal{C}$-spectral sequence are located in the shaded region of the standard (p,q)-diagram (see Fig. 1).

This shows that the $\mathcal{C}$-spectral sequence converges and $\mathcal{C}E^{p,q}_\infty=\mathcal{C}E^{p,q}_{n+1}$. The complex $\{E^{0,*}_0,d_0\}$ deserves a special attention due to its role in various applications. Introduce the alternative notation $\bar\Lambda^q(\mathcal{O})=\mathcal{C}E^{0,q}_0(\mathcal{O})$ and $\bar d=d^{0,q}_0$. By definition $\bar\Lambda^q=\Lambda^q\,/\mathcal{C}\Lambda^q$. Elements of $\bar\Lambda^q$ are called *horizontal forms.* Put also $\bar\Lambda^*=\sum_q\bar\Lambda^q$, i.e., $\bar\Lambda^*=\mathcal{C}E^{0,*}_0$. A local expression for a horizontal form looks as

$$\omega=\sum a_{i_1,\dots,i_q}dx_{i_1}\wedge\dots\wedge dx_{i_q},\quad a_{i_1,\dots,i_q},\in\mathcal{F}(\mathcal{O}).$$

Its peculiarity is that only differentials of independent variables appear in it (compare with (7)). Also,

$$d\omega = \sum_{i_1,\dots,i_q,k} D_k(a_{i_1,\dots,i_q})dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_q}.$$

The complex $\{\bar{\Lambda}^*, \bar{d}\}$ is called the *horizontal* (*de Rham*) *complex* (of $\mathcal{O}$) and its cohomology is called *horizontal* (*de Rham*) cohomology (of $\mathcal{O}$). In the context of differential systems, Bryant and Griffiths [11] use *characteristic cohomology* in same sense. We denote the horizontal cohomology by $\bar{H}^i = \bar{H}^i(\mathcal{O})$, $\bar{H}^* = \sum \bar{H}^i$.

If $\mathcal{O} = \mathcal{E}_\infty$, then horizontal differential n-forms are interpreted naturally as lagrangian densities of variational problems constrained by $\mathcal{E}$ while n-dimensional horizontal cohomology classes are functionals or "actions". Similarly, conserved currents (conserved densities, fluxes) admitted by the equation $\mathcal{E}$ are exact horizontal $(n-1)$-differential forms. Respectively, $(n-1)$-dimensional horizontal cohomology of $\mathcal{E}_\infty$ consists of conservation laws for solutions of $\mathcal{E}$. In other words, $E_1^{0,n}(\mathcal{E}_\infty)$ is the space of functionals constrained by $\mathcal{E}$ and $E_1^{0,n-1}(\mathcal{E}_\infty)$ is the space of conservation laws of $\mathcal{E}$. Spaces $\bar{H}^q(\mathcal{E}_\infty) = E_1^{0,q}(\mathcal{E}_\infty)$ are interpreted in various circumstances as lower order conservation laws, characteristic classes, etc. Horizontal (= characteristic) cohomology is the object of the main interest in the BRST-antifield formalism.

In the situation when the manifold of independent variables is fixed, i.e., the PDE system $\mathcal{E}$ is imposed on sections of a fibred manifold $\pi : E \to M$ (see Sec. 1) another filtration of the de Rham complex on $\mathcal{E}_\infty$ (or $J^\infty(\pi)$) arises. In fact, in that case the projection π induces a projection $\pi_\infty : \mathcal{E}_\infty \to M$. Consider the ideal $J\Lambda^*$ of the algebra $\Lambda^* = \Lambda^*(\mathcal{E}_\infty)$ composed of differential forms vanishing on fibres of the projection π_∞. It is easy to see that $J\Lambda^*$ is generated by the pull-back of $\Lambda^*(M)$ with respect to π_∞, i.e., $J\Lambda^* = \pi_\infty^*(\Lambda^*(M)) \wedge \Lambda^*$. The ideal $J\Lambda^*$ is, obviously, differentially closed and its powers $J^k\Lambda^* = J\Lambda^* \wedge J^{k-1}\Lambda^*$ furnish a filtration of the de Rham complex $\{\Lambda^*, d\}$ of $\mathcal{E}_\infty$:

$$\Lambda^* \supset J^1\Lambda^* = J\Lambda^* \supset J^2\Lambda^* \supset \cdots \supset J^k\Lambda^* \supset \dots$$

The spectral sequence associated with this filtration is nothing but the Leray – Serre spectral sequence of the fibering $\pi_\infty : \mathcal{E}_\infty \to M$ expressed in terms of differential forms. Denote it by $\{\tilde{E}_r^{p,q}, \tilde{d}_r\}$.

Proposition 4.2. *There exists a natural isomorphism $\mathcal{C}E_0^{p,q} = \tilde{E}_0^{q,p}$ and under this isomorphism differentials d_0 and $\tilde{d}_0$ anticommute. So, the term $\mathcal{C}E_0$ acquires the second differential sending $\mathcal{C}E_0^{p,q}$ to $\mathcal{C}E_0^{p+1,q}$ becoming a double complex, called the variational bi-complex.*

Note that for a generic diffiety a local choice of independent variables, for instance, by means of a divided chart (see Sec. 1) allows to introduce into the corresponding $\mathcal{C}$-spectral sequence a local bi-complex structure. So, the variational bi-complex is a local form of the $\mathcal{C}$-spectral sequence.

The first term of the $\mathcal{C}$-spectral sequence is of the most interest in applications. This is mainly due to the interpretation of terms $\mathcal{C}E_1^{0,q} = \bar{H}^q$ which was discussed above. By this reason we shall concentrate in the further discussion around the structure of spaces $E_1^{p,q}(\mathcal{O})$. Nevertheless, the importance of the second term is to be stressed. For instance, various kinds of characteristic classes appear to be elements of suitable spaces $E_2^{p,q}(\mathcal{O})$ for some "universal" $\mathcal{O}$'s.

5. The structure of the term $\mathcal{C}E_1$ for $\mathcal{O} = J^\infty$

With any Cartan form $\omega \in \mathcal{C}\Lambda^1(J^\infty)$ a $\mathcal{C}$-differential operator (see the end of Sec. 3) $\Box_\omega : \varkappa \to \mathcal{F}$ is associated: $\Box_\omega(\chi) := \chi\rfloor\omega, \quad \chi \in \varkappa$. Note that $\mathcal{C}D\rfloor\mathcal{C}\Lambda^1 = 0$ so that the insertion operator $\chi\rfloor$ is well defined.

In a local jet-chart one can identify $\varkappa$ with the $\mathcal{F}$-module of generating functions $\mathcal{F}^m$ (Proposition 3.1). If $\omega = \sum a^i_\sigma \omega^i_\sigma$, $a^i_\sigma \in \mathcal{F}$, then

$$\Box_\omega = (\sum_\sigma a^1_\sigma D_\sigma, \dots, \sum_\sigma a^m_\sigma D_\sigma).$$

This row differential operator acts naturally on the generating function column $\varphi = (\varphi_1, \dots, \varphi_m)^T$.

Proposition 5.1. *The correspondence $\omega \mapsto \Box_\omega$ establishes a natural isomorphism of $\mathcal{F}$-modules $\eta : \mathcal{C}\Lambda^1(J^\infty) \to \mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F})$. This isomorphism extends naturally to isomorphisms*

$$\eta = \eta^{p,q} : \mathcal{C}E_0^{p,q}(J^\infty) \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar\Lambda^q(J^\infty)),$$

where $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(k)}(P;Q)$ stands for the $\mathcal{F}$-module of multi-differential skew-symmetric linear operators of multiplicity k on P with values in Q. Under this isomorphism the differential $d_0^{p,q}$ is identified with $(-1)^p\bar d$, i.e., $d_0^{p,q}(\rho) \leftrightarrow (-1)^p \bar d \circ \eta(p)$.

Below we write $\Delta(\chi_1, \dots, \chi_p) \in \bar\Lambda^q$ for a $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar\Lambda^q)$. In this notation $(\bar d \circ \Delta)(\chi_1, \dots, \chi_p) = \bar d(\Delta(\chi_1, \dots, \chi_p)) \in \bar\Lambda^{q+1}$. Hence, the isomorphism η identifies the complex $\{\mathcal{C}E_0^{p,*}(J^\infty), d_0\}$ and the complex $\{\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar\Lambda^*), (-1)^p\bar d\}$, where $\sum_q \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar\Lambda^q)$ is identified naturally with $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar\Lambda^*)$. It is not difficult to describe completely the cohomology of the latter complex. With this purpose we have to introduce $\mathcal{F}$-modules $L_p(\varkappa)$, $p \geqslant 1$. First we put $L_1(\varkappa) = \hat\varkappa := \mathrm{Hom}_{\mathcal{F}}(\varkappa, \bar\Lambda^n)$. If $p > 1$, then $L_p(\varkappa)$ is the submodule of $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa; \hat\varkappa)$ composed of all skew-symmetric $(p-1)$-differential operators from $\varkappa$ to $\hat\varkappa$ such that

$$\Delta^*_{\chi_1,\dots,\chi_{p-2}} = -\Delta_{\chi_1,\dots,\chi_{p-2}}, \quad \forall \quad \chi_i \in \varkappa,$$

where "$*$" stands for the canonical conjugation of $\mathcal{C}$-differential operators and $\Delta_{\chi_1,\dots,\chi_{p-2}}(\chi) = \Delta(\chi_1, \dots, \chi_{p-2}, \chi)$. Recall that $(\Box^*)_{ij} = (\Box_{ji})^*$ for a matrix $\mathcal{C}$-differential operator $\Box = \|\Box_{ij}\|$ and $\Delta^* = \sum (-1)^\sigma D_\sigma \circ a_\sigma$, if $\Delta = \sum a_\sigma D_\sigma$, $a_\sigma \in \mathcal{F}$.

Proposition 5.2. *The n-th cohomology of the complex*

$$\{\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar\Lambda^*), \ (-1)^p\bar d\}$$

is isomorphic to $L_p(\varkappa)$, $p \geqslant 1$, while all other ones are trivial.

Corollary 5.1 (One Line Theorem)**.** *If $p > 0$, then $\mathcal{C}E_1^{p,q}(J^\infty) = 0$ for $q \neq n$ and $\mathcal{C}E_1^{p,n}(J^\infty) = L_p(\varkappa)$. Moreover, $\mathcal{C}E_1^{0,q}(J^\infty) = \bar H^q(J^\infty)$.*

Therefore, the (p,q)-diagram of the term $\mathcal{C}E_1(J^\infty)$ looks as it is shown on Fig. 2 (the "nontrivial" region is shaded).

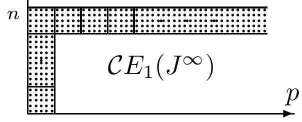

Fig. 2

This configuration implies that

$\mathcal{C}E_r^{0,q}(J^\infty) = \bar H^q(J^\infty)$ for $q < n$, $r \geqslant 1$,

and

$\mathcal{C}E_r^{p,n}(J^\infty) = E_2^{p,n}(J^\infty)$, $r \geqslant 2$.

Also,

$\mathcal{C}E_\infty(J^\infty(E,n)) = H^*(J^1(E,n))$

and

$\mathcal{C}E_\infty(J^\infty(\pi)) = H^*(E)$.

This shows that

$$\bar H^q(J^\infty(E,n)) = H^q(J^1(E,n)) \text{ and } \bar H^q(J^\infty(\pi)) = H^q(E) \text{ for } q < n.$$

Also, $\mathcal{C}E_2^{p,n}(J^\infty(E,n)) = H^{p+n}(J^1(E,n))$ and $\mathcal{C}E_2^{p,n}(J^\infty(\pi)) = H^{p+n}(E)$. In other words, the n-th row of $\mathcal{C}E_1(J^\infty)$ is identified with the complex

$$\bar{H}^n(J^\infty) \xrightarrow{d_1} L_1(\varkappa) \xrightarrow{d_1} \dots \xrightarrow{d_1} L_p(\varkappa) \xrightarrow{d_1} \dots \qquad (8)$$

and p-th cohomology of it is isomorphic to $H^{p+n}(J^1(E,n))$ (resp., $H^{p+n}(E)$) for $J^\infty = J^\infty(E,n)$ (resp., $J^\infty(\pi)$). The differential d_1 of this complex is described as follows.

First, the differential

$$d_1 = d_1^{0,n} : \mathcal{C}E_1^{0,n}(J^\infty) = \bar{H}^n(J^\infty) \to \hat{\varkappa} = \mathcal{C}E_1^{1,n}(J^\infty)$$

is the Euler operator assigning to a functional (action) the corresponding Euler – Lagrange equation. It is natural to denote the cohomology class of a Lagrangian density $Ldx_1 \wedge \dots \wedge dx_n = Ldx$ by $\int L\,dx$. In this notation we have

$$d_1\left(\int L\,dx\right) = l_L^*(1),$$

i.e., the Euler – Lagrange equation corresponding to the functional $\int Ldx$ is $l_L^*(1) = 0$. So, complex (8) may be seen as a "resolvent" of the Euler operator. The description of its cohomology given above shows that it is locally acyclic.

Next, note that $L_2(\varkappa) = \mathcal{C}\mathrm{Diff}^{\mathrm{anti}}(\varkappa, \hat{\varkappa})$, where

$$\mathcal{C}\mathrm{Diff}^{\mathrm{anti}}(\varkappa, \hat{\varkappa}) = \{\Delta \in \mathcal{C}\mathrm{Diff}(\varkappa, \hat{\varkappa}) | \Delta^* = -\Delta\}.$$

Then the differential $d_1 = d_1^{1,n} : \mathcal{C}E_1^{1,n}(J^\infty) = \hat{\varkappa} \to \mathcal{C}\mathrm{Diff}^{\mathrm{anti}}(\varkappa, \hat{\varkappa}) = \mathcal{C}E_1^{2,n}(J^\infty)$ acts as

$$d_1(\varphi) = l_\varphi - l_\varphi^*,\ \varphi \in \bar{\varkappa}.$$

This shows, for instance, that an equation $\varphi = 0$ is locally the Euler – Lagrange one iff $l_\varphi^* = l_\varphi$ and the obstruction to be such globally belongs to $H^{n+1}(J^1(E,n))$ or $H^{n+1}(E)$, respectively. For $p > 1$ the differential $d_1 = d_1^{p,n} : L_p(\varkappa) \to L_{p+1}(\varkappa)$ looks as follows:

$$d_1(\Delta)(\chi_1, \dots, \chi_p) = \textstyle\sum_i (-1)^{p-1} \ni_{\chi_i}(\Delta(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_p)) +$$
$$\textstyle\sum_{i<j} (-1)^{i+j} \Delta(\{\chi_i, \chi_j\}, \chi_1, \dots, \hat{\chi}_i, \dots, \hat{\chi}_j, \dots, \chi_p) +$$
$$\textstyle\sum (-1)^{i-1} [(p-1) l_{\chi_i}^*(\Delta(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_p)) - l^*_{\Delta(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_p)}(\chi_i)]$$

The reader can see that these formulae are rather complicated and contain both "traditional" terms (first two summands) and some unexpected ones (the third summand).

6. The structure of the term $\mathcal{C}E_1$ for $\mathcal{O} = \mathcal{E}_\infty$

Locally a PDE system $\mathcal{E}$ can be given by $F = 0$ with $F = (F_1, \dots, F_l)$, $F_l \in \mathcal{F}(J^\infty)$. But globally it is not so (for instance, for the equation of minimal surfaces) and F must be thought as an element of a suitable (projective) $\mathcal{F}(J^\infty)$-module P. So, further on we suppose that $\mathcal{E} = \{F = 0\}$ with $F \in P$ and locally $F = (F_1, \dots, F_l)$. Moreover, $\mathcal{E}$ will be assumed formally integrable and *regular*. The latter means that the ideal of $\mathcal{E}_\infty$ in J^∞ is generated by F together with all $\square(F)$, $\square \in \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(J^\infty))$.

Consider the universal linearization operator $l_F : \varkappa \to P$ and the map

$$l_F^{\mathrm{Diff}} : \mathcal{C}\mathrm{Diff}(P, \mathcal{F}(J^\infty)) \to \mathcal{C}\mathrm{Diff}(\varkappa, \mathcal{F}(J^\infty))$$

with $l_F^{\mathrm{Diff}}(\Delta) = \Delta \circ l_F$. The equation $\mathcal{E}$ is called *nonoverdetermined*, if the kernel of l_F^{Diff} restricted to $\mathcal{E}_\infty$ is trivial. This condition is guaranteed by nondegeneracy of the main symbol of l_f, i.e., by the fact that the rank of the $l \times m$-matrix with entries $m_{ij} = \sum_\sigma (\partial F_i/\partial u^j_\sigma) p^\sigma$, where $p^\sigma = p_1^{\sigma_1} \cdot \cdots \cdot p_n^{\sigma_1}$ for $\sigma = (\sigma_1, \ldots, \sigma_n)$ and p_i's are some formal variables, is equal to l.

For an arbitrary diffiety $\mathcal{O}$ the $\mathcal{F}(\mathcal{O})$-module $\mathcal{C}^p\Lambda^p(\mathcal{O})$, $p \geqslant 0$, carries a natural $\mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{O}), \mathcal{F}(\mathcal{O}))$-module structure extending the original $\mathcal{F}(\mathcal{O})$-module one. In fact, if $\Delta = X_1 \circ \cdots \circ X_k$ with $X_1, \ldots, X_k \in \mathcal{C}D(\mathcal{O})$ and $\omega \in \mathcal{C}^p\Lambda^p(\mathcal{O})$, then $\Delta(\omega) = L_{X_1}(\ldots L_{X_k}(\omega)\ldots)$. This action is well defined due to the fact that $L_{fX}(\omega) = fL_X(\omega)$ for $X \in \mathcal{C}D(\mathcal{O})$.

An important property of a $\mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{O}), \mathcal{F}(\mathcal{O}))$-module R is that any $\mathcal{C}$-differential operator $\square : P \to Q$ can be extended canonically to a $\mathcal{C}$-differential operator $\square_R : P \otimes_\mathcal{F} R \to Q \otimes_\mathcal{F} R$ (see Sec. 10). This construction is a key one for what follows.

Theorem 6.1 (Two Line Theorem)**.** *If $\mathcal{E}$ is regular and nonoverdetermined, then all terms $\mathcal{C}E_1^{p,q}(\mathcal{E}_\infty)$ with $p > 0$, $q \neq n-1, n$ are trivial. Moreover, the terms $\mathcal{C}E_1^{p,n-1}(\mathcal{E}_\infty)$ and $\mathcal{C}E_1^{p,n}(\mathcal{O})$ are isomorphic to the skew-symmetric parts of the kernel and the cokernel, respectively, of the extended operator $l^*_{[F]}$:*

$$(l^*_{[F]})_{\mathcal{C}^{p-1}\Lambda^{p-1}} : \hat{P} \otimes_{\mathcal{F}(\mathcal{E}_\infty)} \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}_\infty) \to \hat{\varkappa} \otimes_{\mathcal{F}(\mathcal{E}_\infty)} \mathcal{C}^{p-1}\Lambda^{p-1}(\mathcal{E}_\infty).$$

So, the eventually nontrivial terms of the $\mathcal{C}$-spectral sequence of $\mathcal{E}$ for $r \geqslant 1$ are situated in the shaded region of the (p, q)-diagram in Fig. 3.

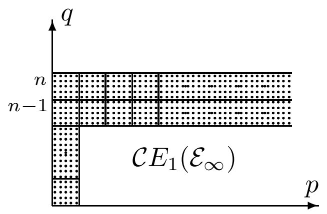

Fig. 3

The standard spectral sequence arguments show that $\mathcal{C}E_r(\mathcal{E}_\infty) = \mathcal{C}E_\infty(\mathcal{E}_\infty)$ for $r \geqslant 3$ and $\bar{H}^q(\mathcal{E}_\infty) = H^q(\mathcal{E}_\infty)$ for $q < n-1$. If $\mathcal{E}$ is formally integrable, then, additionally, $H^*(\mathcal{E}_\infty) = H^*(\mathcal{E}_{(1)})$ (or, even $H^*(\mathcal{E})$, if $\mathcal{E} \subset J^k(\pi)$) with $\mathcal{E}_{(1)}$ being the first prolongation of $\mathcal{E}$. These facts help to estimate various terms of $\mathcal{C}E_2$ and $\mathcal{C}E_3$ of a special interest. For instance, one can see that $\mathcal{C}E_2^{0,n-1}(\mathcal{E}_\infty) = \ker d_1^{0,n-1} = H^{n-1}(\mathcal{E}_{(1)})$ (resp., $H^{n-1}(\mathcal{E})$). Therefore, the cohomology $H^{n-1}(\mathcal{E}_{(1)})$ (respectively, $H^{n-1}(\mathcal{E})$) takes part of the conservation laws space $\bar{H}^{n-1}(\mathcal{E}_\infty) = \mathcal{C}E_1^{0,n-1}(\mathcal{E}_\infty)$. The corresponding conservation laws are not sensitive to deformations of solutions and as such are as a rule negligible. So, the differential $d_1^{0,n-1}$ embeds conservation laws of $\mathcal{E}$ up to that "negligible" part into the term $\mathcal{C}E_1^{1,n-1}(\mathcal{E}_\infty)$ which is isomorphic, according to the two line theorem, to $\ker l^*_{[F]}$. So, the problem of finding all conservation laws admitted by the equation $\mathcal{E}$ is reduced to solving the equation $l^*_{[F]}(\psi) = 0$. Recollecting (Theorem 3.1) that the symmetries of $\mathcal{E}$ are solutions of the equation $l_{[F]}(\varphi) = 0$ one can see that symmetries and conservation laws are in a sense dual concepts which fall in an interaction when operators $l_{[F]}$ and $l^*_{[F]}$ are in a way related one another. For instance, this is the case if $l^*_{[F]} = \pm l_{[F]}$. For such an equation any conservation law determines a symmetry of it. This reveals the nature of the Noether theorem since equations obtained from variational principles have self-adjoint universal linearizations (see Sec. 5), i.e., $l^*_{[F]} = l_{[F]}$. On the other hand, we see that the same relations between symmetries and conservation laws remain valid also for equations in a sense opposite to the Euler–Lagrange ones, i.e., for which $l^*_{[F]} = -l_{[F]}$.

We mention also without entering into details that the isomorphism

$$\mathcal{C}E_1^{1,n}(\mathcal{E}_\infty) = \operatorname{coker} l^*_{[F]}$$

may be viewed as a very generalized Lagrange's multipliers method in the Calculus of Variations.

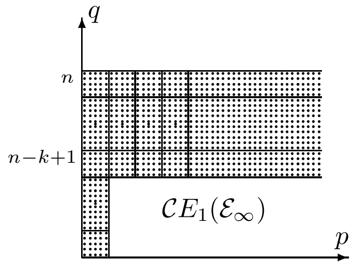

Fig. 4

The above discussion illustrates the $\mathcal{C}$-spectral sequence in action. It is worth also to mention that the hypothesis of the two line theorem is, as a rule, satisfied by most equations of mathematical physics, differential geometry, etc. There are, however, very important exceptions. One of them are Yang–Mills equations. Nevertheless, the techniques used to prove the One or Two Line Theorem are sufficiently efficient to approach the general case. One of the typical results in this direction is reported below.

From the formal theory of PDE's it is known that under a weak regularity condition any linear DO $\Delta : P \to Q$ can be extended to a complex R:

$$0 \to P \xrightarrow{\Delta} Q = Q_1 \xrightarrow{\Delta_1} Q_2 \xrightarrow{\Delta_2} \dots \xrightarrow{\Delta_{p-2}} Q_{p-1} \to 0 \tag{9}$$

of differential operators such that the sequence

$$\mathrm{Diff}(P, A) \xleftarrow{S_0} \mathrm{Diff}(Q, A) \xleftarrow{S_1} \dots \xleftarrow{S_{p-2}} \mathrm{Diff}(Q_{p-1}, A) \leftarrow 0 \tag{10}$$

with $S_i(\square) = \square \circ \Delta_i$ for $\square \in \mathrm{Diff}(Q_{i+1}, A)$ and $A = C^\infty(M)$ being the base algebra is exact. Complex (10) is called a *compatibility complex* for Δ and its length $p-1$ does not exceed n.

The above result is generalized naturally to $\mathcal{C}$-differential operators. Such an operator is of *length* k, if $k-1$ is the minimal length of compatibility complexes it admits.

Theorem 6.2 (p-Line Theorem)**.** *If the operator $l_{[F]}$ is of length p, then the terms $\mathcal{C}E_1^{p,q}(\mathcal{E}_\infty)$ with $p \geqslant 1$ are trivial for $q \leqslant n-p$ (and, obviously, for $q \geqslant n$).*

So, eventually nontrivial terms $\mathcal{C}E_1^{p,q}(\mathcal{O})$ are situated in the shaded region of the (p,q)-diagram on Fig. 4.

These terms can be described in a more explicit way. For instance, let

$$0 \to \varkappa = \hat{Q}_0 \xrightarrow{l_{[F]}} P = Q_1 \xrightarrow{\Delta_1} \dots \xrightarrow{\Delta_{p-1}} Q_{p-1} \to 0$$

be a compatibility complex for $l_{[F]}$. Passing to adjoint operators one gets the complex R^*

$$0 \to \hat{Q}_{p-1} \xrightarrow{\Delta^*_{p-1}} \hat{Q}_{p-2} \xrightarrow{\Delta^*_{p-2}} \dots \xrightarrow{\Delta^*_1} \hat{Q}_1 \xrightarrow{l^*_{[F]}} \hat{Q}_0 \to 0. \tag{11}$$

Then $\mathcal{C}E_1^{1,n-i}(\mathcal{O}) = H^i(R^*)$. Similarly can be described terms $\mathcal{C}E_1^{p,q}(\mathcal{E}_\infty)$ with $p > 1$ and $n-p+1 \leqslant q \leqslant n$.

7. The conception of Secondary Calculus: vectors and differential forms

Three natural operation put in interaction vector fields and differential forms on a manifold: the exterior differential $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$, the insertion $i_X(\omega) = X\rfloor\omega$ with $X \in \mathcal{D}(M)$, $\omega \in \Lambda^*(M)$, and the Lie derivative $L_X(\omega)$. They are related by the infinitesimal Stokes formula $L_X = i_X \circ d + d \circ i_X$. Having at disposal higher symmetries and $\mathcal{C}$-spectral sequence one can observe that for any diffiety $\mathcal{O}$ the insertion operation of a higher symmetry χ into an element $\theta \in \mathcal{C}E_1(\mathcal{O})$ is well defined by passing to quotients as well as that of the Lie derivative of θ along χ. Denoting them by

$$i_\chi \ (\text{or } \chi\rfloor) : \mathcal{C}E_1^{p,q}(\mathcal{O}) \to \mathcal{C}E_1^{p-1,q}(\mathcal{O})$$

and

$$L_\chi : \mathcal{C}E_1^{p,q}(\mathcal{O}) \to \mathcal{C}E_1^{p,q}(\mathcal{O})$$

respectively, one can then prove that they are related by means of the following "infinitesimal Stokes formula":

$$L_\chi = i_\chi \circ d_1 + d_1 \circ i_\chi,$$

where d_1 stands as above for the differential in the first term of the $\mathcal{C}$-spectral sequence. Moreover, these operations can be introduced naturally only in the term $\mathcal{C}E_1(\mathcal{O})$.

For $\mathcal{O} = J^\infty$ operations i_χ and L_χ can be described explicitly in the notation of Sec. 5 as follows.
Insertion $i_\chi = \chi\rfloor$:

$$p = 1 \Rightarrow \chi\rfloor h = \text{ (the cohomology class of) } h(\chi) \text{ in } \bar{H}^n(J^\infty)),$$
$$\text{if } h \in \hat{\varkappa} = \mathcal{C}E_1^{1,n}(J^\infty),$$
$$p > 1 \Rightarrow (\chi\rfloor\Delta)(\chi_1, \ldots, \chi_{p-2}) = \Delta(\chi, \chi_1, \ldots, \chi_{p-2}), \text{ if } \Delta \in L_p(\varkappa).$$

Lie derivative L_χ :

$$p = 0 \Rightarrow L_\chi(\mathcal{L}) = \int \Game_\chi(\omega)\, dx, \text{ if } \mathcal{L} = \int \omega\, dx \in \bar{H}^n(J^\infty) = \mathcal{C}E_1^{0,n}(J^\infty),$$
$$p = 1 \Rightarrow L_\chi(h) = \Game_\chi(h) + l_h^*(\chi), \text{ if } h \in \hat{\varkappa} = \mathcal{C}E_1^{1,n}(J^\infty),$$
$$p > 1 \Rightarrow L_\chi(\Delta)(\chi_1, \ldots, \chi_{p-1}) = \Game_\chi(\Delta(\chi_1, \ldots, \chi_{p-1})) +$$
$$\sum_i \Delta(\chi_1, \ldots, \{\chi_i, \chi\}, \ldots, \chi_{p-1}) + l_\chi^*(\Delta(\chi_1, \ldots, \chi_{p-1})),$$
$$\text{if } \Delta \in L_p(\varkappa) = \mathcal{C}E_1^{p,n}(J^\infty).$$

Hence, we have established the key analogy

$$D(M) \leftrightarrow \operatorname{Sym}\mathcal{O}, \quad \Lambda^p(M) \leftrightarrow \mathcal{C}E_1^{p,*}(\mathcal{O}) = \sum_q \mathcal{C}E_1^{p,q}(\mathcal{O})$$

which becomes an identity for any 0-Dimensional diffiety $\mathcal{O} = M$. This means that the modified "Bohr correspondence principle" $\operatorname{Dim}\mathcal{O} \to 0$ is satisfied, if $\operatorname{Sym}\mathcal{O}$ and $\mathcal{C}E_1^{p,*}(\mathcal{O})$ are interpreted as *secondary vector fields* and *secondary differential forms*, respectively. We used here "secondary" just conventionally in order to stress the observed analogy between the classical "primary" Calculus and the "$\mathcal{C}$-respecting Calculus" on diffieties.

There is a number of different arguments that sustain this analogy. The following is classical: the Euler operator, i.e., $d_1^{0,n}$ plays the same role in Calculus of Variation as the differential $d : C^\infty(M) \to \Lambda^1(M)$ when looking for the extremes of functions.

So, we are led to the suspicion that all natural ingredients of the classical Calculus have *secondary analogues.* The problem to find them is called *the secondarization problem* and by many reasons we put it in parallel with the quantization problem (see, for instance, [63]). Essentially, the secondarization problem is the search of "right definitions" and as such is not a very usual one. The recent history of the $\mathcal{C}$-spectral sequence (variational bi-complex) method shows that it is in no way trivial. Below in this section we reproduce an old examples of secondarization concerning the Poisson bracket. Some new ones are discussed in the rest of these notes.

In the classical situation a Poisson bracket $\{\cdot,\cdot\}$ on a manifold M can be given by means of a homomorphism $\Gamma : \Lambda^1(M) \to D(M)$. Namely, one has to put $\{f, g\} = \Gamma(df)(g)$. We call Γ *Hamiltonian*, if the so-defined bracket is skew-symmetric and satisfies Jacobi identity. So, the Poisson manifold structures on M are in one-to-one correspondence with Hamiltonian homomorphisms.

To secondarize the concept of a Poisson structure one has to secondarize that of the exterior differential d and of a Hamiltonian homomorphism. To secondarize the latter one needs the "secondary $\Lambda^1(M)$", the "secondary $D(M)$" and "secondary homomorphisms". All these ingredients are already at our disposal except for the last one, i.e., secondary homomorphisms. In the next section the reader will see that this is a very delicate point and the concept of a secondary homomorphism looks rather surprising. For the time being we postpone the complete solution of this problem and shall interpret "secondary homomorphisms" for $\mathcal{O} = J^\infty$ as $\mathcal{C}$-differential operators. Numerous known Poisson structures in the theory of integrable systems in favor of this choice.

Thus, the previous discussion leads us to the following definition.

Definition 7.1. A *secondary Poisson structure* on J^∞ is defined to be a Lie algebra structure in $\bar{H}^n(J^\infty) = \mathcal{C}E_1^{0,n}(J^\infty)$ given by a $\mathcal{C}$-differential operator $\Delta : \mathcal{C}E_1^{1,n}(\mathcal{O}) = \hat{\varkappa} \to \varkappa = \operatorname{Sym} J^\infty$:

$$\{\mathcal{L}_1, \mathcal{L}_2\} = L_{\Delta d_1(\mathcal{L}_1)}(\mathcal{L}_2), \tag{12}$$

where $\mathcal{L}_i \in \bar{H}^n(J^\infty)$, $i = 1, 2$, and $d_1 = d_1^{0,n}$ is, as above, the differential in the first term of the $\mathcal{C}$-spectral sequence (= the Euler operator).

As in the classical case a $\mathcal{C}$-differential operator $\Delta : \hat{\varkappa} \to \varkappa$ is called *Hamiltonian*, if the corresponding bracket (12) is a Lie bracket, i.e., a skew-symmetric one satisfying Jacobi identity.

A natural question to characterize explicitly Hamiltonian operators can be answered by making use of some simple formulae of Secondary Calculus.

Theorem 7.1. *A $\mathcal{C}$-differential operator $\Delta : \hat{\varkappa} \to \varkappa$ is Hamiltonian iff it is skew-adjoint, i.e., $\Delta^* = -\Delta$, and*

$$[\ni_{\Delta\varphi}, \Delta] = l_{\Delta\varphi} \circ \Delta + \Delta \circ l^*_{\Delta\varphi}$$

takes place for any polynomial $\sigma = \sigma(x)$ in x of order $\leqslant \deg \Delta + \Phi(\Delta)$ with $\Phi(\Delta)$ being the highest jet order of coefficients of Δ.

This theorem allows, for instance, a complete classification of Poisson structures in field theory for small values of n, m and $\deg \Delta$ and even to prove a kind of Frobenius lemma for them.

8. Secondary modules

Of course, the general concept of a secondary differential operator should be central in Secondary Calculus. It, however, must be preceded by that of a *secondary module* over the "secondary smooth function algebra" of a diffiety $\mathcal{O}$. The latter turned out to be rather delicate and it may happen that the solution presented below will require some polishing.

Fix a diffiety $\mathcal{O}$. A $\mathcal{C}$-complex $K = \{K_i, \Delta_i\}$ over $\mathcal{O}$ is a complex

$$0 \to K_0 \xrightarrow{\Delta_0} \dots K_1 \xrightarrow{\Delta_1} \dots \xrightarrow{\Delta_{i_1}} K_i \xrightarrow{\Delta_i} \dots$$

of $\mathcal{F}(\mathcal{O})$-modules and $\mathcal{C}$-differential operators. With given $\mathcal{C}$-complexes $(K, \Delta) = \{K_i, \Delta_i\}$ and $(\tilde{K}, \tilde{\Delta}) = \{\tilde{K}_i, \tilde{\Delta}_i\}$ one can associate the complex $\{\mathcal{GC}\mathrm{Diff}(K, \tilde{K}), L\}$, where $\mathcal{GC}\mathrm{Diff}(K, \tilde{K}) = \sum_\alpha \mathcal{GC}\mathrm{Diff}^\alpha(K, \tilde{K})$ and

$$\mathcal{GC}\mathrm{Diff}^\alpha(K, \tilde{K}) = \{\square : K \to \tilde{K} \mid \square(K_i) \subset \tilde{K}_{i+\alpha},\, \square|_{K_i} \in \mathcal{C}\mathrm{Diff}(K_i, \tilde{K}_{i+\alpha}),\, \forall i\}$$

while the differential is defined as

$$L(\square) = (-1)^\alpha \square \circ \Delta - \tilde{\Delta} \circ \square \text{ for } \square \in \mathcal{GC}\mathrm{Diff}^\alpha(K, \tilde{K}).$$

Denote its α-th cohomology by $\mathcal{CL}^\alpha(K,\tilde{K})$ and put $\mathcal{CL}(K,\tilde{K}) = \sum_\alpha \mathcal{CL}^\alpha(K,\tilde{K})$. So, elements of $\mathcal{CL}(K,\tilde{K})$ are homotopy classes of graded cochain maps from K to $\tilde{K}$. Therefore, any $h \in \mathcal{CL}(K,\tilde{K})$ generates naturally a map in cohomology which is also denoted by $h : H^*(K) \to H^*(\tilde{K})$. Two $\mathcal{C}$-complexes K and $\tilde{K}$ are said to be *$\mathcal{C}$-homotopy equivalent* if there exist $h \in \mathcal{CL}^0(K,\tilde{K})$ and $\tilde{h} \in \mathcal{CL}(\tilde{K},K)$ such that $\tilde{h}h = \mathrm{id}_{H^*(K)}$, $h\tilde{h} = \mathrm{id}_{H^*(\tilde{K})}$.

Secondary modules turn out to be special $\mathcal{C}$-homotopy equivalence classes of $\mathcal{C}$-complexes. In less solemn terms this means that the cohomology space $V = H^*(K)$ is the main object of our interest but the complex K itself can vary in the limits of the corresponding $\mathcal{C}$-homotopy class. Nevertheless, the cohomological origin of V is indispensable to put it in interaction with other similar objects. This point seems to be of crucial importance in what concerns applications to (quantum) field theory.

Definition 8.1. The "*secondary smooth functions algebra*" of a diffiety $\mathcal{O}$ is the $\mathcal{C}$-homotopy type of the complex $\{\mathcal{GC}\mathrm{Diff}(\bar{\Lambda}^*(\mathcal{O}), \bar{\Lambda}^*(\mathcal{O})), L\}$.

According to the scheme at the end of Sec. 7, $\bar{H}^n(\mathcal{O}) = E_1^{0,*}(\mathcal{O})$ is the secondary analogue of the smooth function algebra $C^\infty(M)$ on a smooth manifold M. The following result shows that the above definition refines this idea.

Theorem 8.1. *A natural isomorphism takes place:*

$$\mathcal{CL}(\bar{\Lambda}^*(\mathcal{O}), \bar{\Lambda}^*(\mathcal{O})) = \bar{H}^*(\mathcal{O}).$$

Put now $\mathcal{S}(\mathcal{O}) = \mathcal{GC}\mathrm{Diff}(\bar{\Lambda}^*(\mathcal{O}), \bar{\Lambda}^*(\mathcal{O}))$ and note that $\bar{d} = \bar{d}(\mathcal{O}) \in \mathcal{S}(\mathcal{O})$.

Definition 8.2. An *$\mathcal{S}$-module* (over $\mathcal{O}$) is a $\mathcal{C}$-complex (K,Δ) supplied with a (left) $\mathcal{S}(\mathcal{O})$-module structure $\mu : \mathcal{S}(\mathcal{O}) \to \mathcal{GC}\mathrm{Diff}(K,K)$ such that

(i) μ is $\mathcal{C}$-differential,
(ii) if $f \in \mathcal{F}(\mathcal{O})$, then $\mu(f) = f$ (multiplication by f operator),
(iii) $\mu(\bar{d}) = \Delta$.

An $\mathcal{S}$-module is, in particular, an unitary $\bar{\Lambda}^*(\mathcal{O})$-module. Besides, the structure homomorphism μ is, in view of (iii), a cochain map of sending $\{\mathcal{S}(\mathcal{O}), L\}$ to $\{\mathcal{GC}\mathrm{Diff}(K,K), L\}$. So, by passing to cohomologies it defines a map $\bar{H}^*(\mathcal{O}) \to \mathcal{CL}(K,K)$ and, therefore, an action $\bar{H}^*(\mathcal{O}) \times \bar{H}^*(K) \to \bar{H}^*(K)$.

The differential Δ is a first order $\mathcal{C}$-differential operator as it results from (iii). $\mathcal{S}$-modules possess a number of "good" properties allowing naturally expected constructions with them.

The concept of a morphism of two $\mathcal{S}$-modules is obvious. We only stress that the corresponding map $K \to \tilde{K}$ is supposed to be a graded $\mathcal{C}$-differential operator. Accordingly, two $\mathcal{S}$-modules (K,μ) and $(\tilde{K},\tilde{\mu})$ are $\mathcal{S}$-homotopy equivalent, if they are $\mathcal{C}$-homotopy equivalent and the cochain maps h and $\tilde{h}$ (see above) realizing this equivalence are additionally $\mathcal{S}$-homomorphisms.

Definition 8.3. A *secondary module* over a diffiety $\mathcal{O}$ is an $\mathcal{S}$-homotopy type of $\mathcal{S}$-modules.

Now we are ready to introduce secondary differential operators. Let (K,Δ,μ), $(\tilde{K},\tilde{\Delta},\tilde{\mu})$ be $\mathcal{S}$-modules and $\Xi : K \to \tilde{K}$ be a graded differential operator (not necessarily $\mathcal{C}$-differential). Put for a $\Box \in \mathcal{S}(\mathcal{O})$

$$\delta_\Box(\Xi) = \Xi \circ \mu(\Box) - (-1)^{\Box \cdot \Xi}\tilde{\mu}(\Box)) \circ \Xi.$$

Definition 8.4. Let $\mathcal{M}, \tilde{\mathcal{M}}$ be secondary modules (K,δ,μ) and $(\tilde{K},\tilde{\Delta},\tilde{\mu})$, respectively. A *secondary differential operator* of order $\leqslant k$ from $\mathcal{M}$ to $\mathcal{M}$ is a map

$H^*(K) \to H^*(\tilde{K})$ induced by a cochain map $\Xi : K \to \tilde{K}$ which is a graded differential operator over $\mathcal{F}(\mathcal{O})$ such that

$$\delta_{\square_1}(\delta_{\square_2}\dots(\delta_{\square_k}(\Xi))\dots) \in \mathcal{GC}\mathrm{Diff}(K, \tilde{K})$$

for any $\square_1, \dots, \square_k \in \mathcal{S}(\mathcal{O})$.

Example 8.1. Consider a secondary module Ω_p represented by the $\mathcal{S}$-module $(\mathcal{C}E_0^{p,*}(\mathcal{O}), d_0)$ (see Sec. 9, Example 9.1, for a description of the action of $\mathcal{S}(\mathcal{O})$ on $\mathcal{C}E_0^{p,*}(\mathcal{O})$). Then $d_1 : E_1^{p,*}(\mathcal{O}) \to E_1^{p+1,*}(\mathcal{O})$ is a first order secondary differential operator.

Note that in virtue of the One Line Theorem (Sec. 5) for $\mathcal{O} = J^\infty$ the differential $d_1 : \mathcal{C}E_1^{0,*}(\mathcal{O}) \to \mathcal{C}E_1^{1,*}(\mathcal{O})$ reduces essentially to the Euler operator assigning to an action the corresponding Euler – Lagrange equation. The Euler operator written explicitly in terms of local jet-coordinates looks as a differential operator of infinite order. But it is just a first order secondary differential operator as the secondary analogue of the classical differential $d : \mathcal{C}^\infty(M) \to \Lambda^1(M)$ should be.

Example 8.2. If $\mathcal{O} = \mathcal{E}_\infty$ and $\chi \in \mathrm{Sym}\,\mathcal{O}$, then the secondary Lie derivative L_χ along χ is a first order secondary differential operator.

For $\mathcal{O} = J^\infty$ secondary “scalar” differential operators, i.e., those acting on the “secondary smooth function algebra” (Definition 8.1), can be described as follows. Fix a local jet-chart (x_i, u_σ^j). An operator $\square : \mathcal{F} \to \mathcal{F}$, $\mathcal{F} = \mathcal{F}(J^\infty)$ is called *vertical* (with respect to the chosen chart), if its local expression does not contain derivations with respect to x_i's. In other words, it looks as

$$\square = \sum_{0<|s|\leqslant k} \sum_{\substack{i_1,\dots,i_s\\ \sigma_1,\dots,\sigma_s}} a_{\sigma_1,\dots,\sigma_s}^{i_1,\dots,i_s} \frac{\partial^s}{\partial u_{\sigma_1}^{i_1}\dots\partial u_{\sigma_s}^{i_s}}.$$

Proposition 8.1. *A vertical differential operator represents a secondary differential operator iff* $[D_i, \square] = 0$, $i = 1, \dots, n$, *and conversely. Any such operator is of the form*

$$\ni_\nabla = \sum_{i,\sigma} \mathcal{L}_\sigma(\nabla_i) \circ \frac{\partial}{\partial u_\sigma^i}$$

with $\nabla = (\nabla_1, \dots, \nabla_m)$ *being a vertical differential operator and*

$$\mathcal{L}_\sigma(\nabla_i) = [D_{i_1}, [D_{i_2}, \dots, [D_{i_s}, \nabla_i]\dots]] \text{ for } \sigma = (i_1, \dots, i_s).$$

Vertical operators ∇'s play the same role as generating functions for evolutionary derivations. However, ∇ is not defined uniquely if $\mathrm{ord}\,\nabla > 0$. In fact, $\ni^2 = 0$, i.e., $\ni_{\ni_\Delta} = 0$ for any vertical Δ.

A shortage of the above approach is that it does not manifest directly tha cohomological nature of secondary operators. How it can be done for evolutionary derivations (Example 8.2) and similar operators is shown in Sec. 11. See also [25], [27] and [26] in this connection. A systematic exposition of this topics will appear somewhere.

9. Secondary modules and flat connections

The class of secondary modules we are going to describe now is distinguished at least by two features. First, the $\mathcal{C}$-spectral sequence construction can be literally repeated for them and analogues of the results reported in Sec. 4 – 6 can be proved. Second, they attach the gauge theory directly to Secondary Calculus.

The following is the $\mathcal{C}$-analogue of the standard notion of a connection.

Definition 9.1. Let P be an $\mathcal{F}(\mathcal{O})$-module. An $\mathcal{F}(\mathcal{O})$-module homomorphism $\nabla : \mathcal{C}D(\mathcal{O}) \to \mathcal{C}\mathrm{Diff}_1(P, P)$ is called a *$\mathcal{C}$-connection* in P if

$$\nabla(X)(fp) = f\nabla(X)(p) + X(f)p$$

for any $f \in \mathcal{F}(\mathcal{O})$, $X \in \mathcal{C}D(\mathcal{O})$, $p \in P$.

Below we follow the standard notation and write ∇_X instead of $\nabla(X)$. A $\mathcal{C}$-connection is said to be *flat*, if $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$ for any $X, Y \in \mathcal{C}D(\mathcal{O})$. Since ∇_X is a $\mathcal{C}$-differential operator, it can be restricted to any integral submanifold of $\mathcal{O}$ by supplying it with a "usual" connection.

Associate with P the graded $\bar{\Lambda}^*(\mathcal{O})$-module $\bar{\Lambda}P = \sum_i \bar{\Lambda}^i P$ with

$$\bar{\Lambda}P = \bar{\Lambda}^*(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} P,\ \bar{\Lambda}^i P = \bar{\Lambda}^i(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} P.$$

Elements of $\bar{\Lambda}P$ are called *P-valued horizontal differential forms.*

A flat $\mathcal{C}$-connection ∇ in P supplies $\bar{\Lambda}P$ with a differential d_∇:

$$d_\nabla(\omega)(X_1, \ldots, X_{p+1}) = \sum_i (-1)^{i-1}\nabla_{X_i}(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) +$$
$$\sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})$$

with $\omega \in \bar{\Lambda}^p P$ and $X_i \in \mathcal{C}D(\mathcal{O})$ for all i. Obviously, $(\bar{\Lambda}P, d_\nabla)$ is a $\mathcal{C}$-complex.

Proposition 9.1. *The $\mathcal{C}$-complex $(\bar{\Lambda}P, d_\nabla)$ admits a unique $\mathcal{S}$-module structure μ such that*

$$\mu(\omega) = \omega^P,\ \mu(i_X) = i_X^P,\ \forall\omega \in \bar{\Lambda}^*(\mathcal{O}),\ X \in \mathcal{C}D(\mathcal{O}),$$

where ω^P stands for the left multiplication by ω in $\bar{\Lambda}P$ and i_X^P denotes the insertion of X into P-valued forms.

This is almost a direct consequence of the fact that the algebra $\mathcal{S}(\mathcal{O})$ is generated by the operators $\bar{d}, i_X, X \in \mathcal{C}D(\mathcal{O})$, and by multiplications by ω, $\omega \in \bar{\Lambda}^*(\mathcal{O})$.

Corollary 9.1. *A flat connection ∇ defines a unique $\mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{O}), \mathcal{F}(\mathcal{O}))$-module structure ν on P such that*

(i) $\nu(f) = f$ *(multiplication by $f \in \mathcal{F}(\mathcal{O})$),*
(ii) $\nu(X) = \nabla_X$, $X \in \mathcal{C}D(\mathcal{O})$,
(iii) $[f, \nu(\Box)] = \nu([f, \Box])$, $\Box \in \mathcal{C}\mathrm{Diff}(\mathcal{F}(\mathcal{O}), \mathcal{F}(\mathcal{O}))$.

So, a flat $\mathcal{C}$-connection ∇ supplies P with *a horizontal module* structure (see [51]).

We distinguish $\mathcal{S}$-module (respectively, horizontal module) structures described in Proposition 9.1 (respectively, Corollary 9.1) by calling them *natural.*

Hence, for a projective $\mathcal{F}(\mathcal{O})$-module P there are natural identifications:

Flat $\mathcal{C}$-connections in P	$\longleftrightarrow$	Natural horizontal module structures in P	$\longleftrightarrow$	Natural $\mathcal{S}$-module structures in $\bar{\Lambda}P$

Example 9.1. If $X \in \mathcal{C}D(\mathcal{O})$ and $\omega \in \mathcal{C}^p\Lambda^p$, then $L_{fX}(\omega) = fL_X(\omega)$ due to $X \rfloor \omega = 0$. This shows that $\mathcal{C}^p\Lambda^p$ is supplied canonically with a flat $\mathcal{C}$-connection $\nabla^{(p)}$ such that $\nabla^{(p)}_X = L_X|_{\mathcal{C}^p\Lambda^p}$, $X \in \mathcal{C}D(\mathcal{O})$. Therefore, $(\mathcal{C}E_0^{p,*}, d_0)$ possesses a canonical $\mathcal{S}$-module structure, i.e., secondary differential p-forms constitute a secondary module. As we have already seen this is a key fact in computation of $\mathcal{C}E_1(\mathcal{O})$ (see also [51]).

Cohomologies involved in the preceding discussion can be, in fact, "computed" as follows. Let P and P' be $\mathcal{F}(\mathcal{O})$-modules supplied with flat $\mathcal{C}$-connections ∇ and

∇', respectively. Then the $\mathcal{F}(\mathcal{O})$-module $\mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P, P')$ can be supplied with a flat $\mathcal{C}$-connection ∇^{Hom} such that

$$\nabla^{\mathrm{Hom}}_X(h) = \nabla'_X \circ h - h \circ \nabla_X, \quad h \in \mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P, P').$$

Below we use $d_{\nabla,\nabla'}$ instead of ambiguous $d_{\nabla^{\mathrm{Hom}}}$. Note also that the $\mathcal{F}(\mathcal{O})$-module $\mathcal{GC}\mathrm{Diff}(\bar{\Lambda}P, \bar{\Lambda}P')$ becomes a complex with respect to the differential $L_{\nabla,\nabla'}$:

$$L_{\nabla,\nabla'}(\square) = \square \circ \bar{d}_\nabla - (-1)^{\square} \bar{d}_{\nabla'} \circ \square.$$

Theorem 9.1. *The cohomology of the complex* $\{\mathcal{GC}\mathrm{Diff}(\bar{\Lambda}P, \bar{\Lambda}P'), L_{\nabla,\nabla'}\}$ *is canonically isomorphic to that of the complex* $\{\bar{\Lambda}\,\mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P, P'), \bar{d}_{\nabla,\nabla'}\}$*:*

$$\begin{aligned} 0 \to &\mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P, P') \xrightarrow{\bar{d}_{\nabla,\nabla'}} \bar{\Lambda}^1(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} \mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P, P') \to \dots \\ &\to \bar{\Lambda}^n(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} \mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P, P') \to 0. \end{aligned}$$

If $P = P'$, then $\mathrm{End}P = \mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P, P')$ and we put $\bar{d}^{\mathrm{End}}_\nabla = \bar{d}_{\nabla,\nabla}$. So, it holds

$$H^*(L_{\nabla,\nabla}) = H^*(\bar{d}^{\mathrm{End}}_\nabla), \tag{13}$$

where $H^*(\delta)$ denotes the cohomology of the complex whose differential is δ. Note that (13) generalizes Theorem 8.1 to arbitrary flat $\mathcal{C}$-connections. In fact, this theorem is a particular case of (13) for $P = \mathcal{F}(\mathcal{O})$ and the canonical flat $\mathcal{C}$-connection ∇ in $\mathcal{F}(\mathcal{O})$ for which $\nabla_X = X$.

Let now μ be the natural $\mathcal{S}$-module structure on P corresponding to a flat $\mathcal{C}$-connection ∇. Then by virtue of (13) the cohomology map induced by μ may be seen as

$$\bar{H}^*(\mathcal{O}) = H^*(\bar{d}) \xrightarrow{H^*(\mu)} H^*(\bar{d}^{\mathrm{End}}_\nabla).$$

Moreover, as it results from the construction of isomorphism (13), $H^*(\mu)$ is identical to the cohomology map induced by the canonical cochain map $\{\bar{\Lambda}^*(\mathcal{O}), \bar{d}\} \to \{\bar{\Lambda}P, \bar{d}_\nabla\}$ with $\omega \mapsto \omega \otimes \mathrm{id}_P$. This implies the result that can be suspected from the very beginning:

Theorem 9.2. *The action of* $\bar{H}^*(\mathcal{O})$ *on* $\bar{H}^*(\bar{d}_\nabla)$ *induced by the natural* $\mathcal{S}$*-module structure on* $\{\bar{\Lambda}P, \bar{d}_\nabla\}$ *coincides with the canonical action of* $\bar{H}^*(\mathcal{O})$ *on* $\bar{H}^*(\bar{d}_\nabla)$*.*

Concluding this section, it is worth to stress the "secondarizability" of basic operations of multi-linear algebra, like Hom, $\otimes$, etc. This is seen especially transparent in the context of natural modules. Indeed, if P_i is an $\mathcal{F}(\mathcal{O})$-module with a flat $\mathcal{C}$-connection ∇_i, $i = 1, 2$, then $P_1 \otimes_{\mathcal{F}(\mathcal{O})} P_2$ and $\mathrm{Hom}_{\mathcal{F}(\mathcal{O})}(P_1, P_2)$ acquire naturally flat connections $\nabla_1 \otimes \nabla_2$ and $\nabla^{\mathrm{Hom}}(\nabla_1, \nabla_2)$, respectively. This allows to define $\otimes$ and Hom operations for corresponding $\mathcal{S}$-modules and, therefore, for secondary modules they represent.

In conclusion we note that the construction presented in this section seemingly responds the question posed by J. Stasheff at the end of his paper in this volume: "Again we see an analog of the Maurer – Cartan equation or of a flat connection, but why?"

10. ∇-$\mathcal{C}$-spectral sequence

The central problem in the theory of secondary modules is to describe in a reasonable way the corresponding cohomologies. Below it is shown that the $\mathcal{C}$-spectral sequence techniques allows to approach this problem for natural secondary modules in a rather efficient way.

Let ∇ be a flat connection in a $\mathcal{F}(\mathcal{O})$-module P. Put $\Lambda P = \Lambda^*(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} P$, $\Lambda^i P = \Lambda^*(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} P$ and denote by d_∇ the standard differential in ΛP associated with ∇, $d_\nabla : \Lambda^i P \to \Lambda^{i+1} P$. Further put $\mathcal{C}^k \Lambda P = \mathcal{C}^k \Lambda^*(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} P$

and $\mathcal{C}^k\Lambda^iP = \mathcal{C}^k\Lambda^i(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} P$. It is easy to see that $\mathcal{C}^k\Lambda P$ is a subcomplex of $\{\Lambda P, d_\nabla\}$. So, the filtration

$$\Lambda P \supset \mathcal{C}\Lambda P \supset \cdots \supset \mathcal{C}^k\Lambda P \supset \ldots$$

generates a spectral sequence. Let us call it ∇-$\mathcal{C}$-spectral sequence and denote by $\{\mathcal{C}_\nabla E_r^{p,q}, d_r^\nabla\}$. Note that

$$\mathcal{C}_\nabla E_0^{p,q} = \mathcal{C}E_0^{p,q}(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} P \tag{14}$$

and that ∇-$\mathcal{C}$-spectral sequence is a "module" over the $\mathcal{C}$-spectral one:

$$\mathcal{C}E_r^{p,q}(\mathcal{O}) \times \mathcal{C}_\nabla E_r^{s,t} \to \mathcal{C}_\nabla E_r^{p+s,q+t}.$$

Moreover, $\mathcal{C}_\nabla E_r$ is a DG-module over the DG-algebra $\mathcal{C}E_r(\mathcal{O})$. By virtue of (14) all eventually nontrivial terms $\mathcal{C}_\Delta E_r^{p,q}$ are located in the region $p \geqslant 0$, $0 \leqslant q \leqslant n$.

The connection ∇ defines canonically a flat $\mathcal{C}$-connection $\bar\nabla$ on P: $\bar\nabla_X = \nabla_X$, $X \in \mathcal{C}D(\mathcal{O})$. Then the covariant differential $\bar d_{\bar\nabla} : P \to \bar\Lambda^1P$ is just the quotient of $d_\nabla : P \to \Lambda^1P$ by $\mathcal{C}\Lambda^1P$ since $\bar\Lambda^1P = \Lambda^1P/\mathcal{C}\Lambda^1P$. This shows also that the complex $\{\mathcal{C}_\nabla E_0^{0,*}, d_0^\nabla\}$ is identified with $\{\bar\Lambda P, \bar d_{\bar\nabla}\}$. So, the cohomology $H^*(\bar d_{\bar\nabla})$ is identified with $\mathcal{C}_\nabla E_1^{0,*}$ and we see that $H^*(\bar d_{\bar\nabla})$ is related with the cohomology $H^*(d_\nabla)$ to which the ∇-$\mathcal{C}$-spectral sequence converges essentially by the same way as the $\mathcal{C}$-spectral one does with respect to the de Rham cohomology $H^*(\mathcal{O})$.

Observe then that diffieties of the most interest, for instance $\mathcal{E}_\infty$, have homotopy types of finite-dimensional manifolds ($\mathcal{E}_{(k)}$ for a sufficiently big k, if $\mathcal{O} = \mathcal{E}_\infty$.) In such a situation computations of the cohomology of the complex $\{\Lambda P, d_\nabla\}$ can be reduced to a finite-dimensional model to which standard methods of algebraic topology can be applied.

Let P and ∇ be as before. Then the complex $\{\mathcal{C}\mathrm{Diff}(Q, \bar\Lambda P), S_\nabla^Q\}$, $S_\nabla^Q(\square) = \bar d_\nabla \circ \square$, is associated with a given $\mathcal{F}(\mathcal{O})$-module Q.

Theorem 10.1. *If P and Q are projective, then $H^i(S_\nabla^Q) = 0$ for $i \neq n$, and $H^n(S_\nabla^Q) = \hat Q \otimes_{\mathcal{F}(\mathcal{O})} P$.*

If $\square \in \mathcal{C}\mathrm{Diff}(Q, R)$, then $\Delta \mapsto \Delta \circ \square$, $\Delta \in \mathcal{C}\mathrm{Diff}(R, \bar\Lambda P)$, is a cochain map

$$\{\mathcal{C}\mathrm{Diff}(R, \bar\Lambda P), S_\nabla^R\} \to \{\mathcal{C}\mathrm{Diff}(Q, \bar\Lambda P), S_\nabla^Q\}.$$

As such it generates a map in cohomology $\square^{*\nabla}$ which for projective P, Q and R looks in view of Theorem 10.1 as

$$\square^{*\nabla} : \hat R \otimes_{\mathcal{F}(\mathcal{O})} P \to \hat Q \otimes_{\mathcal{F}(\mathcal{O})} P. \tag{15}$$

Call this operator the ∇*-adjoint* to $\square$. Obviously,

$$(\square_1 \circ \square_2)^{*\nabla} = \square_2^{*\nabla} \circ \square_1^{*\nabla}.$$

Note that the standard adjoint operator $\square^*$ is ∇-adjoint to $\square$, if ∇ is the canonical flat connection in $P = \mathcal{F}(\mathcal{O})$. The operator

$$\square^{**\nabla} = (\square^*)^{*\nabla} : Q \otimes_{\mathcal{F}(\mathcal{O})} P \to R \otimes_{\mathcal{F}(\mathcal{O})} P$$

is the extension $\square_P$ of $\square$ to $Q \otimes_{\mathcal{F}(\mathcal{O})} P$ mentioned in Sec. 6. Denote by

$$*(Q, R) : \mathcal{C}\mathrm{Diff}_k(Q, R) \to \mathcal{C}\mathrm{Diff}_k(\hat R, \hat Q), \quad \square \mapsto \square^*,$$

the conjugation operation. This is a k-th order $\mathcal{C}$-differential operator. So, the operator

$$*(Q, R)^{**\nabla} : \mathcal{C}\mathrm{Diff}(Q, R) \otimes_{\mathcal{F}(\mathcal{O})} P \to \mathcal{C}\mathrm{Diff}(\hat R, \hat Q) \otimes_{\mathcal{F}(\mathcal{O})} P$$

is well defined. Under the natural isomorphism it can be viewed as

$$*(Q, R)^{**\nabla} : \mathcal{C}\mathrm{Diff}(Q, R \otimes_{\mathcal{F}(\mathcal{O})} P) \to \mathcal{C}\mathrm{Diff}(\hat R, \hat Q \otimes_{\mathcal{F}(\mathcal{O})} P).$$

In other words, it associates with any operator $\Delta \in \mathcal{C}\mathrm{Diff}(Q, R \otimes_{\mathcal{F}(\mathcal{O})} P)$ another operator $\Delta^{\nabla *} \in \mathcal{C}\mathrm{Diff}(\hat{R}, \hat{Q} \otimes_{\mathcal{F}(\mathcal{O})} P)$ which we also call ∇-*adjoint* to Δ. It is easy to see that

$$(\Delta^{\nabla *})^{\nabla *} = \Delta.$$

So, the notion of the adjoint operator splits into two parts when passing to its ∇-generalization. It plays a key role in describing the term $\mathcal{C}_\nabla E_1$ due to Theorem 10.1 and the "∇-extension" of the *fundamental isomorphism* η (Proposition5.1 and its consequence):

$$\{\mathcal{C}_\nabla E_1^{p,*}, d_1^\nabla\} = \{\mathcal{C}\mathrm{Diff}_{(p)}^{\mathrm{alt}}(\varkappa; \bar{\Lambda} P), S_\nabla^{(p)}\},\ p \geqslant 1,$$

where $S_\nabla^{(p)}(\Box) = (-1)^p \bar{d}_{\bar{\nabla}} \circ \Box$.

For $\mathcal{O} = J^\infty$ we have the following ∇-generalizations of the One Line Theorem and the related formulae (Sec. 5 and 7). Define a useful "action" $\ni_\chi(\Delta)$ as

$$\ni_\chi(\Delta)(\chi_1, \ldots, \chi_p) = \ni_\chi(\Delta(\chi_1, \ldots, \chi_p)) + \sum_i \Delta(\chi_1, \ldots, \{\chi_i, \chi\}, \ldots, \chi_p)$$

for $\chi, \chi_i \in \varkappa$ and $\Delta \in \mathcal{C}\mathrm{Diff}_{(p)}^{\mathrm{alt}}(\varkappa; Q)$.

Theorem 10.2. *For $p \geqslant 1$ it holds:*

(i) $\mathcal{C}_\nabla E_1^{p,q} = 0$, *if $q \neq n$,*

(ii) $\mathcal{C}_\nabla E_1^{p,n} = L_p^\nabla(\varkappa)$ *with*

$$L_p^\nabla(\varkappa) = \{\Box \in \mathcal{C}\mathrm{Diff}_{(p-1)}^{\mathrm{alt}}(\varkappa; \hat{\varkappa} \otimes_{\mathcal{F}} P) \mid \Box^{\nabla *}_{\chi_1,\ldots,\chi_{p-2}} = -\Box_{\chi_1,\ldots,\chi_{p-2}}\},$$

(iii) *The P-valued ∇-Euler operator $d_1^{\nabla 0,n} = \mathcal{E}_0^\nabla : H^n(\bar{d}_\nabla) \to \hat{\varkappa} \otimes_{\mathcal{F}} P$ is given by*

$$\mathcal{E}_0^\nabla([\omega]) = l_\omega^{\nabla *}(1), \quad \omega \in \bar{\Lambda}^n \otimes_{\mathcal{F}} P,$$

(iv) *Operators $d_1^{\nabla p,n} = \mathcal{E}_p^\nabla$ for $p \geqslant 1$ are given by*

$$\mathcal{E}_p^\nabla(\Delta)(\chi_1, \ldots, \chi_p) = \textstyle\sum_{i=1}^p (-1)^i \ni_{\chi_i}(\Delta)(\chi_1, \ldots, \hat{\chi}_i, \ldots, \chi_p) +$$
$$(\textstyle\sum_{i=1}^{p-1} (-1)^{s-1} l_{\chi_s}^{*\nabla} \circ \Delta_{\chi_1, \ldots, \hat{\chi}_s, \ldots, \chi_{p-1}} + (-1)^p l^{\nabla *}_{\Delta(\chi_1,\ldots,\chi_{p-1})})(\chi_p),$$

In particular,

$$\mathcal{E}_1^\nabla(\varphi) = l_\varphi - l_\varphi^{\nabla *},$$

(v) *The ∇-Lie derivative along $\chi \in \varkappa$ in $\mathcal{C}_\nabla E_1$ is given by*

$$L_\chi^\nabla = \ni_\chi + l_\chi^{*\nabla}.$$

Formulae for $\mathcal{E}_p^\nabla = d_1^{\nabla p,n}$ and the ∇-Lie derivative are obtained from the corresponding formulae for $d_1^{p,n}$ and L_χ in Sec. 5,7 by means of substitutions $l_\chi^* \mapsto l_\chi^{*\nabla}$ for $\chi \in \varkappa$, $l_\varphi^* \mapsto l_\varphi^{\nabla *}$ for $\varphi \in E_1^{1,n}$ and $L_\chi \mapsto L_\chi^\nabla = [d_1^\nabla, i_\chi]$.

Passing now to the case $\mathcal{O} = \mathcal{E}_\infty$ note that the operator $(l_{[F]}^*)_{\mathcal{C}^{p-1}\Lambda^{p-1}}$ appearing in the Two Line Theorem, Sec. 6, in the notation of this section looks as $l_{[F]}^{*\nabla_{(p-1)}}$, where $\nabla_{(p-1)}$ is the canonical connection (Example (9.1) in $\mathcal{C}^{p-1}\Lambda^{p-1}$.

Theorem 10.3 (∇-Two Line Theorem)**.** *If ∇ is a flat connection in a projective $\mathcal{F}(\mathcal{O})$-module Q, then under the hypothesis of the Two Line Theorem $\mathcal{C}_\nabla E_1^{p,n} = 0$ for $p > 0$, $q \neq n-1, n$, and $\mathcal{C}_\nabla E_1^{p,n-1}$ (resp., $\mathcal{C}_\nabla E_1^{p,n}$) is isomorphic to the skew-symmetric part of* $\ker l_{[F]}^{*(\nabla_{(p-1)} \otimes \bar{\nabla})}$ *(resp.,* $\mathrm{coker}\, l_{[F]}^{*(\nabla_{(p-1)} \otimes \bar{\nabla})}$*).*

Theorem 10.4 (∇-k-Line Theorem)**.** *Let $\mathcal{O} = \mathcal{E}_\infty$ with $\mathcal{E}$ satisfying the hypothesis of the k-Line Theorem, then $\mathcal{C}_\nabla E_1^{p,q} = 0$ for $p > 0$, $q \leqslant n-k$.*

∇-analogue of the complex (11) is

$$0 \to \hat{Q}_{k-1} \otimes_{\mathcal{F}(\mathcal{O})} Q \xrightarrow{\Delta_{k-1}^{*\nabla}} \dots \xrightarrow{\Delta_1^{*\nabla}} \hat{Q}_1 \otimes_{\mathcal{F}(\mathcal{O})} Q \xrightarrow{l_{[F]}^{*\nabla}} \hat{Q} \otimes_{\mathcal{F}(\mathcal{O})} Q \to 0, \qquad (16)$$

where ∇ is a flat connection in Q. Then

$$\mathcal{C}_\nabla E_1^{1,n-i} = (i\text{-th cohomology of complex (16)})$$

and similarly for $\mathcal{C}_\nabla E_1^{p,n-i}$ with $p > 1$, $0 \leqslant i \leqslant n$.

11. Secondarization of multi-vector-valued differential forms

In this section we show how to secondarize multi-vector-valued differential forms. The first step in such a procedure is to bring the concept in question to a "secondarizable form". In our case this is as follow.

Denote $D_i(M)$ the $\mathcal{C}^\infty(M)$-module of i-vector fields on a smooth manifold M and put

$$\Lambda^i D_j(M) = \Lambda^i(M) \otimes_{\mathcal{C}^\infty(M)} D_j(M).$$

Similar meaning have $\Lambda^* D_j$, $\Lambda^i D_*$ and $\Lambda^* D_*$. So, elements of $\Lambda^i D_j(M)$ are *j-vector-valued differential i-forms* on M.

Proposition 11.1. *The $C^\infty(M)$-module of $C^\infty(M)$-linear graded differential operators of order i acting on the graded algebra $\Lambda^*(M)$ coincides with $\Lambda^* D_i(M)$.*

Recall that $\omega \otimes V \in \Lambda^* D_*(M)$ acts on $\Lambda^*(M)$ as $\varrho \mapsto \omega \wedge i_V(\varrho)$. Below multi-vector-valued differential forms are secondarized as differential operators of this type.

The second step is to consider on an arbitrary diffiety $\mathcal{O}$ operators of the same kind by paying attention to the Cartan distribution $\mathcal{C}(\mathcal{O})$ which must be "respected". With this purpose we put

$$\mathcal{C}_r D_i(\mathcal{O}) = \{V \in D_i(\mathcal{O}) | i_V(\mathcal{C}^p \Lambda^*(\mathcal{O})) \subset \mathcal{C}^{p-i+r} \Lambda^*(\mathcal{O})\}$$

and get the filtration

$$0 \subset \mathcal{C}_i D_i \subset \dots \subset \mathcal{C}_r D_i \subset \mathcal{C}_{r-1} D_i \subset \dots \subset \mathcal{C}_0 D_i = D_i \qquad (17)$$

dual to the $\mathcal{C}$-filtration in $\Lambda^*(\mathcal{O})$. So, the graded algebra associated with the filtration (17) acts naturally on that associated with the $\mathcal{C}$-filtration, i.e., on $\mathcal{C}E_0(\mathcal{O})$. More exactly, put

$$D_{r,s}(\mathcal{O}) = \mathcal{C}_s D_{r+s}(\mathcal{O}) / \mathcal{C}_{s+1} D_{r+s}(\mathcal{O}).$$

Then any $\bar{V} \in D_{r,s}(\mathcal{O})$ sends $\mathcal{C}E_0^{p,*}(\mathcal{O})$ to $\mathcal{C}E_0^{p-r,*}(\mathcal{O})$.

Elements of $D_{r,s}(J^\infty)$ in terms of jet coordinates are represented by formal series of the form

$$\sum a^{i_1 \dots i_r}_{\sigma_1 \dots \sigma_r, j_1 \dots j_s} \frac{\partial}{\partial u^{i_1}_{\sigma_1}} \wedge \dots \wedge \frac{\partial}{\partial u^{i_r}_{\sigma_r}} \wedge D_{j_1} \wedge \dots \wedge D_{j_s}.$$

The following assertion is an analogue of Proposition 11.1.

Proposition 11.2. *$\mathcal{F}(\mathcal{O})$-module of $\mathcal{F}(\mathcal{O})$-linear differential operators acting on $\mathcal{C}E_0(\mathcal{O})$ coincides with $\mathcal{C}E_0(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} D_{*,*}(\mathcal{O})$, $D_{*,*} = \sum_{p,q} D_{p,q}$.*

According to the concept of a secondary module, any kind of secondary operators should be homotopy classes of (graded) cochain maps. A graded operator $\Delta : \mathcal{C}E_0(\mathcal{O}) \to \mathcal{C}E_0(\mathcal{O})$ is a cochain map of the complex $\{\mathcal{C}E_0(\mathcal{O}), d_0\}$ into itself iff $L_0(\delta) := [d_0, \Delta] = 0$.

Proposition 11.3. *If $\Delta \in \mathcal{C}E_0(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} D_{*,q}(\mathcal{O})$ with $q > 0$, then $L_0(\Delta) \neq 0$.*

On the other hand, it is easy to see that L_0 sends $\Lambda^{s,t} \otimes D_{p,0}$ to $\Lambda^{s,t+1} \otimes D_{p,0}$ and $L_0^2 = 0$. So, we obtain the complex:

$$\begin{aligned} 0 \quad \rightarrow \quad & \Lambda^{s,0} \otimes_{\mathcal{F}(\mathcal{O})} D_{p,0}(\mathcal{O}) \xrightarrow{L_0} \ldots \xrightarrow{L_0} \Lambda^{s,t} \otimes_{\mathcal{F}(\mathcal{O})} D_{p,0}(\mathcal{O}) \xrightarrow{L_0} \ldots \\ \xrightarrow{L_0} \quad & \Lambda^{s,n} \otimes_{\mathcal{F}(\mathcal{O})} D_{p,0}(\mathcal{O}) \rightarrow 0 \end{aligned} \tag{18}$$

with $\Lambda^{s,t} = \mathcal{C}E_0^{s,t}(\mathcal{O})$.

Definition 11.1. *Secondary p-vector-valued differential s-forms* on $\mathcal{O}$ are cohomologies of complex (18), i.e., of $\{\mathcal{C}E_0^{s,*}(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} D_{p,0}(\mathcal{O}), L_0\}$.

In particular, *secondary p-vector fields* on $\mathcal{O}$ are cohomologies of the complex $\{\bar{\Lambda}^*(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} D_{p,0}(\mathcal{O}), L_0\}$. Similarly, *secondary vector-valued differential s-forms* on $\mathcal{O}$ are cohomologies of $\{\mathcal{C}E_0^{s,*}(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} D_{1,0}(\mathcal{O})\}$.

Denote by $H^*_{\Lambda^s D_p}(\mathcal{O})$ the cohomology of (18). Similar meaning have $H^*_{\Lambda^* D}$, $H^*_{D_*}$, etc.

In the classical situation various types of tensors from $\Lambda^* D_*(M)$ are subject of a number of *natural operations* like insertions, multiplications, brackets, etc. Without going to exact definitions we state the following general result.

Proposition 11.4. *All natural operations with tensors of the type $\Lambda^* D_*$ are secondarizable.*

The next problem is how to compute the cohomology $H^*_{\Lambda^s D_p}(\mathcal{O})$? The method used with this purpose for the $\mathcal{C}$-spectral sequence which is the subcase $p = 0$ of the above problem works as well in the general case (see Sec. 5 and 6). Its key point is the *fundamental isomorphism* η which should be extended properly. A reasonable description of this extension requires more space than we have at our disposal here. So, we shall limit ourself further on with the case $p = 1$ only and refer the reader to the forthcoming systematic exposition in [66].

Let now $\mathcal{O} = J^\infty$ and consider the $\mathcal{F}(J^\infty)$-module $\bar{\mathcal{J}}(\varkappa)$ of infinite horizontal jets of $\varkappa$ (see Sec. 3). Recall the horizontal Spencer jet-complex (see [59, 31]):

$$0 \rightarrow \bar{\mathcal{J}}(\varkappa) \xrightarrow{S} \bar{\Lambda}^1(J^\infty) \otimes_{\mathcal{F}} \bar{\mathcal{J}}(\varkappa) \xrightarrow{S} \ldots \xrightarrow{S} \bar{\Lambda}^n(J^\infty) \otimes_{\mathcal{F}} \bar{\mathcal{J}}(\varkappa) \rightarrow 0$$

with $S : \omega \otimes \bar{\jmath}_\infty(\chi) \mapsto \bar{d}\omega \otimes \bar{\jmath}_\infty(\chi)$, $\omega \in \bar{\Lambda}^*(J^\infty)$, $\chi \in \varkappa$.

Proposition 11.5. *There exists a natural isomorphism of $\mathcal{F}(J^\infty)$-modules $\eta : D_{1,0}(J^\infty) \rightarrow \bar{\mathcal{J}}(\varkappa)$ such that $\eta(\ni_\chi) = \bar{\jmath}_\infty(\chi)$. Moreover, this isomorphism together with that of Sec. 5 gives rise to isomorphisms:*

$$\eta : \Lambda^{p,q}(J^\infty) \otimes_{\mathcal{F}(\mathcal{O})} D_{1,0}(J^\infty) \rightarrow \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar{\Lambda}^q(J^\infty) \otimes_{\mathcal{F}} \bar{\mathcal{J}}(\varkappa)).$$

Note that the Spencer differential S induces the differential $\square \mapsto S \circ \square$ in the module $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \bar{\Lambda}^*(J^\infty) \otimes_{\mathcal{F}} \bar{\mathcal{J}}(\varkappa))$. So, $H^*_{\Lambda^* D}(J^\infty)$ is isomorphic to the cohomology of the so-obtained complex. Its cohomology can be computed easily since, the Spencer jet-complex is acyclic in positive dimensions and its 0-cocycles are of the form $\bar{\jmath}_\infty(\chi)$. This way one gets the following result. Below $H^{p,q}_{\Lambda^* D}(\mathcal{O})$ denotes the q-th cohomology of the complex $\mathcal{C}E_0^{p,*}(\mathcal{O}) \otimes_{\mathcal{F}(\mathcal{O})} D_{1,0}(\mathcal{O}), L_0\}$.

Theorem 11.1. *It holds:*

(i) $H^{p,0}_{\Lambda^* D}(J^\infty) = \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa; \varkappa)$,
(ii) $H^{p,q}_{\Lambda^* D}(J^\infty) = 0$ *for* $q > 0$.

So, secondary vector-valued differential p-forms on J^∞ are identified with the skew-symmetric p-differential operators on $\varkappa$ which take values also in $\varkappa$.

Since $H_{\Lambda^* D_*}(\mathcal{O})$ consists of homotopy classes of co-chain maps of $\{\mathcal{C}E_0(\mathcal{O}), d_0\}$ into itself a natural action

$$H^*_{\Lambda^* D_*}(\mathcal{O}) \times \mathcal{C}E_1(\mathcal{O}) \rightarrow \mathcal{C}E_1(\mathcal{O})$$

is defined. In particular, this action for secondary vector-valued forms looks as

$$H^{r,s}_{\Lambda^* D}(\mathcal{O}) \times \mathcal{C}E_1^{p,q}(\mathcal{O}) \to \mathcal{C}E_1^{p+r-1,q+s}(\mathcal{O}).$$

In view of Corollary 5.1 and Theorem 11.1 this action on J^∞ is reduced to

$$\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(r)}(\varkappa;\varkappa) \times L_p(\varkappa) \to L_{p+r-1}(\varkappa).$$

Denote the result of this action of $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(r)}(\varkappa;\varkappa)$ on $\square \in L_p(\varkappa)$ by $\Delta \rfloor \square$. Then the following formula holds:

$$(\Delta \rfloor \square)(\chi_1, \ldots, \chi_{p+r-2}) = \sum_{|I|=r} (-1)^{(I,J)} \square(\Delta(\chi_I), \chi_J) + \sum_{|I|=r-1} (-1)^{(I,J)} \Delta^*_{\chi_I}(\square(\chi_J)), \tag{19}$$

where χ_I stands for $\chi_{i_1}, \ldots, \chi_{i_r}$ with $I = (i_1, \ldots, i_r)$, $1 \leqslant i_1 < \cdots < i_r \leqslant p + r - 2$, etc. and $(-1)^{(I,J)}$ denotes the sign of the corresponding permutation. This one-by-line action is illustrated on Fig. 5.

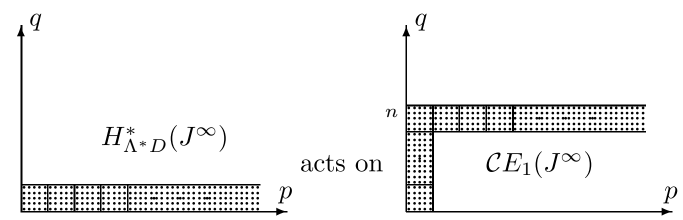

Fig. 5

As an illustration of Proposition 11.4 we show now on what the secondary Frölicher−Nijenhuis bracket on J^∞ looks like. According to the identifications made above this bracket can be considered as a pairing

$$\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa;\varkappa) \times \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(r)}(\varkappa;\varkappa) \xrightarrow{[\cdot,\cdot]^{FN}} \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p+r)}(\varkappa;\varkappa).$$

So, if $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa;\varkappa)$ and $\nabla \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(r)}(\varkappa;\varkappa)$, then

$$[\Delta, \nabla]^{FN}(\chi_1, \ldots, \chi_{p+r}) = \sum (-1)^{(I,J)} \{\Delta(\chi_I), \nabla(\chi_J)\} +$$
$$\sum (-1)^{(\alpha,I,J)+p} \nabla(\{\chi_\alpha, \Delta(\chi_I)\}, \chi_J) - \sum (-1)^{(\alpha,I,J)} \Delta(\{\chi_\alpha, \nabla(\chi_I)\}, \chi_J) +$$
$$\sum (-1)^{(\alpha,\beta,I,J)} \nabla(\Delta(\{\chi_\alpha, \chi_\beta\}, \chi_I), \chi_J)) + \sum (-1)^{(\alpha,\beta,I,J)} \Delta(\nabla(\{\chi_\alpha, \chi_\beta\}, \chi_I), \chi_J)$$

Here $\{\cdot,\cdot\}$ is the bracket (6) and $\alpha < \beta$ is assumed. Subsets I, J of $\{1, 2, \ldots, p + r\}$ are supposed ordered and $(-1)^{(\alpha,\beta,I,J)}$ means the sign of the corresponding permutation, etc.

Computations of $H^*_{\Lambda^* D_*}$ for $\mathcal{O} = \mathcal{E}_\infty$ proceed essentially along the same lines as in sec.6 by taking into account a kind of duality between "operators" and "operated". This means that instead of the sequence (9) for $\Delta = l_{[F]}$ one has to consider in the notation of Sec. 6 the *exact* sequence

$$0 \to \bar{\mathcal{J}}(Q_0) \xrightarrow{l^J_{[F]}} \bar{\mathcal{J}}(Q_1) \xrightarrow{\Delta^J_1} \ldots \xrightarrow{\Delta^J_{k-1}} \bar{\mathcal{J}}(Q_{k-1}) \to 0 \tag{20}$$

with $Q_0 = \varkappa \mid_{\mathcal{E}_\infty}$, $Q_1 = P \mid_{\mathcal{E}_\infty}$.

Maps composing (20) are canonical jet-prolongations of operators $l_{[F]}$ and Δ_i (see [31]). The operator $l^J_{[F]}$ can be extended naturally to a cochain map of jet-Spencer complexes:

$$\bar{\mathcal{J}}(Q_0) \otimes \bar{\Lambda}^*(\mathcal{E}_\infty) \to \bar{\mathcal{J}}(Q_1) \otimes \bar{\Lambda}^*(\mathcal{E}_\infty).$$

The kernel of this map is isomorphic in view of (20) to

$$\bar{\Lambda}^*(\mathcal{E}_\infty) \otimes_{\mathcal{F}(\mathcal{E}_\infty)} D_{1,0}(\mathcal{E}_\infty).$$

This proves that $H^{0,q}_{\Lambda^* D}(\mathcal{E}_\infty) = 0$ for $q \geqslant k$. Similar considerations involving once again appropriate flat $\mathcal{C}$-connections prove the following result.

Theorem 11.2 (k-Line Theorem From the Bottom)**.** *If an equation $\mathcal{E}$ satisfies the hypothesis of the k-Line Theorem ("from the top"), then $H^{r,s}_{\Lambda^* D}(\mathcal{E}_\infty) = 0$ if $s \geqslant k$ (and, obviously, if $s < 0$).*

In the whole the situation is illustrated below.

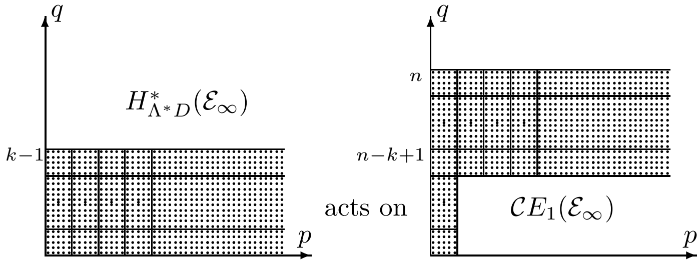

Fig. 6

Similar situation takes place for $H^{r,s}_{\Lambda^* D_p}(\mathcal{E}_\infty)$ for $p > 1$ and, even more, for all *Functors of Differential Calculus over Commutative Algebras and their Representative Objects*. See, for instance, [53, 31, 69] to get an idea of what such functors are. To prove this general assertion one needs only a formalization of the *Differential Calculus Functors Algebra* which will appear elsewhere.

Note, finally, that the cohomology $H^*_{\Lambda* D}$ coincides essentially with Krasil'shchik's $\mathcal{C}$-cohomology [27]. His approach is based on algebraic Frölicher – Nijenhuis machinery and, so, is quite different from ours.

12. Secondarization versus quantization

As we already have seen, the objects of Secondary Calculus are formed by suitable cohomology classes whose cohomological nature can not be forgotten. To express this idea better, we need a more precise terminology. Let $\mathcal{M}$ be a secondary module represented by an $\mathcal{S}$-module $\{K, \Delta\}$, i.e., $\mathcal{M}$ is the homotopy equivalence class of $\{K, \Delta\}$. Call $H^*(\Delta)$ the *body* of $\mathcal{M}$. Obviously, $\mathcal{M}$ is not reduced to its body because an isomorphism of cohomology spaces does not, generally, guarantee a homotopy equivalence of the corresponding complexes. A physicist could describe such a situation by saying that a secondary module consists of a *body* surrounded by an *aura*. While the quantities we are interested in take part of the body, the aura is indispensable to put them in interaction. This is, obviously, parallel to the idea of an particle together with the field it emanates.

Hence, in frames of Secondary Calculus cohomology classes of one kind are to be "observed" by means of cohomologies of another kind. Moreover, this observability mechanism becomes classical in the limit "Dim $\to 0$" and the substitution "Dim $\to$ 0" for "$h \to 0$" is suggested when looking for principles of Quantum Physics. In other words, Secondary Calculus seems to be as indispensable for Quantum Physics as (Primary) Calculus is for the Classical. In this limit any body loses the proper aura. Mathematically this means that $d_0^\nabla \equiv 0$ for 0-Dimensional diffieties, while physical meaning is that information costs nothing in the classical limit. So, from mathematical point of view "*quantum*" means "*cohomological*" in the sense made precise above.

In other words, this "privilege principle" gives a status of universality to what happened around the BRST.

In these proceedings (see [43]) Jim Stasheff writes that a decade ago "some people thought that phrase (i.e., 'cohomological physics') was a bit much". Alas.

13. Bibliography and comments

Elements of the geometry of infinite jet spaces and infinitely prolonged equations adopted to the purposes of Secondary Calculus are summarized in review article [58] and can be found in a developed form in books [68, 31] and [32], see also [1] and [60]. A systematic use of the algebraic language of the differential calculus over commutative algebras in these texts becomes almost indispensable in the Secondary Calculus context. See [40, 38] and [44] for alternative approaches.

The concept of a diffiety emerges very naturally from a category-theoretic setting for PDE's given in [61]. See also [34].

Necessary results from formal theory of PDE's can be found in H. Goldschmidt's works [17, 18] and Spencer's review article [42]. See [31] for an adapted to our purposes exposition and [38] for an alternative point of view.

We use the adjective "higher" for nonclassical (non-Lie) symmetries of PDE's whose fundamentals were elaborated for the first time in [56] and [54] (see also [58, 60] and [31]). In subsequent books by N. Ibragimov [23] and P. Olver [37] "Lie – Bäcklund transformations" and "generalized symmetries", respectively, were used for the same concept. In these books, however, not much attention is paid to the fact that higher symmetries are, in fact, suitable cosets or, better, cohomology classes of vector fields. This is why they are not very friendly to Secondary Calculus.

An updated exposition of higher symmetries inscribed into the context of Secondary Calculus is given in [32]. There nonlocal and integro-differential symmetries and conservation laws are also treated. Graded ("super") generalizations are studied in [27, 39, 28, 26] and [25].

The $\mathcal{C}$-spectral sequence was introduced by the author in [54], where basic interpretations of its terms were made and 1-, 2- and p-line theorems were announced. Interrelations between higher symmetries and the first term of the $\mathcal{C}$-spectral sequence which brought to light the idea of Secondary Calculus (see Sec. 7) were observed a little later in [57] (see also [58]). It turned out impossible to publish in USSR a detailed exposition of these and related results that time. It happened six years later in USA thanks to the help of D. Spencer and G.-C. Rota. Meantime an important paper by T. Tsujishita [46] appeared. In this paper, some author's results were reinterpreted in the local "environment" of the variational bi-complex (see Sec. 4) and a natural interpretation of various kinds of characteristic classes (foliations, vector fields, etc.) in its terms were proposed. Earlier in [47] W. Tulczijew interpreted his "Lagrange complex" in the form of "One Line Theorem" for $\mathcal{O} = J^{\infty}\pi$. An extensive work of divulgation and systematization concerning the variational bi-complex made by I. Anderson is summarized in his recent book [2] where the reader will find also an alternative version of the history of the problem.

Recently R. Bryant and Ph. Griffiths introduced the notion of the characteristic cohomology of an exterior differential system and developed an analogue of the $\mathcal{C}$-spectral method in this context. See [11, 12, 14]. We note that characteristic cohomologies become horizontal ones when converting the original exterior system into the corresponding PDE's.

A disadvantage of the variational complex interpretation of the $\mathcal{C}$-spectral sequence method is that it requires from the very beginning a fixation of independent variables. This creates various inconveniences in treating global and singular problems and impedes the use of the natural algebraic language of (Primary) Calculus.

Perhaps, this is why not all ingredients, for instance, of [59] were reinterpreted till now in terms of variational bi-complex. To respect, however, the "general relativity principle" we report also Tsujishita's point of view ([44, Footnote 17]) on the subject: "such (i.e., algebraic) formalism however obscure the point of our framework, which is very simple in essence."

The method of computing the first term of the $\mathcal{C}$- and ∇-$\mathcal{C}$-spectral sequences exposed in Sec. 5, 6 and 10 is an important improvement of original author's method [54, 59] due to works by M. Marvan [35], T. Tsujishita [45], D. Guessler [15] and A. Verbovetsky [51]. It turns out much simpler to deal directly with the compatibility complex than with the corresponding δ-Spencer cohomologies which controls its length. See the contribution by A. Verbovetsky to these proceedings for further details, examples and references.

Alternative approaches were proposed by G. Barnich, F. Brandt and M. Henneaux [8] and by R. Bryant and Ph. Griffiths [11]. While the former method makes use of the Tate – Koszul complex, the latter deals with infinitely prolonged exterior systems. It seems rather clear what are relationships between these two approaches and the presented in our notes. This, however, is to be made more precise. In this connections we send the reader to contributions by M.'Henneaux [22] and F. Brandt [10] in these proceedings where these constructions are presented "in action".

Literature dedicated to applications of the $\mathcal{C}$-spectral sequence (variational bi-complex) method is growing rapidly now. Computations of symmetries and conservation laws take a significant part of it. We mention collection [64], the first in which this method was applied successfully to nonintegrable equations. It should be stressed that the use of basic formulae of Secondary Calculus makes such computations much more efficient (see, for instance, [50, 41]). Further references see in [32] and [2].

There are not so much works dealing with the $\mathcal{C}$-spectral sequence in the whole, i.e., with a description of all its terms. Works [24, 3, 15], [12], [14, 13] as well as the paper by A. Verbovetsky in this volume give an idea of what is happening in that direction.

Aspects of the $\mathcal{C}$-spectral sequence in connection with PDE's quotiening procedure, group action, invariance properties and characteristic classes were considered in [46, 47, 4, 52, 65].

The list of publications dedicated consciously to Secondary Calculus apart of the $\mathcal{C}$-spectral sequence (variational bi-complex) matters is rather short. Hamiltonian formalism in field theory from this point of view is considered in [55, 5, 6, 26]. Works [27, 28, 39, 29, 30] initiate the study of graded or super generalizations. Secondary higher order differential operators were introduced in [67] and [19]. A general secondarization algorithm will be presented in [66]. One of its key points is computation of various kinds of horizontal cohomologies as it was sketched in Sec. 5, 6, 9, 10. The state of art in this problem can be found in Verbovetsky's paper [51].

Finally, it should be stressed that efficient concrete computations in Secondary Calculus require a solid computer support to overcome as a rule very tedious symbolic computations. A number of personal experiences and, first of all, that of P.Kersten is all what we have at our disposal now.

14. Acknowledgments

The authors is obliged to D.M. Gessler, I.S. Krasil'shchik and A.M. Verbovetsky for numerous open and friendly discussions accelerated his understanding of what Secondary Calculus is. Some of ideas presented in these notes matured during few author's visits to the E. Schrödinger Institute in Vienna in years 1993 – 95.

References

[1] D. V. Alekseevski, V. V. Lychagin, A. M. Vinogradov, *Basic ideas and concepts of differential geometry,* Encyclopedia of Math. Sciences, **28** (1991), Springer-Verlag, Berlin.

[2] I. M. Anderson, *The variational bicomplex*, Academic Press, Boston, to appear.

[3] I. M. Anderson, sec. Kamran, *Conservation laws and the variational bicomplex for second order scalar hyperbolic equations in the plane.* in: P. H. M. Kersten and I. S. Krasil'shchik, eds., "Geometric and Algebraic Structures in Differential Equations", Kluwer Acad. Publ., Dordrecht, 1995, 135-144.

[4] I. M. Anderson, J. Pohjanpelto, *The Cohomology of Invariant Variational Bicomplexes* Kluwer Acad. Publ., Dordrecht, 1995, 3–20.

[5] A. M. Astashov, *Normal forms for Hamiltonian operators in the field theory.* Dokl. AN SSSR **270** (1983) 1033–1037 (in Russian).

[6] A. M. Astashov, *On generalization of the Darboux theorem for the Hamiltonian field theory,* Mat. zametki **36** (1984) 537–547 (in Russian).

[7] A. M. Astashov, A. M. Vinogradov, *On the structure of Hamiltonian operator in field theory,* J. Geom. and Phys., 3 (1986), no. 2, 263-287.

[8] G. Barnich, F. Brandt, and M. Henneaux, *Local BRST cohomology in the antifield formalism: I. General theorems*, Comm. Math. Phys. **174** (1995), 57–92, E-print hep-th/9405109.

[9] G. W. Bluman, S.Kumei, *Symmetries and Differential Equations,* Appl. Math. Sci., **81**, Springer-Verlag, New York, 1989.

[10] F. Brandt, *Gauge covariant algebra and local BRST cohomology.*, This volume

[11] R. L. Bryant, Ph. A. Griffiths, *Characteristic cohomology of differential systems,* I: *General theory,* J. Amer. Math. Soc. **8** (1995), 507–596, URL: http://www.math.duke.edu/˜bryant.

[12] R. L. Bryant, Ph. A. Griffiths, *Characteristic cohomology of differential systems, II: Conservation laws for a class of parabolic equations*, Duke Mathematical Journal **78** (1995), 531–676. MR 96d:58158.

[13] R. L. Bryant, Ph. A. Griffiths, L.Hsu *Hyperbolic exterior differential systems and their conservation laws, Part II*, Mathematica Selecta, **1** (1995). CMP 1 354 599.

[14] R. L. Bryant, Ph. A. Griffiths, L. Hsu *Hyperbolic exterior differential systems and their conservation laws, Part I*, Mathematica Selecta, **1** (1995), 21–112. CMP 1 327 228

[15] D. M. Gessler, *On the Vinogradov $\mathcal{C}$-spectral sequence for determined systems of differential equations*, Differential Geom. Appl. **7** (1997), no. 4.

[16] D. M. Gessler, *The "three-line" theorem for the Vinogradov $\mathcal{C}$-spectral sequence of the Yang-Mills equation*, Preprint SISSA 71/95/FM, 1995.

[17] H. Goldschmidt, *Existence theorems for analytic linear partial differential equations.* Ann. Math. **86** (1967), 246-270.

[18] H. Goldschmidt, *Integrability criteria for systems of non-linear partial differential equations.* J.Differential Geometry bf 1 (1967), 269-307.

[19] V. N. Gusyatnikova ,A .M. Vinogradov, V. A. Yumaguzhin V. A., *Secondary differential operators,* J. Geom. and Phys., v. 2, No 2, (1985), 23-65.

[20] M. Henneaux, B. Knaepen, and C. Schomblond, *Characteristic cohomology of p-form gauge theories*, Comm. Math. Phys. **186** (1997), 137–165, E-print hep-th/9606181.

[21] M. Henneaux, C. Teitelboim,*Quantization of Gauge Systems,* Princeton University Press, 1992.

[22] M. Henneaux, *Consistent interactions between gauge fields: the cohomological approach.*, This volume

[23] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics.* Riedel, Boston, 1985.

[24] N. G. Khorkova, *On the $\mathcal{C}$-spectral sequence of differential equations* Diff. Geom. Appl. **3** (1993), 219–243.

[25] I. S. Krasil'shchik, *Cohomology background in the geometry of PDE*, Proc. Conf. Secondary Calculus and Cohomology Physics (M. Henneaux, I. S. Krasil'shchik, and A. M. Vinogradov, eds.), Contemporary Mathematics, Amer. Math. Soc., Providence, RI, 1998, this volume.

[26] I. S. Krasil'shchik, *Hamiltonian formalism and supersymmetry for nonlinear differential equations.* Erwin Schrödinger International Inst. for Math. Physics, Wien. Preprint no. 257, 1995.

[27] I. S. Krasil'shchik, *Some new cohomological invariants for nonlinear differential equations*, Differential Geom. Appl. **2** 1992), 307–350.

[28] I. S. Krasil'shchik, P. H. M. Kersten, *Deformations and recursion operators for evolution equations*, Geometry in Partial Differential Equations (A. Prastaro and Th. M. Rassias, eds.), World Scientific, Singapore, 1994, pp. 114–154.

[29] I. S. Krasil'shchik, P. H. M. Kersten, *Graded differential equations and their deformations: A computational theory for recursion operators*, Acta Appl. Math. **41** (1994), 167–191.
[30] I. S. Krasil'shchik, P. H. M. Kersten, *Graded Frölicher-Nijenhuis brackets and the theory of recursion operators for super differential equations*, The Interplay between Differential Geometry and Differential Equations (V. V. Lychagin, ed.), Amer. Math. Soc. Transl. (2), Amer. Math. Soc., Providence, RI, 1995, pp. 143–164.
[31] I. S. Krasil'shchik, V. V. Lychagin, A. M. Vinogradov, *Geometry of jet spaces and nonlinear partial differential equations*, Gordon and Breach, New York, 1986.
[32] I. S. Krasil'shchik, A. M. Vinogradov (eds.), *Symmetries and conservation laws for differential equations of mathematical physics*, Monograph, Faktorial Publ., Moscow, 1997 (Russian; English transl. to be published by the Amer. Math. Soc.).
[33] I. S. Krasil'shchik, A. M. Vinogradov, *Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws, and Bäcklund transformations*, Acta Appl. Math. **15** (1989), 161–209.
[34] M. Marvan, *A note on the category of partial differential equations.* in:"Differential Geometry and its Applications", Proc.Conf.Brno Czechoslovakia, Aug. 24-30, 1986.
[35] M. Marvan, *On the $\mathcal{C}$-spectral sequence with "general" coefficients*, Differential Geometry and its Application, Proc. Conf. Brno, 1989, World Scientific, Singapore, 1990, pp. 361–371.
[36] J. McCleary, *A user's guide to spectral sequences* Wilmington (USA, Delaware): Publish or Perish, 1986.
[37] P. J. Olver, *Applications of Lie groups to differential equations*, 2nd ed., Graduate Texts in Mathematics, n.107, Springer-Verlag, New York, 1993.
[38] J.-F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups.* Gordon and Breach, New York, 1978.
[39] G. H. M. Roelofs, *Prolongation structures of supersymmetric systems,* Dept. of Appl.Math. Univ. of Twente, Enschede (1993).
[40] D. J. Saunders, *The Geometry of Jet Bundles.* Cambridge Univ. Press, Cambridge, 1989.
[41] S. I. Senashov, A. M. Vinogradov, *Symmetries and conservation laws of 2-dimensional ideal plasticity,* Proc. Edinburgh Math.Soc., **31** (1988), 415–439.
[42] D. C. Spencer, *Over determined systems of linear partial differential equations.* Bull. Amer. Math. Soc. **75** (1969), 179-239.
[43] J. D. Stasheff, *The (secret?) homological algebra of the Batalin – Vilkovisky approach* This volume.
[44] T. Tsujishita, *Formal geometry of systems of differential equations*, Sugaku Expositions **2** (1989), 1–40.
[45] T. Tsujishita, *Homological method of computing invariants of systems of differential equations*, Differential Geom. Appl. **1** (1991), 3–34.
[46] T. Tsujishita, *On variation bicomplexes associated to differential equations*, Osaka J. Math. **19** (1982), 311–363.
[47] W. M. Tulczyjew, *Cohomology of the Lagrange complex.* Ann. Scuola Norm. Sup. Pisa (1988), 217-227.
[48] W. M. Tulczyjew, *The Euler-Lagrange resolution.* Lecture Notes in Math. **836**, Springer-Verlag, New-York (1980), 22-48.
[49] W. M. Tulczyjew, *The Lagrange complex.* Bull. Soc. Math. France **105** (1977), 419-431.
[50] A. M. Verbovetsky, *Local non-integrability of long-short wave interaction equations.* Acta Appl. Math. **15** (1989), 121-136.
[51] A. M. Verbovetsky, *Notes on the horizontal cohomology.* This volume.
[52] A. M. Verbovetsky, A. M. Vinogradov, D. M. Gessler, *Scalar differential invariants and characteristic classes of homogeneous geometric structures*, Math. Notes **51** (1992), 543–549.
[53] A. M. Vinogradov, *The logic algebra for the theory of linear differential operators*, Soviet Math. Dokl., **13** (1972), 1058–1062.
[54] A. M. Vinogradov, *A spectral sequence associated with a nonlinear differential equation and algebro-geometric foundations of Lagrangian field theory with constraints*, Soviet Math. Dokl. **19** (1978), 144–148.
[55] A. M. Vinogradov, *Hamiltonian structures in field theory,* Soviet. Math. Dokl. **19** (1978), 790–794.
[56] A. M. Vinogradov, *The theory of higher infinitesimal symmetries of nonlinear partial differential equations*, Soviet Math. Dokl.,**20** (1979), 985-990.
[57] A. M. Vinogradov, *Some homological systems associated with differential calculus over commutative algebras,* Russian Math. Surveys, **34** (1979), n.6, pp.250–255.
[58] A. M. Vinogradov, *Geometry of nonlinear differential equations,* J., Sov. Math., **17** (1981)), 1624–1649.

[59] A. M. Vinogradov, *The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws.* I. *The linear theory.* II. *The nonlinear theory*, J. Math. Anal. Appl. **100** (1984), 1–129.
[60] A. M. Vinogradov, *Local symmetries and conservation laws*, Acta Appl. Math.,**3**, (1984), 21-78.
[61] A. M. Vinogradov, *Category of partial differential equations*, Lecture Notes in Math., 1108 (1984), Springer-Verlag, Berlin, 77-102.
[62] A. M. Vinogradov, *The category of differential equations and its significance for physics* , In: Krupka D. (Ed.), Proc. Conf. Diff. Geom. Appl. (Brno, 1984), (1984), J.E. Purkyně Univ., Brno, Czechoslovakia, 289–301.
[63] A. M. Vinogradov, *From symmetries of partial differential equations towards secondary (“quantized”) calculus*, J. Geom. Phys. **14** (1984), 146–194.
[64] A. M. Vinogradov, Ed.,*Symmetries of Partial Differential Equations,* 1989 (reprint from *Acta Appl. Math.*, **15,** 1989, sec.1–2, and **16** (1989), n.1–2). 151–152, (in Russian). English translation: Russian Math. Survey **44** (1989), n.3, 220–221.
[65] A. M. Vinogradov, *Scalar differential invariants, diffieties, and characteristic classes,* In: Francaviglia M. (Ed.), Mechanics, Analysis and Geometry: 200 Years after Lagrange, (1991), Elsvier, Amsterdam, 379–416.
[66] A. M. Vinogradov, *Secondary Calculus: multi-vector-valued differential forms* . To appear.
[67] A. M. Vinogradov, V. N.Gusyatnikova, V. A.Yumaguzhin, *Secondary differential operators*, Dokl.Acad. Nauk SSSR,**283** (1985), n.4, 801–805 (in Russian).
[68] A. M. Vinogradov, I. S. Krasil’shchik, V. V.Lychagin, *Introduction to Geometry of Nonlinear Partial Differential Equations,* “Nauka”, Moscow, 1986 (in Russian)
[69] M. M. Vinogradov, *Principal functors of differential calculus in graded algebras,* Uspehi mat.nauk., **44,** (1989), n.3, 267–268.

[1] Dipartimento di Ingegneria dell’Informazione e Matemetica Applicata, Universitá di Salerno — Italia., [2] Istituto Nazionale di Fisica Nucleare, sez. Napoli – Salerno, Italia, [3] The Diffiety Institute, Pereslavl-Zalessky, Russia

E-mail address: vinograd@ponza.dia.unisa.it