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SYMMETRIES OF EQUATIONS CONTAINING A SMALL PARAMETER

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Abstract For differential equations containing a small parameter ε symmetries are found in a form of series in powers of ε. The recurrent relation on summands of the series is obtained; the first summand is just a symmetry of the unperturbed equation (i.e., for an equation with $\varepsilon = 0$). Conservation laws of the unperturbed equation are not conserved in case $\varepsilon \neq 0$ and the rate of their decay is explicitly determined. Illustrative examples include the Burgers equation and a system of magnetohydrodynamics equations.

1 Introduction

Most well known model equations of mathematical physics, such as Korteweg-de Vries, Burgers, etc. are distinguished by their not-so-usual differential and algebraic properties. They have rich symmetry algebras and/or numerous conservation laws and so on. These properties are, however, regrettably unstable: few survive any perturbations of the equation. It is quite a disappointing situation from the point of view of physical applications: in physics, there is no such thing as an individual equation. Rather, the thing of prime interest is an underlying phenomenon, so there are 'clasters' of equation describing it in varying assumptions concerning physically meaningful quantifiers; small perturbations of the ideal state equation is a universal way to take into account different changing characteristics.

This paper is an attempt to use symmetries and conservation laws of the ideal (non-perturbed) equation when the latter is perturbed. In the first section the recurrent relations for an expansion of a symmetry by a small parameter is obtained for an evolution equation with one spatial variable. Its truncated solution is used to construct approximate solutions of the perturbed equation starting with a solution of the unperturbed one. The Burgers equation is used for various examples.

In the second section we estimate the rate of 'decay' of an ideal state conservation law in presence of a dissipative-like perturbations. Different rates of such a

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decay for different conservation laws produce quasistable solutions. The general argument is illustrated by a system of third order nonlinear magnetohydrodynamics equations with two dependent and three independent variables describing incompressible magnetofluids; this example may be found in [2] in more detail.

2 Symmetries

Consider an equation

$$u_t = F(x,t,u,u_x,\ldots) + \varepsilon G(x,t,u,u_x,\ldots), \tag{1}$$

where ε is a small parameter. Its symmetry may be chosen in the form

$$S = S(\varepsilon, x, t, u_x, u_{xx}, \ldots).$$

Then it must satisfy the symmetry equation

$$\partial_t S + \{F,S\} + \varepsilon\{G,S\}, \tag{2}$$

where

$$\{F,S\} = (?_F - \ell_F)S = (\ell_S - ?_S)F.$$

Here, by definition,

$$\ell_F = \sum_{n=0}^{\infty} \frac{\partial F}{\partial u_n} D^n$$

$$?_F = \sum_{n=0}^{\infty} D^n(F) \frac{\partial}{\partial u_n}$$

and the total derivative D with respect to x is given by

$$D = \sum_{i=0}^{\infty} u_{n+1} \frac{\partial}{\partial u_n};$$

u_n stands for $u_{\underbrace{xxx\ldots x}_{n}}$ If we expand

$$S(\varepsilon, x, t, u, \ldots) = \sum_{k=0}^{\infty} S_i(x,t,u,\ldots)\varepsilon^i$$

and substitute this expansion into (2), we obtain ε-less system of recurrent relations for S_i:

$$\partial_t S_i + \{F, S_i\} + \{G, S_{i-1}\}. \tag{3}$$

Here $S_i = 0$ is assumed for $i < 0$. Thus S_0 is a usual symmetry of the unperturbed equation $u_t = F$:

$$\frac{\partial S_0}{\partial t} + \{F, S_0\} = 0.$$

Even in case when the unperturbed equation possesses a rich nontrivial symmetry algebra, the perturbed equation (1) may has no symmetries. But if we calculate S_i, say, up to $i = k$, then we can obtain a certain k-approximation to a solution of the equation (1) in the following way.

Let $u = f(x,t)$ be a solution of the unperturbed equation $u_t = F$. Consider a problem

$$\begin{aligned} u_\tau = \sum_{i=0}^{k} S_i(x,t,u,u_x\ldots)\varepsilon^i, \\ u|_{\tau=0} = f(x,t). \end{aligned} \tag{4}$$

If $v(\tau,\varepsilon,x,t)$ is a solution of this problem, then

$$\begin{aligned} &\frac{\partial}{\partial \tau}|_{\tau=0}\,(v_t - F - \varepsilon G) = \\ &\frac{\partial}{\partial t}\sum_{i=0}^{k} S_i(x,t,u,u_x\ldots)\varepsilon^i + \{F, \sum_{i=0}^{k} S_i(x,t,u,u_x\ldots)\varepsilon^i\} + \{G, \sum_{i=0}^{k} S_i(x,t,u,u_x\ldots)\varepsilon^i\} = \\ &\varepsilon^{k+1}\{G, S_k\}. \end{aligned} \tag{5}$$

If, moreover, $\{G, S_k\}$ is bounded, then (5) means that v is a solution of (1) up to $O(\varepsilon^{k+1})$.

Note that if S is a genuine symmetry of (1), then for any solution of the problem $u_\tau = S$, $u|_{\tau=0} = f$ and for any τ_0 the function $u|_{\tau=\tau_0}$ is a solution of (1) as well as f. This procedure is known as a solution generating method. Hence, the solution of the problem (4) leads to a method of generating of approximate solutions.

Remark 1 If the addend G commutes with all symmetries from $sym\,\mathcal{E}$, then the symmetry algebra of the perturbed equation includes the symmetry algebra of the unperturbed one:

$$\{G, sym\,\mathcal{E}\} = 0 \Longrightarrow sym\,\mathcal{E} \subset sym\,\mathcal{E}_\varepsilon$$

2.1 Examples

2.1.1 u_t symmetry

The chain of symmetry equations (3) has few chances to have a solution. Here is one trivial example when it do has a solution (and not as a series, but as a finite sum).

Suppose both F and G do not depend on t explicitly, $\partial_t F = 0$ and $\partial_t = 0G$. Then (3) comes to the system

$$\{F, S_0\} = 0, \quad \{F, S_i\} = \{S_{i-1}, G\},\ i > 0.$$

One can take $S_0 = F$ in the first equation of the system, $S_1 = G$ in the second one and $S_i = 0$ for $i > 1$. The resulting symmetry is $S = F + \varepsilon G$, which coincides with an obvious symmetry u_t restricted to the equation $u_t = F + \varepsilon G$.

2.1.2 Burgers equation.

Burgers equation has a form

$$u_t = u_{xx} + \varepsilon u u_x.$$

In this case $F = u_{xx}$, $G = uu_x$ and the symmetry equation is of the form

$$\frac{\partial S_i}{\partial t} + \left(\sum_{j=0}^{\infty} u_{j+2} \frac{\partial}{\partial u_n} - D^2 \right) S_i = \left(uD + u_1 - \sum_{j=0}^{\infty} D^j (uu_1) \frac{\partial}{\partial u_j} \right) S_{i-1} \qquad (6)$$

The unperturbed equation in this case is simply the heat equation $u_t = u_{xx}$. It has a lot of symmetries. For instance, take $S_0 = u \equiv u_0$. (This is a common symmetry for all linear equations.) Then we have the following equation on S_1:

$$\frac{\partial S_1}{\partial t} + \left(\sum_{j=0}^{\infty} u_{j+2} \frac{\partial}{\partial u_j} - D^2 \right) S_1 = \left(uD + u_1 - \sum_{j=0}^{\infty} D^j (uu_1) \frac{\partial}{\partial u_j} \right) u \equiv uu_1 \qquad (7)$$

Suppose $S_1 = S_1(x, t, u_0)$. Then (7) comes to

$$\frac{\partial S_1}{\partial t} + u_2 \frac{\partial S_1}{\partial u} - D^2 S_1 = uu_1$$

or

$$(S_1)_t - (S_1)_{xx} - 2u_1 (S_1)_{xu} - u^2 (S_1)_{uu} = uu_1.$$

This equation has no solutions, but the structure of the previous one prompts to take

$$S_1 = S_1(x, t, u, w), \text{ where } Dw = u \text{ or } w = \int u\, dx.$$

It follows then that

$$w_t = \frac{\partial}{\partial t} \int u\, dx = \int u_t\, dx = \int (u_2 + \varepsilon uu_1)\, dx = u_1 + \frac{1}{2} u^2$$

In other words we choose the *covering* over Burgers equation (see [1]), so that

$$D_x = D + u \frac{\partial}{\partial w}, \quad D_t = \sum_{i=0}^{\infty} D^i (u_2 + \varepsilon uu_1) + (u_1 + \frac{\varepsilon}{2} u^2) \frac{\partial}{\partial w}$$

Denote $w = u_{-1}$ for convenience. The symmetry equation for such an S is of the form

$$\frac{\partial S_1}{\partial t} + \left(\sum_{j=-1}^{\infty} u_{j+2}\frac{\partial}{\partial u_j} - D^2\right) S_1 = \left(uD + u_1 - \sum_{j=-1}^{\infty} D^j(uu_1)\frac{\partial}{\partial u_j}\right) S_0. \qquad (8)$$

For $S_0 = u$ it has a solution $S_1 = -\frac{1}{2}uw$. Moreover, if $S_i = Auw^i$, then S_{i+1} may be chosen as $-\frac{A}{2(i+1)}uw^{i+1}$. Starting with $S_0 = u$ we obtain

$$S = u(1 - \frac{\varepsilon w}{2} + \frac{1}{2}\left(\frac{\varepsilon w}{2}\right)^2 - \frac{1}{6}\left(\frac{\varepsilon w}{2}\right)^3 \ldots) = ue^{-\frac{\varepsilon w}{2}}$$

Of course, this result may be obtained in a more straightforward way, i.e., without expansion in series, see [1].

2.1.3 Solution generated symmetries

Here is one more example for Burgers equation. Since the heat equation is a linear one, any of its solutions $u = f(x,t)$ is a symmetry. Take $S_0 = x$ which is an obvious solution. Assume $S_1 = S_1(x,t,u)$. Then

$$S_1 = \frac{\alpha}{4}(2t - x^2)u + \beta xu + \delta u + g(x,t),$$

where α, β, δ are arbitrary constants and $g(x,t)$ is an arbitrary solution of the heat equation. No S_2 of the form $S_2(x,t,u)$ exists here. But the solution generating process (4) with the help of

$$S = S_0 + \varepsilon S_1 = x + \varepsilon(\frac{\alpha}{4}(2t - x^2)u + \beta xu + \delta u + g(x,t))$$

is a success up to $O(\varepsilon^2)$.

3 Conservation laws

First we introduce notions and notations which are necessary to formulate results.

Let $\mathbf{E} = (\mathbf{E_1}, \ldots, \mathbf{E_l}) = \mathbf{0}$ be a N-th order system of l nonlinear differential equations on m-vector function $(f^1, \ldots, f^m)$ of $n+1$ independent variables $(x_0 = t, x_1, \ldots, x_n)$.

We interpret the equation as a submanyfold in a jet space $J^N(\pi)$, where $\pi : \mathbb{R}^m \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is a trivial bundle. If x_i and u^j are base and fiber coordinates of π, then $u^j = f^j(x_0, x_1, \ldots, x_n)$, $j = 1, \ldots, m$ are sections of π denoted by $j_0(\mathbf{f})$. The bundle $\pi_N : J^N(\pi) \longrightarrow \mathbb{R}^{n+1}$ has $\mathbf{x}$ for base coordinates and $u^j_\sigma,\ j=1,\dots,m,\ \sigma=(i_o,\dots,i_n),\ |\sigma|\le N$ for its fiber coordinates. The sections $j_N(\mathbf{f})$ of π_N are given by the formula

$$u^j_\sigma = \frac{\partial^{|\sigma|}}{\partial \mathbf{x}^\sigma} f^j(\mathbf{x}) = \frac{\partial^{|\sigma|}}{\partial \mathbf{x_0^{i_0}} \cdots \partial \mathbf{x_n^{i_n}}} \mathbf{f^j}(\mathbf{x})$$

Now $\{\mathbf{E}(\mathbf{x},\mathbf{u},\dots,\mathbf{u}_\sigma)=\mathbf{0}\}\subset \mathbf{J^N}(\pi)$ defines a submanyfold in the jet space and we denote this submanyfold by $\mathcal{E}$. Solutions of $\mathbf{E}=0$ are such $\mathbf{f}(x)$ that $j_N(\mathbf{f})(I\!R^{n+1})\subset\mathcal{E}$. Introduce the total differentiations D_i with respect to x_i:

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\,\sigma} u^j_{\sigma+1_i}\frac{\partial}{\partial u^j_\sigma} \quad \text{where} \quad 1_k = (\underbrace{0,\dots,0}_{k-1},1,0,\dots,0)$$

All differential prolongations of $\mathbf{E}$ i.e., the differential ideal $\mathcal{J}$ generated by $\mathbf{E}$ and the total differentiations D_i, define the submanyfold $\mathcal{E}^\infty\subset\mathcal{J}^\infty(\pi)$.

Consider some $l\times m$ matrix $\ell_{\mathbf{E}}$ (l is the codimension of $\mathbf{E}$) in $J^N(\pi)$), $(\ell_{\mathbf{E}})_{rs} = \sum\limits_\sigma \frac{\partial E_s}{\partial u^r_\sigma} D_\sigma$,
or

$$\ell_{\mathbf{E}} = \begin{pmatrix} \sum\limits_\sigma \frac{\partial E_1}{\partial u^1_\sigma} D_\sigma & \dots & \sum\limits_\sigma \frac{\partial E_1}{\partial u^r_\sigma} D_\sigma & \dots & \sum\limits_\sigma \frac{\partial E_1}{\partial u^m_\sigma} D_\sigma \\ \vdots & & \vdots & & \vdots \\ \sum\limits_\sigma \frac{\partial E_s}{\partial u^1_\sigma} D_\sigma & \dots & \sum\limits_\sigma \frac{\partial E_s}{\partial u^r_\sigma} D_\sigma & \dots & \sum\limits_\sigma \frac{\partial E_s}{\partial u^m_\sigma} D_\sigma \\ \vdots & & \vdots & & \vdots \\ \sum\limits_\sigma \frac{\partial E_l}{\partial u^1_\sigma} D_\sigma & \dots & \sum\limits_\sigma \frac{\partial E_l}{\partial u^r_\sigma} D_\sigma & \dots & \sum\limits_\sigma \frac{\partial E_l}{\partial u^m_\sigma} D_\sigma \end{pmatrix} \tag{9}$$

A *conservation law* for the equation $\mathbf{E}$ is a differential $n-$ form $\omega = \sum\limits_{i=0}^n \omega_i \hat{dx_i}$, such that $d\omega=0$ on $\mathcal{E}^\infty$; here $\hat{dx_i} = dx_0\wedge\ldots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\ldots\wedge dx_n$ and ω_i's are some functions on $J^\infty(\pi)$. (In physics terminology ω_0 is called a conserved density, while $(-\omega_1,\omega_2,\dots,(-1)^n\omega_n)$ is called a flux.)

The method for finding of conservation laws is as follows. Let $\ell^*_{\mathbf{E}}$ be a formal conjugate of $\ell_{\mathbf{E}}$,

$$\ell^*_{\mathbf{E}} = \begin{pmatrix} \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_1}{\partial u^1_\sigma} & \dots & \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_r}{\partial u^1_\sigma} & \dots & \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_l}{\partial u^1_\sigma} \\ \vdots & & \vdots & & \vdots \\ \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_1}{\partial u^s_\sigma} & \dots & \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_r}{\partial u^s_\sigma} & \dots & \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_l}{\partial u^s_\sigma} \\ \vdots & & \vdots & & \vdots \\ \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_1}{\partial u^m_\sigma} & \dots & \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_r}{\partial u^m_\sigma} & \dots & \sum\limits_\sigma (-1)^{|\sigma|} D_\sigma\circ\frac{\partial E_l}{\partial u^m_\sigma} \end{pmatrix} \tag{10}$$

Solutions ψ of the equation

$$\ell^*_{\mathbf{E}}(\psi)|_{\mathcal{E}^\infty} = 0 \tag{11}$$

are so called *generating functions* of conservation laws. They are connected to conservation laws themselves in a following way. By the definition of a conservation law, the equation $d\omega|_{\mathcal{E}^\infty} = 0$ holds, which is equivalent to $d\omega = \mathcal{O}(\mathbf{E})dx_0 \wedge \ldots \wedge dx_n$, where $\mathcal{O}(\mathbf{E}) \in \mathcal{J}$ $\mathcal{O}(\mathbf{E}) = \sum_{\sigma, r} \mathcal{O}_\sigma D_\sigma(E_r)$. Now $\mathcal{O}^*(1)$ is the generating function of this conservation law or a solution of (11) (* stand for formal conjugation). It remains however to find the conservation law itself and to check whether it is trivial (*trivial* by definition are conservation laws ω which are exact, i.e., $\omega = dw$ for some $(n-1)$– differential form w).

The general procedure for finding the conservation law, starting with its generating function, is connected to C-spectral sequence of the equation; however, for a low order operator $\mathcal{O}$ (as in example discussed here) it is usually not hard to discover the conservation law corresponding to any given $\mathcal{O}^*(1)$.

The main theorem is as follows (see [2])

THEOREM 1 *Consider an equation $\mathbf{E}(\eta)$ depending on a small parameter η in such a way that $\mathbf{E}_0 = \mathbf{E}(0)$ is a non–dissipative system. Let $\omega = \sum\limits_0^n \omega_i \hat{dx_i}$ be the conservation law of $\mathbf{E}_0$. Then the decay velocity of the conserved quantity $\langle\omega_0\rangle$ in presence of dissipation is given by*

$$\frac{d}{dt}\langle\omega_0\rangle\Big|_{\mathbf{E}(\eta)} = -\eta\langle\mathcal{O}^*(1)\cdot\frac{\partial\mathbf{E}}{\partial\eta}\Big|_{\eta=0}\rangle \quad \text{up} \quad \text{to } O(\eta^2)$$

3.1 Example: Conservation laws of MHD-equation

The 3-dimensional MHD–equation, describing incompressible magnetofluids in dimensionless variables may be taken in the following form, [2]:

$$\begin{cases} \Delta u_t + u_x\Delta u_y - u_y\Delta u_x + v_y\Delta v_x - v_x\Delta v_y = \nu\Delta^2 u \\ v_t + u_x v_y - u_y v_x = \eta\Delta v \end{cases} \tag{12}$$

Here subscripts mean partial differentiation: $u_x = \frac{\partial u}{\partial x}$ and so on. The (12) equation will be denoted $E(\nu, \eta)$ onward. The ideal state is described by $E(0,0)$ and denoted by E_0.

To obtain conservation laws of the ideal state one starts by solving

$$\ell^*_{E_0}\mathbf{f}|_{\mathcal{E}_0^\infty} = 0 \tag{13}$$

Here $\mathbf{f} = \begin{pmatrix} S \\ T \end{pmatrix}$ is a possible generating function of a would be conservation law; components S and T are some functions on $J^\infty(I\!R^3, I\!R^2)$. We remind that * stands for formal conjugation. The universal linearization operator ℓ_{E_0} for (12) is given by the formula

$$\ell_{E_0} = \begin{pmatrix} D_t\Delta + u_x\Delta D_y + \Delta u_y \cdot D_x - & v_y\Delta D_x - v_x\Delta D_y + \\ u_y\Delta D_x - \Delta u_x \cdot D_y & \Delta v_x \cdot D_y - \Delta v_y \cdot D_x \\ & \\ v_y D_x - v_x D_y & D_t + u_x D_y - u_y D_x \end{pmatrix} \tag{14}$$

Now $\ell^*_{E_0}$ is of the form

$$\ell^*_{E_0} = \begin{pmatrix} -D_t\Delta - u_x D_y\Delta + u_y D_x\Delta + & -v_y D_x + v_x D_y \\ 2(u_{yy} - u_{xx})D_x D_y + 2u_{xy}(D_x^2 - D_y^2) & \\ & \\ v_x D_y\Delta - v_y D_x\Delta - & -D_t - u_x D_y + u_y D_x \\ 2(v_{yy} - v_{xx})D_x D_y - 2v_{xy}(D_x^2 - D_y^2) & \end{pmatrix} \tag{15}$$

We restrict ourselves to low order conservation laws, that is to such a $\mathbf{f} = \begin{pmatrix} S \\ T \end{pmatrix}$ in (13) that S and T are functions on $J^0(I\!R^3, I\!R^2)$ and $J^2(I\!R^3, I\!R^2)$ respectively. This choice may be understood by considering the structure of $\ell^*_{E_0}$ matrix: its second column is a first order operator while the first column is of third order. Solving the equation (13), which depends polinomially on higher derivatives u_σ, v_σ, is very tedious but straightforward job. We simply list the results.

The kernel of $\ell^*_{E_0}|_{\mathcal{E}_0^\infty}$ is linearly generated by

$$\begin{pmatrix} h(t) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x^2 + y^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} p(t)x \\ 0 \end{pmatrix}, \quad \begin{pmatrix} q(t)y \\ 0 \end{pmatrix}, \tag{16}$$

$$\begin{pmatrix} u \\ \Delta u \end{pmatrix}, \quad \begin{pmatrix} f(v) \\ f'(v)\Delta v \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \Phi'(v) \end{pmatrix} \tag{17}$$

where h, p, q, f and Φ are arbitrary functions. Most of them produce trivial conservation laws. There are three non–trivial conserved densities (two of them depending on arbitrary functions): the total energy E (magnetic plus kinetic energy), generalized 'cross helicity' H_c and mean magnetic potential A,

$$\begin{aligned} E &= \tfrac{1}{2}\langle u_x^2 + u_y^2 + v_x^2 + v_y^2 \rangle \\ H_c &= \langle f'(v) \cdot (u_x v_x + u_y v_y) \rangle \\ A &= \langle \Phi(v) \rangle \end{aligned} \tag{18}$$

Their generating functions are placed on the second line of (17) in respective order. Recall that f and Φ are arbitrary functions of v.

Let us calculate the decay rates of these conservation laws

Once dissipation coefficients ν or η are allowed to have small but finite values, quantities (17) are conserved no more. In accordance with general formulas of section 2 their decay rates are

$$
\begin{aligned}
\frac{dE}{dt} &= -\nu \int_S u\Delta^2 u \, dxdy - \eta \int_S (\Delta v)^2 \, dxdy = -\int_S [\nu(\Delta u)^2 + \eta(\Delta v)^2] \, dxdy \ ; \\
\frac{dH_c}{dt} &= \frac{1}{2} \int_S [\nu f(v)\Delta^2 u + \eta f'(v)\Delta u \Delta v] \, dxdy = \\
&= -\frac{1}{2}(\nu + \eta) \int_S f'(v)\Delta u \Delta v \, dxdy - \frac{1}{2}\nu \int_S f''(v)\Delta u(v_x^2 + v_y^2) \, dxdy \ ; \\
\frac{dA}{dt} &= -\eta \int_S \Phi'(v)\Delta v \, dxdy = \eta \int_S \Phi''(v)(v_x^2 + v_y^2) \, dxdy
\end{aligned}
\tag{19}
$$

One can see that the decay of E is monotonic but those of H_c and A are not necessarily so. Such an inequality in decay rates leads to a distinct physical phenomenon of 'self–organization' or quasi–stable states of plasma. Depending on initial conditions competing processes called 'selective decay' or 'dynamic alignment' occur: in selective decay energy decays relatively to mean potential, and in dynamic alignment energy decays relatively to cross helicity (velocity and magnetic field being aligned). There are also some more delicate possibilities of self–organization.

There exist a very simple procedure for finding solutions of the above described behavior. It was suggested in [3], and is known as 'Taylor trick'; it allows us to predict and calculate quasistable states. The procedure is as follows.

Taking into consideration their comparative decay rates let us minimize E with H_c and A as constrains. We put $\delta(E + \lambda H_c + \mu A) = 0$, A and H_c presumed constant, λ and μ being Lagrange multipliers. The Euler–Lagrange equations are

$$
\left\{
\begin{array}{l}
\Delta[u - F(v)] = 0 \\
\Delta v = f(v)\Delta u + g(v) \ ,
\end{array}
\right.
\tag{20}
$$

where $F' = f$ and $g = \pm\Phi'$.

The system (20) generally is not compatible with (12). But it is compatible if $\eta = \nu$ which is in particular true in the ideal case $\eta = \nu = 0$. In this case, combining (12) and (20) we get

$$
\left\{
\begin{array}{l}
\Delta[u - F(v)] = 0 \\
\Delta v = \frac{ff'}{1-f^2}(v_x^2 + v_y^2) + \frac{g}{1-f^2} \\
v_t = u_y v_x - u_x v_y \\
(u_{xy} - f v_{xy})(v_x^2 - v_y^2) + [(u_{yy} - f v_{yy}) - (u_{xx} - f v_{xx})] v_x v_y = 0
\end{array}
\right.
\tag{21}
$$

Solutions of (21) describe the quasistationary states with remarkable accuracy as it was demonstrated numerically for special types of f and Φ in [4].

Remark 2 The first and the last equations of ((21) form the closed system

$$
\left\{
\begin{array}{l}
\Delta w = 0 \\
z_t + w_x z_y - w_y z_x = 0 \ ,
\end{array}
\right.
$$

where $w = u - F(v)$ and $z = v_x^2 + v_y^2$.

Remark 3 The second equation in ((21) may be written in a closed form $\Delta R = \Psi(R)$ *where* $R = R(v)$, $R' = \sqrt{1 - f^2}$

Remark 4 The case of $u = F(v)$ *in (21) is a generalization of dynamic alignment studied in [3] (aligned are gradients of* u *and* v*). It implies stationary solutions*

$$\begin{cases} u = F(v) \\ v_t = 0 \\ \Delta R = \Psi(R) \ , \end{cases}$$

where $R'(v) = \sqrt{1 - f^2(v)}$ *as in previous remark.*

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