

**Notes on the horizontal
cohomology**

by

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Notes on the horizontal cohomology

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ABSTRACT. This paper is devoted to the horizontal (“characteristic”) cohomology of systems of differential equations. Recent results on computing the horizontal cohomology via the compatibility complex are generalized. New results on the Vinogradov \mathcal{C} -spectral sequence and Krasil’shchik’s \mathcal{C} -cohomology are obtained. As an application of general theory, the examples of an evolution equation and a p -form gauge theory are explicitly worked out.

INTRODUCTION

Consider a system of differential equations

$$F_s(x_i, u^j, \dots, u_\sigma^j, \dots) = 0, \quad (*)$$

with independent variables x_i , unknown functions u^j , and $u_\sigma^j = \partial^\sigma u^j / \partial x_{i_1} \cdots \partial x_{i_r}$ being their derivatives, $\sigma = i_1 \dots i_r$. Let \mathcal{F} be the algebra of functions in the variables x_i , u^j , and u_σ^j . Two functions are said to be equivalent if they coincide when equations (*) hold. Denote the quotient algebra by \mathcal{F}_F . The horizontal de Rham complex for system (*) is the exterior algebra generated by elements of \mathcal{F}_F and differentials dx_i , with the differential being the total exterior differential $\bar{d}(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_i (df/dx_i) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. The cohomology of this complex, called the *horizontal* (“characteristic”) cohomology, has been studied and used extensively in the literature (see the bibliography). This cohomology plays a central role in the Lagrangian formalism (because the horizontal cohomology group in degree n is the space of actions of variational problems constrained by equations (*)). The horizontal cohomology group in degree $n - 1$ consists of conservation laws for the system under consideration. This fact is basic to finding conservation laws via the Vinogradov \mathcal{C} -spectral sequence. The horizontal cohomology in degrees less than $n - 1$ has attracted recently a great deal of interest in the context of gauge theories (for equations without gauge symmetries this cohomology is trivial by the famous “two-line theorem” of Vinogradov [28, 30]), where it appears as a means of calculating the BRST cohomology [3].

For computing the horizontal cohomology there is a general method based on the Vinogradov \mathcal{C} -spectral sequence. It can be outlined as follows. The horizontal cohomology is the term $E_1^{0, \bullet}$ of the Vinogradov \mathcal{C} -spectral sequence and thereby related to the terms $E_1^{p, \bullet}$, $p > 0$. For each p , such a term is also a horizontal cohomology but with some nontrivial coefficients. The crucial observation is that the corresponding module of coefficients is supplied with a filtration such that the differential of the associated graded complex is linear over the functions. Hence, the cohomology can be computed algebraically. For $p = 1$, this has been done in [23] and the main result is: The term $E_1^{1, \bullet}$ coincides with the homology of the complex dual to the compatibility complex for the linearization operator of the system under consideration. In this paper, this result is generalized to $p > 1$. We compute also Krasil’shchik’s \mathcal{C} -cohomology (see [9, 10, 11, 12, 13, 14]). In particular, the “ k -line

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theorem” is proved for both cases. It should be also stressed that our techniques are capable not only of obtaining vanishing results, but facilitate the computation of nonzero part as well.

The author’s thinking about computing the horizontal cohomology was especially stimulated by works of I. S. Krasil’shchik [9] and M. Marvan [19] where the horizontal cohomology with coefficients in the Cartan forms and zero-curvature representations was calculated. By trying to understand their calculations, the author arrived at the construction of the present paper.

The ability to calculate the horizontal cohomology is not unique to the method described above. There is also an approach based on the Koszul - Tate resolution. The discussion of this approach does not enter into the scope of the present paper. It is desired here merely to point out that both methods take as a starting point the compatibility complex and furnish equivalent results although the precise relationships between them have yet to be uncovered.

The paper is organized as follows. We begin with a short introduction to the \mathcal{C} -differential calculus on differential equations. In the opening subsection of Section 2, we discuss the notion of a \mathcal{C} -module. A \mathcal{C} -module is, by definition, a module equipped with an action of \mathcal{C} -differential operators. Thus, \mathcal{C} -modules serve as modules of coefficients for horizontal de Rham complexes. Such complexes are dealt with in Section 2.2. In Section 2.3, we relate the horizontal cohomology to the cohomology of the compatibility complex. The results of this section play a key role in the explicit calculation of the horizontal cohomology. Finally, in Section 3 we turn to the examples of computing the Vinogradov \mathcal{C} -spectral sequence and Krasil’shchik’s \mathcal{C} -cohomology.

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1. \mathcal{C} -DIFFERENTIAL CALCULUS

1.1. Jet spaces. In this section a brief discussion of the \mathcal{C} -differential calculus on differential equations is given, further details being available in [17, 15, 29, 20].

Let $\pi: E \rightarrow M$ be an $n+m$ -dimensional vector bundle over an n -dimensional base manifold M , and let $\pi_\infty: J^\infty(\pi) \rightarrow M$ be the infinite jet bundle of local sections of π . Denote by $j_\infty(s)(x)$ the infinite jet of a local section s of the bundle π at point $x \in M$. Thus each section $s: M \rightarrow E$ gives rise to the section $j_\infty(s): M \rightarrow J^\infty(\pi)$.

A coordinate system (x_i, u^j) , $i = 1, \dots, n$, $j = 1, \dots, m$, of $\pi: E \rightarrow M$ induces coordinates $(x_i, u^j, u_{i_1}^j, \dots, u_{i_\sigma}^j, \dots)$ on $J^\infty(\pi)$ such that

$$u_{i_\sigma}^j(j_\infty(s)(x)) = \frac{\partial^{|\sigma|} s^j}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_r}},$$

where $\sigma = i_1 i_2 \dots i_r$ is a multi-index, $|\sigma| = r$.

Denote the algebra of functions on $J^\infty(\pi)$ by $\mathcal{F} = \mathcal{F}(\pi)$. Functions on $J^\infty(\pi)$ are smooth functions of a *finite* number of coordinates. Let $\alpha: W \rightarrow J^\infty(\pi)$ be a vector bundle and $\Gamma(\alpha)$ be the set of local sections of α . Naturally, $\Gamma(\alpha)$ is a module over the algebra \mathcal{F} . In the sequel, by a module we always mean a module of this form. Recall that homomorphisms of modules are in one-to-one correspondence with morphisms of bundles, so we can and will not distinguish between bundles and modules.

On an infinite jet space, there exists the \mathcal{C} -differential calculus. In coordinate language, this is the total derivatives calculus. Let us describe \mathcal{C} -differential operators, corresponding jets, and forms in more details.

Consider two \mathcal{F} -modules P and P_1 . A differential operator $\Delta: P \rightarrow P_1$ is called \mathcal{C} -differential, if it can be restricted to the manifolds of the form $j_\infty(s)(M)$, where s is a local section of π . In other words, Δ is a \mathcal{C} -differential operator, if the equality $j_\infty(s)^*(\varphi) = 0$, $\varphi \in P$, implies $j_\infty(s)^*(\Delta(\varphi)) = 0$.

In local coordinates, \mathcal{C} -differential operators have the form

$$\begin{pmatrix} \sum_{\sigma} a_{i_1}^{\sigma} D_{\sigma} & \cdots & \sum_{\sigma} a_{i_{m_1}}^{\sigma} D_{\sigma} \\ \cdots & \cdots & \cdots \\ \sum_{\sigma} a_{i_{m_2}}^{\sigma} D_{\sigma} & \cdots & \sum_{\sigma} a_{i_{m_2 m_1}}^{\sigma} D_{\sigma} \end{pmatrix},$$

where $a_{i_j}^{\sigma} \in \mathcal{F}$, $D_{\sigma} = D_{i_1} \circ \cdots \circ D_{i_r}$, for $\sigma = i_1 \dots i_r$, and $D_i = \partial/\partial x_i + \sum_{j,\sigma} u_{\sigma}^j \partial/\partial u_{\sigma}^j$ is the i -th total derivative operator.

We shall denote the \mathcal{F} -module of \mathcal{C} -differential operators from P to P_1 by $\mathcal{C}\text{Diff}(P, P_1)$. In this module, there exists a filtration by the modules $\mathcal{C}\text{Diff}_k(P, P_1)$ that consist of \mathcal{C} -differential operators of order $\leq k$.

Next, define the *module of horizontal jets*. Let P be an \mathcal{F} -module. Say that two elements of P are horizontally equivalent up to order $k \leq \infty$ at point $\theta \in J^\infty(\pi)$, if their total derivatives up to order k coincide at θ . The horizontal jet space $\bar{J}_\theta^k(P)$ is P modulo this relation, and the collection $\bar{J}^k(P) = \bigcup_{\theta \in J^\infty(\pi)} \bar{J}_\theta^k(P)$ constitutes the *horizontal jet bundle* $\bar{J}^k(P) \rightarrow J^\infty(\pi)$. We denote the module of sections of horizontal jet bundle by $\bar{\mathcal{J}}^k(P)$.

As with the usual jet bundles, there exist the natural \mathcal{C} -differential operators

$$\bar{j}_k: P \rightarrow \bar{\mathcal{J}}^k(P),$$

and the natural projections $\nu_{k,l}: \bar{\mathcal{J}}^k(P) \rightarrow \bar{\mathcal{J}}^l(P)$ such that $\nu_{k,l} \circ \bar{j}_k = \bar{j}_l$. For any \mathcal{C} -differential operator $\Delta: P \rightarrow P_1$ of order $\leq k$, there is a unique \mathcal{F} -homomorphism $\varphi_\Delta: \bar{\mathcal{J}}^k(P) \rightarrow P_1$ such that $\Delta = \varphi_\Delta \circ \bar{j}_k$. The composition

$$\bar{j}_l \circ \Delta \in \mathcal{C}\text{Diff}_{k+l}(P, \bar{\mathcal{J}}^l(P_1))$$

is called the l -th prolongation of Δ and the corresponding homomorphism from $\bar{\mathcal{J}}^{k+l}(P)$ to $\bar{\mathcal{J}}^l(P_1)$ is denoted by φ_Δ^l . In particular, Δ gives rise to the mapping $\varphi_\Delta^\infty: \bar{\mathcal{J}}^\infty(P) \rightarrow \bar{\mathcal{J}}^\infty(P_1)$.

Let $D(\pi)$ be the \mathcal{F} -module of vector fields on $J^\infty(\pi)$. Define the *module of Cartan vector fields* as the intersection $\mathcal{C}\text{D}(\pi) = D(\pi) \cap \mathcal{C}\text{Diff}_1(\mathcal{F}, \mathcal{F})$. In coordinates, a Cartan vector field is $\sum_i f_i D_i$, $f_i \in \mathcal{F}$. A vector field $X \in D(\pi)$ is called *vertical* if $X(j_\infty(s)^*(h)) = 0$ for all functions $h \in C^\infty(M)$. Locally a vertical vector field has the form $\sum_{j,\sigma} f_{j,\sigma} \partial/\partial u_{\sigma}^j$. Denote the module of vertical vector fields by $D^\vee(\pi)$. Clearly, $D(\pi) = \mathcal{C}\text{D}(\pi) \oplus D^\vee(\pi)$.

Further, consider the module $\Lambda^k(\pi)$ of differential k -forms on $J^\infty(\pi)$. A form $\omega \in \Lambda^k(\pi)$ is called a *Cartan form*, if ω satisfies $j_\infty(s)^*(\omega) = 0$ for every local section s of π . Equivalently, Cartan forms are forms that vanish on the Cartan vector fields. The set of all Cartan forms defines an ideal

$$\mathcal{C}\Lambda^*(\pi) = \bigoplus_{k \geq 0} \mathcal{C}\Lambda^k(\pi)$$

in the ring

$$\Lambda^*(\pi) = \bigoplus_{k \geq 0} \Lambda^k(\pi)$$

of all forms on $J^\infty(\pi)$. In coordinates, this ideal is generated by the Cartan 1-forms

$$\omega_\sigma^j = dw_\sigma^j - \sum_i w_{\sigma i}^j dx_i$$

for all j and σ . The quotient algebra

$$\bar{\Lambda}^*(\pi) = \Lambda^*(\pi) / \mathcal{C}\Lambda^*(\pi)$$

is called the algebra of *horizontal forms* on $J^\infty(\pi)$. In coordinates, a horizontal k -form is a sum of terms of the form

$$f dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad f \in \mathcal{F}(\pi).$$

The exterior derivative

$$d: \Lambda^k(\pi) \rightarrow \Lambda^{k+1}(\pi)$$

gives rise to the *horizontal differential*

$$\bar{d}: \bar{\Lambda}^k(\pi) \rightarrow \bar{\Lambda}^{k+1}(\pi),$$

since the ideal of Cartan forms $\mathcal{C}\Lambda^*(M)$ is stable with respect to d : $d(\mathcal{C}\Lambda^*(\pi)) \subset \mathcal{C}\Lambda^*(\pi)$. Thus we get the *horizontal de Rham complex*

$$0 \rightarrow \mathcal{F}(\pi) \xrightarrow{\bar{d}} \bar{\Lambda}^1(\pi) \xrightarrow{\bar{d}} \bar{\Lambda}^2(\pi) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n(\pi) \rightarrow 0.$$

More general, consider the filtration in the de Rham complex on the jet space $J^\infty(\pi)$

$$\cdots \subset \mathcal{C}^{k+1}\Lambda^* \subset \mathcal{C}^k\Lambda^* \subset \cdots \subset \mathcal{C}^2\Lambda^* \subset \mathcal{C}\Lambda^* \subset \Lambda^*,$$

where $\mathcal{C}^k\Lambda^*$ is the k -th exterior power of the Cartan ideal $\mathcal{C}\Lambda^*$. Evidently, $d(\mathcal{C}^k\Lambda^*) \subset \mathcal{C}^k\Lambda^*$, so that we obtain a spectral sequence converging to the de Rham cohomology of $J^\infty(\pi)$. The zero term of the spectral sequence is $E_0^{p,q} = \mathcal{C}^p\Lambda^{p+q} = \mathcal{C}\Lambda^1 \wedge \cdots \wedge \mathcal{C}\Lambda^1 \otimes \bar{\Lambda}^q$. This spectral sequence is called the Vinogradov \mathcal{C} -spectral sequence [27, 28]. Further details on this important spectral sequence can be found in [30, 17, 31, 21, 22, 23, 5, 1, 2, 26, 18] and below in this paper as well.

Given a \mathcal{C} -differential operator $\Delta \in \mathcal{C}\text{Diff}_k(P, P_1)$, we define the (l -th) symbol $\sigma(\Delta)$ of Δ by the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S^{k+l}(\bar{\Lambda}^1) \otimes P & \longrightarrow & \bar{\mathcal{J}}^{k+l}(P) & \xrightarrow{\nu_{k+l, k+l-1}} & \bar{\mathcal{J}}^{k+l-1}(P) & \longrightarrow & 0 \\ & & \downarrow \sigma(\Delta) & & \downarrow \varphi_\Delta^{k+l} & & \downarrow \varphi_\Delta^{k+l-1} & & \\ 0 & \longrightarrow & S^k(\bar{\Lambda}^1) \otimes P_1 & \longrightarrow & \bar{\mathcal{J}}^k(P_1) & \xrightarrow{\nu_{k, k-1}} & \bar{\mathcal{J}}^{k-1}(P_1) & \longrightarrow & 0. \end{array}$$

The rows of the diagram are exact, with inclusions $S^k(\bar{\Lambda}^1) \otimes P \rightarrow \bar{\mathcal{J}}^k(P)$ given by $df_1 \cdots df_k \otimes p \mapsto [\dots [\bar{j}_k, f_1], \dots, f_k](p)$, where $p \in P$, $[\cdot, f_i]$ is the commutator with the operator of multiplication by $f_i \in \mathcal{F}$.

Let us consider the pullback of π along the projection $J^\infty(\pi) \rightarrow M$ and denote the module of sections of this vector bundle by \varkappa . It is readily seen that for any point $\theta = j_\infty(s)(x) \in J^\infty(\pi)$ one has

$$T_\theta(J^\infty(\pi)) = J_x^\infty(\pi) = \bar{J}_\theta^\infty(\varkappa). \quad (1.1)$$

This yields the canonical isomorphism

$$D^\vee(\pi) = \bar{\mathcal{J}}^\infty(\varkappa). \quad (1.2)$$

The dual isomorphism reads

$$\mathcal{C}\Lambda^1(\pi) = \mathcal{C}\text{Diff}(\varkappa, \mathcal{F}). \quad (1.3)$$

In coordinates, the form ω_σ^j under this isomorphism is the operator (\dots, D_σ, \dots) , with D_σ on j -th place.

It is clear that the Cartan k -forms can be identified with multilinear skew-symmetric \mathcal{C} -differential operators in k arguments.

1.2. Differential equations. Pick up a system of k -th order partial differential equations

$$F_s(x_i, u^j, \dots, u_\sigma^j, \dots) = 0, \quad s = 1, \dots, l. \quad (1.4)$$

We shall consider $F = (F_1, \dots, F_l)$ as an element of an \mathcal{F} -module P . The system $F = 0$ defines a subbundle $\mathcal{E} \rightarrow M$ of the jet bundle $J^k(\pi) \rightarrow M$. A solution to the system of differential equations is a section $s: M \rightarrow E$ such that $j_k(s)(M) \subset \mathcal{E}$. One can define the *infinite prolongation* \mathcal{E}^∞ of \mathcal{E} by equations $F_s = 0, \dots, D_\sigma F_s = 0, \dots$. This system is equivalent to (1.4) in the sense that both have the same set of solutions. For brevity, \mathcal{E} and \mathcal{E}^∞ will be referred to as “equation”.

We define the function algebra $\mathcal{F}(\mathcal{E})$ on \mathcal{E}^∞ to be the restriction of the algebra $\mathcal{F}(\pi)$ on $J^\infty(\pi)$. It is straightforward to show that one can pull all ingredients of the \mathcal{C} -differential calculus on $J^\infty(\pi)$, discussed in the previous subsection, back to \mathcal{E}^∞ . Thus, we have only one thing to do here: to generalize formulae (1.2) and (1.3).

Pick a point $\theta = j_\infty(s)(x) \in \mathcal{E}^\infty$. In view of (1.1), the tangent space $T_\theta(\mathcal{E}^\infty)$ is isomorphic to a subspace $R_\theta \subset \bar{J}^\infty(\mathcal{X})$. In coordinates, the subspace $T_\theta(\mathcal{E}^\infty) \subset T_\theta(J^\infty(\pi))$ is given by the equations $\sum_{j,\sigma} (\partial D_\tau(F_s)/\partial u_\sigma^j) du_\sigma^j = 0$, τ being a multi-index. Therefore R_θ is defined by

$$\sum_{j,\sigma} \frac{\partial D_\tau(F_s)}{\partial u_\sigma^j} w_\sigma^j = 0, \quad (1.5)$$

where $w_\sigma^j = \pi_\infty^*(u_\sigma^j)$ are coordinates on $\bar{J}^\infty(\mathcal{X})$. Clearly, system (1.5) can be rewritten in the form

$$\sum_{j,\sigma} D_\tau \left(\frac{\partial F_s}{\partial u_\sigma^j} w_\sigma^j \right) = 0$$

or $D_\tau(\ell_F(w^j)) = 0$, where

$$\ell_F = \begin{pmatrix} \sum_\sigma \frac{\partial F_1}{\partial u_\sigma^1} D_\sigma & \dots & \sum_\sigma \frac{\partial F_1}{\partial u_\sigma^m} D_\sigma \\ \dots & \dots & \dots \\ \sum_\sigma \frac{\partial F_l}{\partial u_\sigma^1} D_\sigma & \dots & \sum_\sigma \frac{\partial F_l}{\partial u_\sigma^m} D_\sigma \end{pmatrix}$$

is the *operator of universal linearization* for F .

Now our discussion can be summarized as follows.

Proposition 1.1. *For any differential equation \mathcal{E} ,*

1. *The module $D^\vee(\mathcal{E})$ is isomorphic to the kernel of the homomorphism $\varphi_{\ell_F}^\infty: \bar{J}^\infty(\mathcal{X}) \rightarrow \bar{J}^\infty(P)$.*
2. *The module $\mathcal{C}\Lambda^1(\mathcal{E})$ is isomorphic to $\mathcal{C}\text{Diff}(\mathcal{X}, \mathcal{F})$ modulo the submodule consisting of the operators of the form $\nabla \circ \ell_F$, $\nabla \in \mathcal{C}\text{Diff}(P, \mathcal{F})$.*

We use the notation $D^\vee(\mathcal{E})$, $\mathcal{C}^k \Lambda^s(\mathcal{E})$, and so on, for the corresponding modules on \mathcal{E}^∞ .

Remark 1.2. The constructions and results covered in this paper are valid not only for equations \mathcal{E}^∞ , but for arbitrary diffieties [31, 17] as well. Recall, that a diffiety is an infinite-dimensional manifold furnished with an involutive finite-dimensional distribution that locally is of the form \mathcal{E}^∞ endowed with the distribution of Cartan fields.

2. HORIZONTAL COHOMOLOGY

2.1. \mathcal{C} -modules on differential equations. Fix an equation \mathcal{E}^∞ . Let $\mathcal{F} = \mathcal{F}(\mathcal{E})$ be the algebra of functions on \mathcal{E}^∞ . An \mathcal{F} -module Q is called a \mathcal{C} -module, if Q is endowed with a left module structure over the ring $\mathcal{C}\text{Diff}(\mathcal{F}, \mathcal{F})$, i.e., for any scalar \mathcal{C} -differential operator $\Delta \in \mathcal{C}\text{Diff}_k(\mathcal{F}, \mathcal{F})$ there exists an operator $\Delta_Q \in \mathcal{C}\text{Diff}_k(Q, Q)$ with

1. $(\sum_i f_i \Delta_i)_Q = \sum_i f_i (\Delta_i)_Q$, $f_i \in \mathcal{F}$,
2. $(\text{id}_{\mathcal{F}})_Q = \text{id}_Q$,
3. $(\Delta_1 \circ \Delta_2)_Q = (\Delta_1)_Q \circ (\Delta_2)_Q$.

In other words, a \mathcal{C} -module is a module equipped with a *flat horizontal connection*, i.e., with an action on Q of the module $\mathcal{C}\mathcal{D} = \mathcal{C}\mathcal{D}(\mathcal{E})$ of Cartan vector fields, $X \mapsto \nabla_X$, that is \mathcal{F} -linear:

$$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y, \quad f, g \in \mathcal{F}, \quad X, Y \in \mathcal{C}\mathcal{D},$$

satisfies the Leibnitz rule:

$$\nabla_X(fq) = X(f)q + f\nabla_X(q), \quad q \in Q, \quad X \in \mathcal{C}\mathcal{D}, \quad f \in \mathcal{F},$$

and is a Lie algebra homomorphism:

$$[\nabla_X, \nabla_Y] = \nabla_{[X, Y]}.$$

The coordinate description of a flat horizontal connection looks as

$$\nabla_{D_i}(s_j) = \sum_k \Gamma_{ij}^k s_k, \quad \Gamma_{ij}^k \in \mathcal{F},$$

where s_j are basis elements of Q .

Remark 2.1. Let Q be the module of sections of a vector bundle $\tau: W \rightarrow \mathcal{E}^\infty$, $Q = \Gamma(\tau)$. A flat horizontal connection on Q defines a completely integrable n -dimensional linear distribution on W that is projected onto the Cartan distribution on \mathcal{E}^∞ . Thus, geometrically, a \mathcal{C} -module is the module of sections of a linear covering (see [17, 16]).

In coordinates the covering has the form

$$\widetilde{D}_i = D_i + \sum_{j,k} \Gamma_{ij}^k w^j \frac{\partial}{\partial w_k},$$

where w^i are fiber coordinate on W .

Here are basic examples of \mathcal{C} -modules.

Example 2.2. The simplest example of a \mathcal{C} -module is $Q = \mathcal{F}$ with the usual action of horizontal operators.

Example 2.3. The module of vertical vector fields $Q = \mathcal{D}^\vee = \mathcal{D}^\vee(\mathcal{E})$ with the connection

$$\nabla_X(Y) = [Y, X], \quad X \in \mathcal{C}\mathcal{D}, \quad Y \in \mathcal{D}^\vee$$

is a \mathcal{C} -module.

Example 2.4. Next example is the modules of Cartan forms $Q = \mathcal{C}^k \Lambda^k = \mathcal{C}^k \Lambda^k(\mathcal{E})$. A vector field $X \in \mathcal{C}\mathcal{D}$ acts on $\mathcal{C}^k \Lambda^k$ as the Lie derivative L_X . It is easily seen that in coordinates we have

$$(D_i)_{\mathcal{C}^k \Lambda^k}(\omega_\sigma^j) = \omega_{\sigma i}^j$$

Example 2.5. The infinite jet module $Q = \bar{\mathcal{J}}^\infty(P)$ of an \mathcal{F} -module P is a \mathcal{C} -module via

$$\nabla_X(f\bar{j}_\infty(p)) = X(f)\bar{j}_\infty(p),$$

where $X \in \mathcal{CD}$, $f \in \mathcal{F}$, $p \in P$.

Example 2.6. Let us dualize the previous example. It is clear that for any \mathcal{F} -module P the module $Q = \mathcal{CDiff}(P, \mathcal{F})$ is a \mathcal{C} -module. The action of horizontal operators is the composition.

Example 2.7. More generally, let $\Delta: P \rightarrow P_1$ be a \mathcal{C} -differential operator and $\varphi_\Delta^\infty: \bar{\mathcal{J}}^\infty(P) \rightarrow \bar{\mathcal{J}}^\infty(P_1)$ be the corresponding prolongation of Δ . Obviously, φ_Δ^∞ is a morphism of \mathcal{C} -modules, i.e., a homomorphism over the ring $\mathcal{CDiff}(\mathcal{F}, \mathcal{F})$, so that $\ker \varphi_\Delta^\infty$ and $\text{coker } \varphi_\Delta^\infty$ are \mathcal{C} -modules.

Dually, the operator Δ gives rise to the morphism of \mathcal{C} -modules $\mathcal{CDiff}(P_1, \mathcal{F}) \rightarrow \mathcal{CDiff}(P, \mathcal{F})$, $\nabla \mapsto \nabla \circ \Delta$. Thus the kernel and cokernel of this map are \mathcal{C} -modules.

Example 2.8. Given two \mathcal{C} -modules Q_1 and Q_2 , we can define \mathcal{C} -module structures on $Q_1 \otimes_{\mathcal{F}} Q_2$ and $\text{Hom}_{\mathcal{F}}(Q_1, Q_2)$ by

$$\begin{aligned} \nabla_X(q_1 \otimes q_2) &= \nabla_X(q_1) \otimes q_2 + q_1 \otimes \nabla_X(q_2), \\ \nabla_X(f)(q_1) &= \nabla_X(f(q_1)) - f(\nabla_X(q_1)), \end{aligned}$$

where $X \in \mathcal{CD}$, $q_1 \in Q_1$, $f \in \text{Hom}_{\mathcal{F}}(Q_1, Q_2)$.

For instance, we have \mathcal{C} -module structures on $Q = \bar{\mathcal{J}}^\infty(P) \otimes_{\mathcal{F}} \mathcal{C}^k \Lambda^k$ and $Q = \mathcal{CDiff}(P, \mathcal{C}^k \Lambda^k)$ for any \mathcal{F} -module P .

Example 2.9. Let \mathfrak{g} be a Lie algebra and $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(W)$ a linear representation of \mathfrak{g} . Each \mathfrak{g} -valued horizontal form $\omega \in \bar{\Lambda}^1(\mathcal{E}) \otimes_{\mathbb{R}} \mathfrak{g}$ that satisfies the horizontal Maurer-Cartan condition $\bar{d}\omega + \frac{1}{2}[\omega, \omega] = 0$ defines on the module Q of sections of the trivial vector bundle $\mathcal{E}^\infty \times W \rightarrow \mathcal{E}^\infty$ the following \mathcal{C} -module structure:

$$\nabla_X(q)_a = X(q)_a + \rho(\omega(X))(q_a),$$

where $X \in \mathcal{CD}$, $q \in Q$, $a \in \mathcal{E}^\infty$. Such \mathcal{C} -modules are called *zero-curvature representations* over \mathcal{E}^∞ (cf. [19]). Take the example of the KdV equation (in the form $u_t = uu_x + u_{xxx}$) and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. Then there exists a one-parameter family of Maurer-Cartan forms $\omega(\lambda) = A_1(\lambda) \bar{d}x + A_2(\lambda) \bar{d}t$, λ being a parameter:

$$A_1(\lambda) = \begin{pmatrix} 0 & -(\lambda + u) \\ \frac{1}{6} & 0 \end{pmatrix}, \quad A_2(\lambda) = \begin{pmatrix} -\frac{1}{6}u_x & -u_{xx} - \frac{1}{3}u^2 + \frac{1}{3}\lambda u + \frac{2}{3}\lambda^2 \\ \frac{1}{18}u - \frac{1}{9}\lambda & \frac{1}{6}u_x \end{pmatrix}.$$

This is the zero-curvature representation used in the inverse scattering method.

In coordinates, if the form ω is given by $\omega = \sum_i A_i dx_i$, $A_i \in \mathfrak{g}$, then for any \mathcal{C} -differential operator Δ the coordinate description of the operator Δ_Q can be obtained by replacing all occurrences of D_i with $D_i + \text{ad } A_i$.

Remark 2.10. In parallel with left \mathcal{C} -modules one can consider *right* \mathcal{C} -modules, i.e., right modules over the ring $\mathcal{CDiff}(\mathcal{F}, \mathcal{F})$. There is a natural way to pass from left \mathcal{C} -modules to right ones and back. Namely, for any left module Q set

$$B(Q) = Q \otimes_{\mathcal{F}} \bar{\Lambda}^n(\mathcal{E}),$$

with the right action of $\mathcal{CDiff}(\mathcal{F}, \mathcal{F})$ on $B(Q)$ given by

$$\begin{aligned} (q \otimes \omega)f &= fq \otimes \omega = q \otimes f\omega, \quad f \in \mathcal{F}, \\ (q \otimes \omega)X &= -\nabla_X(q) \otimes \omega - q \otimes L_X\omega, \quad X \in \mathcal{CD}. \end{aligned}$$

One can easily verify that B determines an equivalence between the categories of left \mathcal{C} -modules and right \mathcal{C} -modules.

Take a \mathcal{C} -module Q . By definition, for a scalar \mathcal{C} -differential operator $\Delta: \mathcal{F} \rightarrow \mathcal{F}$ there exists the operator $\Delta_Q: Q \rightarrow Q$. In fact one has more:

Proposition 2.11. *Let P, S be \mathcal{F} -modules. Then there exists a unique mapping*

$$\mathcal{C}\text{Diff}_k(P, S) \rightarrow \mathcal{C}\text{Diff}_k(P \otimes_{\mathcal{F}} Q, S \otimes_{\mathcal{F}} Q), \quad \Delta \mapsto \Delta_Q,$$

such that the following conditions hold:

1. if $P = S = \mathcal{F}$ then the mapping is given by the \mathcal{C} -module structure on Q ,
2. $(\sum_i f_i \Delta_i)_Q = \sum_i f_i (\Delta_i)_Q$, $f_i \in \mathcal{F}$,
3. if $\Delta \in \mathcal{C}\text{Diff}_0(P, S) = \text{Hom}_{\mathcal{F}}(P, S)$ then $\Delta_Q = \Delta \otimes_{\mathcal{F}} \text{id}_Q$,
4. if R is another \mathcal{F} -module and $\Delta_1: P \rightarrow S$, $\Delta_2: S \rightarrow R$ are \mathcal{C} -differential operators, then $(\Delta_2 \circ \Delta_1)_Q = (\Delta_2)_Q \circ (\Delta_1)_Q$.

Proof. The uniqueness is obvious. To prove the existence consider the family of operators $\Delta(p, s^*): \mathcal{F} \rightarrow \mathcal{F}$, $p \in P$, $s^* \in S^* = \text{Hom}_{\mathcal{F}}(S, \mathcal{F})$, $\Delta(p, s^*)(f) = s^*(\Delta(fp))$, $f \in \mathcal{F}$. Clearly, the operator Δ is defined by the family $\Delta(p, s^*)$. The following statement is also obvious.

Lemma 2.12 ([30]). *For the family of operators $\Delta[p, s^*] \in \mathcal{C}\text{Diff}_k(\mathcal{F}, \mathcal{F})$, $p \in P$, $s^* \in S^*$, we can find an operator $\Delta \in \mathcal{C}\text{Diff}_k(P, S)$ such that $\Delta[p, s^*] = \Delta(p, s^*)$, if and only if*

$$\begin{aligned} \Delta[p, \sum_i f_i s_i^*] &= \sum_i f_i \Delta[p, s_i^*], \\ \Delta[\sum_i f_i p_i, s^*] &= \sum_i \Delta[p_i, s^*] f_i. \end{aligned}$$

In view of this lemma, the family of operators

$$\Delta_Q[p \otimes q, s^* \otimes q^*](f) = q^*(\Delta(p, s^*)_Q(fq))$$

uniquely determines the operator Δ_Q . □

2.2. The horizontal de Rham complex. Consider a complex of \mathcal{C} -differential operators $\cdots \rightarrow P_{i-1} \xrightarrow{\Delta_i} P_i \xrightarrow{\Delta_{i+1}} P_{i+1} \rightarrow \cdots$. Multiplying it by a \mathcal{C} -module Q and taking into account Proposition 2.1, we obtain the complex

$$\cdots \rightarrow P_{i-1} \otimes Q \xrightarrow{(\Delta_i)_Q} P_i \otimes Q \xrightarrow{(\Delta_{i+1})_Q} P_{i+1} \otimes Q \rightarrow \cdots$$

Applying this construction to the horizontal de Rham complex, we get *horizontal de Rham complex with coefficients in Q* :

$$0 \rightarrow Q \xrightarrow{\bar{d}_Q} \bar{\Lambda}^1 \otimes_{\mathcal{F}} Q \xrightarrow{\bar{d}_Q} \cdots \xrightarrow{\bar{d}_Q} \bar{\Lambda}^n \otimes_{\mathcal{F}} Q \rightarrow 0,$$

where $\bar{\Lambda}^i = \bar{\Lambda}^i(\mathcal{E})$. The differential $\bar{d} = \bar{d}_Q$ can also be defined by

$$\begin{aligned} (\bar{d}q)(X) &= \nabla_X(q), \quad q \in Q, \\ \bar{d}(\omega \otimes q) &= \bar{d}\omega \otimes q + (-1)^p \omega \wedge \bar{d}q, \quad \omega \in \bar{\Lambda}^p. \end{aligned}$$

One easily sees that a morphism $f: Q_1 \rightarrow Q_2$ of \mathcal{C} -modules gives rise to a chain mapping of the de Rham complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q_1 & \xrightarrow{\bar{d}} & \bar{\Lambda}^1 \otimes_{\mathcal{F}} Q_1 & \xrightarrow{\bar{d}} & \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n \otimes_{\mathcal{F}} Q_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q_2 & \xrightarrow{\bar{d}} & \bar{\Lambda}^1 \otimes_{\mathcal{F}} Q_2 & \xrightarrow{\bar{d}} & \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n \otimes_{\mathcal{F}} Q_2 \longrightarrow 0. \end{array}$$

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bar{S}^k \otimes P & \longrightarrow & \bar{\mathcal{J}}^k(P) & \longrightarrow & \bar{\mathcal{J}}^{k-1}(P) \longrightarrow 0 \\
& & \downarrow \bar{\delta} & & \downarrow \bar{S} & & \downarrow \bar{S} \\
0 & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{S}^{k-1} \otimes P & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^{k-1}(P) & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^{k-2}(P) \longrightarrow 0 \\
& & \downarrow \bar{\delta} & & \downarrow \bar{S} & & \downarrow \bar{S} \\
0 & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{S}^{k-2} \otimes P & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^{k-2}(P) & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^{k-2}(P) \longrightarrow 0 \\
& & \downarrow \bar{\delta} & & \downarrow \bar{S} & & \downarrow \bar{S} \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

DIAGRAM 2.1.

The cohomology of the horizontal de Rham complex with coefficients in Q is said to be *horizontal cohomology* and is denoted by $\bar{H}^i(Q)$.

Let us discuss some examples of the horizontal de Rham complexes.

Example 2.13. The horizontal de Rham complex with coefficients in $\bar{\mathcal{J}}^\infty(P)$

$$0 \rightarrow \bar{\mathcal{J}}^\infty(P) \xrightarrow{\bar{d}} \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P) \xrightarrow{\bar{d}} \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P) \xrightarrow{\bar{d}} \dots \xrightarrow{\bar{d}} \bar{\Lambda}^n \otimes \bar{\mathcal{J}}^\infty(P) \rightarrow 0$$

turns out to be the project limit of the *horizontal Spencer complexes*

$$0 \rightarrow \bar{\mathcal{J}}^k(P) \xrightarrow{\bar{S}} \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^{k-1}(P) \xrightarrow{\bar{S}} \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^{k-1}(P) \xrightarrow{\bar{S}} \dots \xrightarrow{\bar{S}} \bar{\Lambda}^n \otimes \bar{\mathcal{J}}^{k-n}(P) \rightarrow 0, \quad (2.1)$$

where $\bar{S}(\omega \otimes \bar{j}_l(p)) = \bar{d}\omega \otimes \bar{j}_{l-1}(p)$. As usual Spencer complexes, they are exact in positive degrees and

$$H^0(\bar{\Lambda}^\bullet \otimes \bar{\mathcal{J}}^{k-\bullet}(P)) = P.$$

The proof is standard but we sketch it now because it involves issues which will be needed further. Consider the short exact sequence of the complexes in Diagram 2.1, with $\bar{S}^k = S^k(\bar{\Lambda}^1)$ being the symmetric power of $\bar{\Lambda}^1$. The left column, called the *horizontal Spencer δ -complex*, is a complex of homomorphisms. The operator $\bar{\delta}: \bar{\Lambda}^s \otimes \bar{S}^r \otimes P \rightarrow \bar{\Lambda}^{s+1} \otimes \bar{S}^{r-1} \otimes P$ is defined by $\bar{\delta}(\omega \otimes u \otimes p) = (-1)^s \omega \wedge i(u) \otimes p$, where $i: \bar{S}^r \rightarrow \bar{\Lambda}^1 \otimes \bar{S}^{r-1}$ is the natural inclusion. Dropping the multiplier P and considering the sequence at a point of \mathcal{E}^∞ , we get the de Rham complex with polynomial coefficients. This proves that the Spencer δ -complex is exact. Hence the second and third columns have the same cohomology, so that the cohomology of (2.1) can be computed at $k = 0$ and the desired statement is proved. \square

Remark 2.14. Here we encounter the phenomenon mentioned in the Introduction: the existence of a filtration in the \mathcal{C} -module of coefficients, such that the associated graded complex is a complex of \mathcal{F} -homomorphisms, makes it possible to calculate the horizontal cohomology *locally*.

Now, let us multiply the previous diagram by a \mathcal{C} -module Q (possibly of infinite rank). Arguing as above, we see that the complex

$$0 \rightarrow \bar{\mathcal{J}}^\infty(P) \hat{\otimes} Q \xrightarrow{\bar{d}} \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P) \hat{\otimes} Q \xrightarrow{\bar{d}} \dots \xrightarrow{\bar{d}} \bar{\Lambda}^n \otimes \bar{\mathcal{J}}^\infty(P) \hat{\otimes} Q \rightarrow 0$$

is exact in positive degrees and

$$H^0(\bar{\Lambda}^\bullet \otimes \bar{\mathcal{J}}^\infty(P) \widehat{\otimes} Q) = P \otimes Q.$$

Here

$$\bar{\mathcal{J}}^\infty(P) \widehat{\otimes} Q = \text{proj lim } \bar{\mathcal{J}}^k(P) \otimes Q.$$

Example 2.15. The dualization of the previous example is the following. The coefficient module is $\mathcal{C}\text{Diff}(P, \mathcal{F})$. The corresponding horizontal de Rham complex multiplied by a \mathcal{C} -module Q has the form

$$0 \rightarrow \mathcal{C}\text{Diff}(P, \mathcal{F}) \otimes Q \xrightarrow{\bar{d}} \mathcal{C}\text{Diff}(P, \bar{\Lambda}^1) \otimes Q \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \mathcal{C}\text{Diff}(P, \bar{\Lambda}^n) \otimes Q \rightarrow 0.$$

As in the previous example, it is easily shown that

$$\begin{aligned} H^i(\mathcal{C}\text{Diff}(P, \bar{\Lambda}^\bullet) \otimes Q) &= 0 \quad \text{for } i < n, \\ H^n(\mathcal{C}\text{Diff}(P, \bar{\Lambda}^\bullet) \otimes Q) &= \hat{P} \otimes Q, \end{aligned}$$

where $\hat{P} = \text{Hom}_{\mathcal{F}}(P, \bar{\Lambda}^n)$.

One can use this fact to define the notion of adjoint operator as follows (cf. [23, 24]). Any horizontal differential operator

$$\Delta: P \rightarrow P_1$$

gives rise to the morphism of complexes

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{C}\text{Diff}(P_1, \mathcal{F}) & \longrightarrow & \mathcal{C}\text{Diff}(P, \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{C}\text{Diff}(P_1, \bar{\Lambda}^1) & \longrightarrow & \mathcal{C}\text{Diff}(P, \bar{\Lambda}^1) \\ \downarrow & & \downarrow \\ \mathcal{C}\text{Diff}(P_1, \bar{\Lambda}^2) & \longrightarrow & \mathcal{C}\text{Diff}(P, \bar{\Lambda}^2) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathcal{C}\text{Diff}(P_1, \bar{\Lambda}^n) & \longrightarrow & \mathcal{C}\text{Diff}(P, \bar{\Lambda}^n) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

and hence the map of the cohomology groups $\Delta^*: \hat{P}_1 \rightarrow \hat{P}$, which is the *adjoint operator*. The reader will have no difficulty in showing that

1. $(\Delta_1 \circ \Delta_2)^* = \Delta_2^* \circ \Delta_1^*$;
2. if $\Delta \in \mathcal{C}\text{Diff}_k(P, P_1)$ then $\Delta^* \in \mathcal{C}\text{Diff}_k(\hat{P}_1, \hat{P})$.

In coordinates, we have $(\sum_{\sigma} f_{\sigma} D_{\sigma})^* = (-1)^{|\sigma|} \sum_{\sigma} D_{\sigma} \circ f_{\sigma}$ for a scalar operator and $\|\Delta_{ij}\|^* = \|\Delta_{ji}^*\|$ for a matrix one.

Example 2.16. The choice $Q = Q_0 \otimes \mathcal{C}^p \Lambda^p$, with Q_0 a zero-curvature representation, leads to the p -th *gauge complex* (cf. [19]).

Example 2.17. The zero term $E_0^{p,q}$ of the Vinogradov \mathcal{C} -spectral sequence consists of the complexes

$$0 \rightarrow E_0^{p,0} \rightarrow E_0^{p,1} \rightarrow \dots \rightarrow E_0^{p,n} \rightarrow 0, \quad p = 0, 1, 2, \dots,$$

which are the horizontal de Rham complexes with coefficients in $\mathcal{C}^p \Lambda^p$.

Example 2.18. Take the \mathcal{C} -module

$$Q = \bigoplus_p D^v(\mathcal{C}^p \Lambda^p) = \bigoplus_p \text{Hom}_{\mathcal{F}}(\mathcal{C} \Lambda^1, \mathcal{C}^p \Lambda^p).$$

The horizontal de Rham complex with coefficient in Q can be written as

$$0 \rightarrow D^v \rightarrow D^v(\Lambda^1) \rightarrow D^v(\Lambda^2) \rightarrow \dots$$

It is simply a calculation to verify that this is the complex introduced by Krasil'shchik [9]. The cohomology of this complex is called the \mathcal{C} -cohomology. It contains important invariants of differential equations, in particular, symmetries and deformations of the Cartan structure (= recursion operators) (see [9, 10, 11, 12, 13, 14]).

The calculation of the cohomology of the complexes from the last two examples is our main concern in this paper.

2.3. Compatibility complex. Recall that a complex of \mathcal{C} -differential operators $\dots \rightarrow P_{i-1} \xrightarrow{\Delta_i} P_i \xrightarrow{\Delta_{i+1}} P_{i+1} \rightarrow \dots$ is called *formally exact*, if the complex

$$\dots \rightarrow \bar{\mathcal{J}}^{k_i+k_{i+1}+l}(P_{i-1}) \xrightarrow{\varphi_{\Delta_i}^{k_i+k_{i+1}+l}} \bar{\mathcal{J}}^{k_{i+1}+l}(P_i) \xrightarrow{\varphi_{\Delta_{i+1}}^{k_{i+1}+l}} \bar{\mathcal{J}}^l(P_{i+1}) \rightarrow \dots,$$

with $\text{ord } \Delta_j \leq k_j$, is exact for any l .

Consider a \mathcal{C} -differential operator $\Delta: P_0 \rightarrow P_1$ and the corresponding \mathcal{C} -module $\mathcal{R}_\Delta = \ker \varphi_\Delta^\infty$ (cf. example 2.7). Suppose that there exists a formally exact complex

$$P_0 \xrightarrow{\Delta} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} \dots \quad (2.2)$$

Then the cohomology of this complex coincides with the horizontal cohomology with coefficients in \mathcal{R}_Δ :

Theorem 2.19.

$$\bar{H}^i(\mathcal{R}_\Delta) = H^i(P_\bullet).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P_0) & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P_1) & \longrightarrow & \bar{\Lambda}^2 \otimes \bar{\mathcal{J}}^\infty(P_2) \longrightarrow \dots \\ & & \uparrow \bar{d} & & \uparrow \bar{d} & & \uparrow \bar{d} \\ 0 & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P_0) & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P_1) & \longrightarrow & \bar{\Lambda}^1 \otimes \bar{\mathcal{J}}^\infty(P_2) \longrightarrow \dots \\ & & \uparrow \bar{d} & & \uparrow \bar{d} & & \uparrow \bar{d} \\ 0 & \longrightarrow & \bar{\mathcal{J}}^\infty(P_0) & \longrightarrow & \bar{\mathcal{J}}^\infty(P_1) & \longrightarrow & \bar{\mathcal{J}}^\infty(P_2) \longrightarrow \dots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

The horizontal maps are induced by the operators Δ_i . All the sequences are exact except for the terms in the left column and the bottom row. Now the standard spectral sequence arguments completes the proof. \square

Let us multiply the previous diagram by a \mathcal{C} -module Q . This yields

$$\bar{H}^i(\mathcal{R}_\Delta \hat{\otimes} Q) = H^i(P_\bullet \otimes Q), \quad (2.3)$$

where $\mathcal{R}_\Delta \hat{\otimes} Q = \text{proj lim } \mathcal{R}_\Delta^l \otimes Q$, with $\mathcal{R}_\Delta^l = \ker \varphi_\Delta^{k+l}$, $\text{ord } \Delta \leq k$.

We can dualize our discussion. Namely, consider the diagram

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longleftarrow & \mathcal{C}\text{Diff}(P_0, \bar{\Lambda}^{n-2}) & \longleftarrow & \mathcal{C}\text{Diff}(P_1, \bar{\Lambda}^{n-2}) & \longleftarrow & \mathcal{C}\text{Diff}(P_2, \bar{\Lambda}^{n-2}) \longleftarrow \dots \\ & & \downarrow \bar{d} & & \downarrow \bar{d} & & \downarrow \bar{d} \\ 0 & \longleftarrow & \mathcal{C}\text{Diff}(P_0, \bar{\Lambda}^{n-1}) & \longleftarrow & \mathcal{C}\text{Diff}(P_1, \bar{\Lambda}^{n-1}) & \longleftarrow & \mathcal{C}\text{Diff}(P_2, \bar{\Lambda}^{n-1}) \longleftarrow \dots \\ & & \downarrow \bar{d} & & \downarrow \bar{d} & & \downarrow \bar{d} \\ 0 & \longleftarrow & \mathcal{C}\text{Diff}(P_0, \bar{\Lambda}^n) & \longleftarrow & \mathcal{C}\text{Diff}(P_1, \bar{\Lambda}^n) & \longleftarrow & \mathcal{C}\text{Diff}(P_2, \bar{\Lambda}^n) \longleftarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

As above, we readily obtain

$$\bar{H}^i(\mathcal{R}_\Delta^*) = H_{n-i}(\hat{P}_\bullet)$$

and, more generally,

$$\bar{H}^i(\mathcal{R}_\Delta^* \otimes Q) = H_{n-i}(\hat{P}_\bullet \otimes Q), \quad (2.4)$$

where $\mathcal{R}_\Delta^* = \text{Hom}(\mathcal{R}_\Delta, \mathcal{F})$. The homology in the right-hand side of these formulae is the homology of the complex

$$\hat{P}_0 \xleftarrow{\Delta^*} \hat{P}_1 \xleftarrow{\Delta_1^*} \hat{P}_2 \xleftarrow{\Delta_2^*} \hat{P}_3 \xleftarrow{\Delta_3^*} \dots,$$

dual to the complex (2.2).

Now, suppose we are given a \mathcal{C} -differential operator $\Delta \in \mathcal{C}\text{Diff}_k(P_0, P_1)$. How do we find a formally exact complex of the form (2.2)? To this end consider for each positive integer k_1 the mapping $\varphi_\Delta^{k+k_1} : \bar{\mathcal{J}}^{k+k_1}(P_0) \rightarrow \bar{\mathcal{J}}^{k_1}(P_1)$. Without loss of generality it can be assumed that for $k_1 = 0$ this mapping is surjective. Fix the integer $k_1 > 0$. By definition, put $P_2 = \text{coker } \varphi_\Delta^{k+k_1}$ and let φ_{Δ_1} be the natural projection $\bar{\mathcal{J}}^{k_1}(P_1) \rightarrow P_2$. It is clear that $\Delta_1 \circ \Delta = 0$. Further, starting from an integer k_2 and the operator Δ_1 , we construct as above an operator Δ_2 , such that $\Delta_2 \circ \Delta_1 = 0$. Continuing this process, we obtain a complex of the form (2.2).

Is this complex formally exact? To settle the question, let us discuss the notion of involutiveness of a \mathcal{C} -differential operator. Consider the Spencer δ -complex of the module P_0 (see Diagram 2.1)

$$0 \rightarrow \bar{S}^r \otimes P_0 \xrightarrow{\bar{\delta}} \bar{\Lambda}^1 \otimes \bar{S}^{r-1} \otimes P_0 \xrightarrow{\bar{\delta}} \bar{\Lambda}^2 \otimes \bar{S}^{r-2} \otimes P_0 \xrightarrow{\bar{\delta}} \dots \quad (2.5)$$

Let $g^{k+l} \subset \bar{S}^{k+l} \otimes P_0$ be the symbolic module of the operator Δ , i.e., $g^{k+l} = \ker \sigma(\Delta)$. It is easily shown that the subcomplex of complex (2.5)

$$0 \rightarrow g^{k+l} \xrightarrow{\bar{\delta}} \bar{\Lambda}^1 \otimes g^{k+l-1} \xrightarrow{\bar{\delta}} \bar{\Lambda}^2 \otimes g^{k+l-2} \xrightarrow{\bar{\delta}} \dots \quad (2.6)$$

is well defined. Cohomology of this complex in the term $\bar{\Lambda}^i \otimes g^{k+l-i}$ is denoted by $\bar{H}^{k+l,i}(\Delta)$ and is said to be *horizontal Spencer δ -cohomology of the operator Δ* . Note that $\bar{H}^{k+l,0}(\Delta) = \bar{H}^{k+l,1}(\Delta) = 0$. The operator Δ is called *involutive* (in the sense of Cartan), if $\bar{H}^{k+l,i}(\Delta) = 0$ for all $i \geq 0$.

Theorem 2.20. *If the operator Δ is involutive, then the complex of the form (2.2) constructed as described above is formally exact for all positive integers k_1, k_2, k_3, \dots .*

Proof. We must prove that the sequences

$$\bar{S}^{k_{i-1}+k_i+l} \otimes P_{i-1} \rightarrow \bar{S}^{k_i+l} \otimes P_i \rightarrow \bar{S}^l \otimes P_{i+1}$$

are exact for all $l \geq 1$. The proof is by induction on i and l , with the inductive step involving the standard spectral sequence arguments applied to the commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \bar{S}^l \otimes P_{i+1} & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^1 \otimes \bar{S}^{l-1} \otimes P_{i+1} & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^2 \otimes \bar{S}^{l-2} \otimes P_{i+1} \xrightarrow{\bar{\delta}} \dots \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \bar{S}^{k_i+l} \otimes P_i & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^1 \otimes \bar{S}^{k_i+l-1} \otimes P_i & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^2 \otimes \bar{S}^{k_i+l-2} \otimes P_i \xrightarrow{\bar{\delta}} \dots \\
& & \uparrow & & \uparrow & & \uparrow \\
& & \vdots & & \vdots & & \vdots \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \bar{S}^{K+l} \otimes P_0 & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^1 \otimes \bar{S}^{K+l-1} \otimes P_0 & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^2 \otimes \bar{S}^{K+l-2} \otimes P_0 \xrightarrow{\bar{\delta}} \dots \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & g^{K+l} & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^1 \otimes g^{K+l-1} & \xrightarrow{\bar{\delta}} & \bar{\Lambda}^2 \otimes g^{K+l-2} \xrightarrow{\bar{\delta}} \dots \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

where $K = k + k_1 + k_2 + \dots + k_i$. \square

Definition 2.21. A formally exact complex of the form (2.2) is called the *compatibility complex* for the operator Δ .

For a discussion of the compatibility complex see [4, 23, 25].

The condition of involutiveness is not necessary for the existence of the compatibility complex. Really, the δ -Poincaré lemma (see, for example, [15, 4]) says that for any \mathcal{C} -differential operator Δ there exists an integer $l_0 = l_0(m, n, k)$, where $m = \text{rank } P$, such that $\bar{H}^{k+l, i}(\Delta) = 0$ for $l \geq l_0$ and $i \geq 0$. Taking into account this fact, we see from the proof of the previous theorem that for sufficiently large integer k_1 the compatibility complex exists for any \mathcal{C} -differential operator Δ .

The following proposition is obvious.

Proposition 2.22. *In the compatibility complex each operator Δ_{i+1} is a compatibility operator for the preceding operator Δ_i , i.e., for any operator ∇ and an integer $l \geq 0$ such that $\nabla \circ \Delta_i = 0$ and $\text{ord } \nabla \geq k_{i+1} - l$, there exists an operator \square such that $\nabla^{(l)} = \square \circ \Delta_{i+1}$, where $\nabla^{(l)} = \bar{j}_l \circ \nabla$ is the l -th prolongation of ∇ .*

Example 2.23. The de Rham complex is the compatibility complex for the operator $\bar{d}: \mathcal{F} \rightarrow \bar{\Lambda}^1$. The proof is trivial.

Example 2.24. Fix an \mathcal{F} -linear scalar product of index i on the module $\bar{\Lambda}^1$. For an integer $p \geq 1$ consider the operator $\Delta = \bar{d} * \bar{d}: \bar{\Lambda}^p \rightarrow \bar{\Lambda}^{n-p}$, where $*$: $\bar{\Lambda}^k \rightarrow \bar{\Lambda}^{n-k}$ is the Hodge star operator. Let us show that the complex

$$\bar{\Lambda}^p \xrightarrow{\Delta} \bar{\Lambda}^{n-p} \xrightarrow{\bar{d}} \bar{\Lambda}^{n-p+1} \xrightarrow{\bar{d}} \bar{\Lambda}^{n-p+2} \xrightarrow{\bar{d}} \dots \xrightarrow{\bar{d}} \bar{\Lambda}^n \rightarrow 0$$

is the compatibility complex for the operator Δ . Indeed, we must prove that the image of the map $\sigma(\Delta): \bar{S}^{l+2} \otimes \bar{\Lambda}^p \rightarrow \bar{S}^l \otimes \bar{\Lambda}^{n-p}$ coincides with the image of the map $\sigma(\bar{d}): \bar{S}^{l+1} \otimes \bar{\Lambda}^{n-p-1} \rightarrow \bar{S}^l \otimes \bar{\Lambda}^{n-p}$ for all $l \geq 0$. Since $\Delta * = \bar{d} * \bar{d} * = \bar{d}(*\bar{d} * + (-1)^{pn+n+i}\bar{d})$, it is sufficient to show that the map $\sigma(*\bar{d} * + (-1)^{pn+n+i}\bar{d}): \bar{S}^{l+1} \otimes \bar{\Lambda}^{n-p-1} \rightarrow \bar{S}^l \otimes \bar{\Lambda}^{n-p}$ is an epimorphism. Take an element $\xi \in \bar{\Lambda}^1$. One has $\sigma(\bar{d})(\xi^{l+1} \otimes \omega) = l\xi^l \otimes \xi \wedge \omega$. Denote by $A_{n-p-1}: \bar{\Lambda}^{n-p-1} \rightarrow \bar{\Lambda}^{n-p}$ the map of exterior multiplication by ξ . We have $A_{n-p}^* = (-1)^{pn+n+i} * A_{p-1}^*$, so it will suffice to check that the map $A_{n-p-1} + A_{n-p}^*$ is an epimorphism. But this is obvious: $\bar{\Lambda}^{n-p} = \text{im } A_{n-p-1} \oplus (\text{im } A_{n-p-1})^\perp = \text{im } A_{n-p-1} \oplus (\ker A_{n-p})^\perp = \text{im } A_{n-p-1} \oplus \text{im } A_{n-p}^* = \text{im}(A_{n-p-1} + A_{n-p}^*)$.

3. APPLICATIONS TO COMPUTING COHOMOLOGICAL INVARIANTS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

3.1. Main theorems. Let $\mathcal{E} = \{F = 0\}$, $F \in P_1$, be an equation,

$$P_0 = \varkappa \xrightarrow{\ell_F} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} P_4 \xrightarrow{\Delta_4} \dots$$

the compatibility complex for the operator of universal linearization, and

$$\hat{P}_0 = \hat{\varkappa} \xleftarrow{\ell_F^*} \hat{P}_1 \xleftarrow{\Delta_1^*} \hat{P}_2 \xleftarrow{\Delta_2^*} \hat{P}_3 \xleftarrow{\Delta_3^*} \hat{P}_4 \xleftarrow{\Delta_4^*} \dots$$

the dual complex. Take a \mathcal{C} -module Q .

Theorem 3.1. $\bar{H}^i(\text{D}^\vee(Q)) = H^i(P_\bullet \otimes Q)$, $\bar{H}^{n-i}(\mathcal{C}\Lambda^1 \otimes Q) = H_i(\hat{P}_\bullet \otimes Q)$.

Proof. The statement follows immediately from (2.3), (2.4) and Proposition 1.1. \square

Let $Q = \mathcal{C}^p \Lambda^p$. The previous theorem gives, first, a method for computing of the cohomology groups $\bar{H}^i(\text{D}^\vee(\mathcal{C}^p \Lambda^p))$, which are Krasil'shchik's \mathcal{C} -cohomology groups (see Example 2.18). Second, since the term $E_1^{p,q} = \bar{H}^q(\mathcal{C}^p \Lambda^p)$ of the Vinogradov \mathcal{C} -spectral sequence is a direct summand in the cohomology group $\bar{H}^q(\mathcal{C}\Lambda^1 \otimes \mathcal{C}^{p-1} \Lambda^{p-1})$, we have a description for the first term of the Vinogradov \mathcal{C} -spectral sequence. Thus:

Corollary 3.2. $\bar{H}^i(\text{D}^\vee(\mathcal{C}^p \Lambda^p)) = H^i(P_\bullet \otimes \mathcal{C}^p \Lambda^p)$.

Corollary 3.3. *The term $E_1^{p,q}$ of the Vinogradov \mathcal{C} -spectral sequence is the skew-symmetric part of the group $H_{n-q}(\hat{P}_\bullet \otimes \mathcal{C}^{p-1} \Lambda^{p-1})$.*

It is useful to describe the isomorphisms given by these corollaries in an explicit form. We discuss the Vinogradov \mathcal{C} -spectral sequence, the case of Krasil'shchik's \mathcal{C} -cohomology is similar.

Consider an operator $\nabla \in \mathcal{C}\text{Diff}(\varkappa, \bar{\Lambda}^q \otimes \mathcal{C}^{p-1} \Lambda^{p-1})$ that represents an element of $E_1^{p,q}$. This means that

$$\bar{d} \circ \nabla = \nabla_1 \circ \ell_F$$

for an operator $\nabla_1 \in \mathcal{C}\text{Diff}(P_1, \bar{\Lambda}^{q+1} \otimes \mathcal{C}^{p-1} \Lambda^{p-1})$. Applying the operator \bar{d} to both sides of this formula and using Proposition 2.22, we get

$$\bar{d} \circ \nabla_1 = \nabla_2 \circ \Delta_1$$

¹I thank D. Gessler for drowning my attention to this example (cf. [6]).

for an operator $\nabla_2 \in \mathcal{CDiff}(P_2, \bar{\Lambda}^{q+2} \otimes \mathcal{C}^{p-1}\Lambda^{p-1})$. Continuing this process, we obtain operators $\nabla_i \in \mathcal{CDiff}(P_i, \bar{\Lambda}^{q+i} \otimes \mathcal{C}^{p-1}\Lambda^{p-1})$, $i = 1, 2, \dots, n - q$, such that

$$\bar{d} \circ \nabla_{i-1} = \nabla_i \circ \Delta_{i-1}.$$

For $i = n - q$ this formula means that the operator $\nabla_{n-q} \in \mathcal{CDiff}(P_{n-q}, \bar{\Lambda}^n \otimes \mathcal{C}^{p-1}\Lambda^{p-1})$ represents an element of the module $\hat{P}_{n-q} \otimes \mathcal{C}^{p-1}\Lambda^{p-1}$ that lies in the kernel of the operator Δ_{n-q-1}^* . This is the element that gives rise to the homology class in $H_{n-q}(\hat{P}_\bullet \otimes \mathcal{C}^{p-1}\Lambda^{p-1})$ corresponding to the chosen element of $E_1^{p,q}$.

It follows from our results that if there is an integer k such that $P_k = P_{k+1} = P_{k+2} = \dots = 0$, i.e., the compatibility complex has the form

$$P_0 \xrightarrow{\ell_F} P_1 \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} P_3 \xrightarrow{\Delta_3} \dots \xrightarrow{\Delta_{k-2}} P_{k-1} \rightarrow 0,$$

then

1. $E_1^{p,q} = 0$ for $p > 0$ and $q \leq n - k$,
2. $H^i(\mathcal{D}^\vee(\mathcal{C}^p\Lambda^p)) = 0$ for $i \geq k$.

This result is known as the *k-line theorem*.

What are the values of the integer k for differential equations encountered in mathematical physics? The existence of a compatibility operator Δ_1 is usually due to the existence of dependencies between the equations under consideration: $\Delta_1(F) = 0$. The majority of systems that occur in practice consist of independent equations and for them $k = 2$. In this case the *two-line theorem* holds:

Theorem 3.4 (the two-line theorem). *Let an equation \mathcal{E} be such that the compatibility complex for ℓ_F is of length two. Then:*

1. $E_1^{p,q} = 0$ for $p > 0$ and $q \leq n - 2$,
2. $E_1^{p,n-1} \subset \ker(\ell_F^*)_{\mathcal{C}^{p-1}\Lambda^{p-1}}$ for $p > 0$,
3. $E_1^{p,n} \subset \text{coker}(\ell_F^*)_{\mathcal{C}^{p-1}\Lambda^{p-1}}$ for $p > 0$,
4. $H^i(\mathcal{D}^\vee(\mathcal{C}^p\Lambda^p)) = 0$ for $i \geq 2$,
5. $H^0(\mathcal{D}^\vee(\mathcal{C}^p\Lambda^p)) = \ker(\ell_F)_{\mathcal{C}^p\Lambda^p}$,
6. $H^1(\mathcal{D}^\vee(\mathcal{C}^p\Lambda^p)) = \text{coker}(\ell_F)_{\mathcal{C}^p\Lambda^p}$. □

Further, we meet with the case $k > 2$ in gauge theories, when the dependencies $\Delta_1(F) = 0$ are given by the second Noether theorem. For usual irreducible gauge theories, like electromagnetism, Yang-Mills models, and Einstein's gravity, the Noether identities are independent, so that the operator Δ_2 is trivial and, thus, $k = 3$. Finally, for an L -th stage reducible gauge theory, one has $k = L + 3$.

3.2. Example: Evolution equations. Consider an evolution equation $\mathcal{E} = \{F = u_t - f(x, t, u_i) = 0\}$, with independent variables x, t and dependent variable u ; u_i denotes the set of variables corresponding to derivatives of u with respect to x .

Natural coordinates for \mathcal{E}^∞ are (x, t, u_i) . The total derivatives operators D_x and D_t on \mathcal{E}^∞ have the form

$$D_x = \frac{\partial}{\partial x} + \sum_i u_{i+1} \frac{\partial}{\partial u_i}, \quad D_t = \frac{\partial}{\partial t} + \sum_i D_x^i(f) \frac{\partial}{\partial u_i}.$$

The operator of universal linearization is given by

$$\ell_F = D_t - \ell_f = D_t - \sum_i \frac{\partial f}{\partial u_i} D_x^i.$$

The adjoint of ℓ_F is

$$\ell_F^* = -D_t - \ell_f^* = D_t - \sum_i (-1)^i D_x^i \circ \frac{\partial f}{\partial u_i}.$$

Clearly, for an evolution equation the two-line theorem holds, hence Krasil'shchik's \mathcal{C} -cohomology $\bar{H}^q(D^v(\mathcal{C}^p\Lambda^p))$ is trivial for $q \geq 2$ and the first term $E_1^{p,q}$ of the Vinogradov \mathcal{C} -spectral sequence is trivial for $q \neq 1, 2$ and $p > 0$. Now, assume that the order of the equation \mathcal{E} is greater than or equal to 2, i.e., $\text{ord } \ell_f \geq 2$. Then one has more:

Theorem 3.5. *For any evolution equation of order ≥ 2 , one has*

1. $\bar{H}^0(D^v(\mathcal{C}^p\Lambda^p)) = 0$ for $p \geq 2$,
2. $E_1^{p,1} = 0$ for $p \geq 3$.

Proof. It follows from Theorem 3.4 that $\bar{H}^0(D^v(\mathcal{C}^p\Lambda^p)) = \ker(\ell_F)_{\mathcal{C}^p\Lambda^p}$ and $E_1^{p,1} = \ker(\ell_F^*)_{\mathcal{C}^{p-1}\Lambda^{p-1}}$. Hence to prove the theorem it suffices to check that equations

$$(D_t - \ell_f)(\omega) = 0 \quad \text{and} \quad (D_t + \ell_f^*)(\omega) = 0, \quad (3.1)$$

with $\omega \in \mathcal{C}^p\Lambda^p$, has no nontrivial solutions for $p \geq 2$.

To this end, consider the symbol of (3.1). Denote $\sigma(D_x) = \theta$. The symbol of ℓ_F has the form $\sigma(\ell_f) = f\theta^k$, $k \geq 2$. An element $\omega \in \mathcal{C}^p\Lambda^p$ can be identified with a multilinear \mathcal{C} -differential operator, so the symbol of ω is a homogeneous polynomial in p variables $\sigma(\omega) = \delta(\theta_1, \dots, \theta_p)$. Either of the two equations (3.1) yields

$$[f(\theta_1^k + \dots + \theta_p^k) \pm f(\theta_1 + \dots + \theta_p)^k](\delta) = 0.$$

The conditions $k \geq 2$ and $p \geq 2$ obviously imply that $\delta = 0$. This completes the proof. \square

Remark 3.6. In the work [7] this proof has been generalized for determined systems of evolution equations with arbitrary number of independent variables.

3.3. Example: Abelian p -form theories. Let M be a (pseudo-)Riemannian manifold and $\pi: E \rightarrow M$ the p -th exterior power of the cotangent bundle over M , so that a section of π is a p -form on M . Evidently, on the jet space $J^\infty(\pi)$ there exists a unique horizontal form $A \in \bar{\Lambda}^p(J^\infty(\pi))$ such that $j_\infty^*(\omega)(A) = \omega$ for all $\omega \in \Lambda^p(M)$. Consider the equation $\mathcal{E} = \{F = 0\}$, with $F = d_*\bar{d}A$. Our aim is to calculate the terms of the Vinogradov \mathcal{C} -spectral sequence $E_1^{i,q}$ for $q \leq n - 2$. We shall assume that $1 \leq p < n - 1$ and that the manifold M is topologically trivial.

Obviously, we have $P_0 = \mathcal{X} = \bar{\Lambda}^p$, $P_1 = \bar{\Lambda}^{n-p}$, and $\ell_F = \bar{d}_*\bar{d}: \bar{\Lambda}^p \rightarrow \bar{\Lambda}^{n-p}$. Taking into account Example 2.24, we see that the compatibility complex for ℓ_F has the form

$$\begin{array}{ccccccc} \bar{\Lambda}^p & \xrightarrow{\ell_F} & \bar{\Lambda}^{n-p} & \xrightarrow{\bar{d}} & \bar{\Lambda}^{n-p+1} & \xrightarrow{\bar{d}} & \dots \xrightarrow{\bar{d}} & \bar{\Lambda}^n & \longrightarrow & 0 \\ \parallel & & \parallel & & \parallel & & & \parallel & & \\ P_0 & & P_1 & & P_2 & & & P_{k-1} & & \end{array} \quad (3.2)$$

Thus $k = p + 2$ and the k -line theorem yields $E_1^{i,q} = 0$ for $i > 0$ and $q < n - p - 1$. Since the Vinogradov \mathcal{C} -spectral sequence converges to the de Rham cohomology of \mathcal{E}^∞ , which is trivial, we also get $E_1^{0,q} = 0$ for $0 < q < n - p - 1$, and $\dim E_1^{0,0} = 1$, i.e., $\bar{H}^1 = \bar{H}^2 = \dots = \bar{H}^{n-p-2} = 0$ and $\dim \bar{H}^0 = 1$. Next, consider the terms $E_1^{i,q}$ for $n - p - 1 \leq q < 2(n - p - 1)$ and $i > 0$. In view of Corollary 3.3 one has

$$E_1^{i,q} \subset \bar{H}^{q-(n-p-1)}(\mathcal{C}^{i-1}\Lambda^{i-1}) = E_1^{i-1, q-(n-p-1)}$$

because the complex dual to the compatibility complex (3.2) has the form

$$\begin{array}{ccccccc} \bar{\Lambda}^{n-p} & \xleftarrow{\ell_F^*} & \bar{\Lambda}^p & \xleftarrow{\bar{d}} & \bar{\Lambda}^{p-1} & \xleftarrow{\bar{d}} & \dots \xleftarrow{\bar{d}} & \mathcal{F} & \longleftarrow & 0. \\ \parallel & & \parallel & & \parallel & & & \parallel & & \\ \hat{P}_0 & & \hat{P}_1 & & \hat{P}_2 & & & \hat{P}_{p+1} & & \end{array}$$

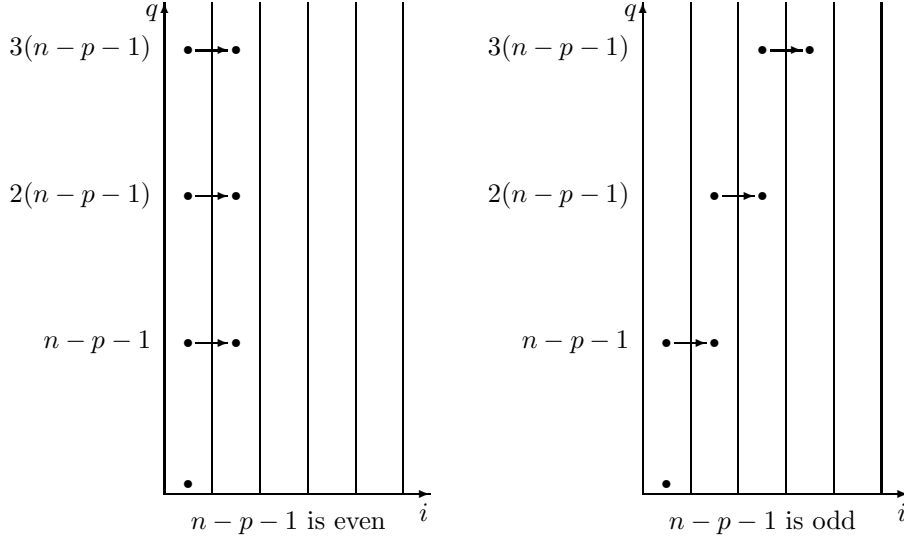


DIAGRAM 3.1.

(Throughout, it is assumed that $q \leq n-2$.) Thus we obtain $E_1^{i,q} = 0$ for $n-p-1 < q < 2(n-p-1)$, $i > 0$ and $\dim E_1^{1,n-p-1} = 1$. Again, taking into account that the spectral sequence converges to the trivial cohomology, we get $E_1^{0,q} = 0$ for $n-p-1 < q < 2(n-p-1)$ and $\dim E_1^{0,n-p-1} = 1$. In addition, the map $d_1^{0,n-p-1}: E_1^{0,n-p-1} \rightarrow E_1^{1,n-p-1}$ is an isomorphism. Explicitly, one readily obtains that the one-dimensional space $E_1^{0,n-p-1}$ is generated by the element $*\bar{d}A \in \bar{\Lambda}^{n-p-1}$ and the map $d_1^{0,n-p-1}$ takes this element to the operator $*\bar{d}: \mathcal{X} = \bar{\Lambda}^p \rightarrow \bar{\Lambda}^{n-p-1}$, which generates the space $E_1^{1,n-p-1}$.

Further, let us consider the terms $E_1^{i,q}$ for $2(n-p-1) \leq q < 3(n-p-1)$. Arguing as before, we see that all these terms vanish unless $q = 2(n-p-1)$ and $i = 0, 1, 2$, with $\dim E_1^{1,2(n-p-1)} = 1$ and $\dim E_1^{i,2(n-p-1)} \leq 1$, $i = 0, 2$. To compute the terms $E_1^{i,2(n-p-1)}$ for $i = 0$ and $i = 2$, we have to consider two cases: $n-p-1$ is even and $n-p-1$ is odd (see Diagram 3.1).

In the first case, the map $d_1^{1,2(n-p-1)}: E_1^{1,2(n-p-1)} \rightarrow E_1^{2,2(n-p-1)}$ is trivial. Indeed, the operator $(*\bar{d}A) \wedge *\bar{d}: \mathcal{X} = \bar{\Lambda}^p \rightarrow \bar{\Lambda}^{2(n-p-1)}$, which generates the space $E_1^{1,2(n-p-1)}$, under the mapping $d_1^{1,2(n-p-1)}$ is the antisymmetrization of the operator $\omega_1 \times \omega_2 \mapsto (*\bar{d}\omega_1) \wedge (*\bar{d}\omega_2)$, $\omega_i \in \mathcal{X} = \bar{\Lambda}^p$. But this operator is symmetric, so that $d_1^{1,2(n-p-1)} = 0$. Consequently, $E_1^{2,2(n-p-1)} = 0$ and $\dim E_1^{0,2(n-p-1)} = 1$. This settles the case when $n-p-1$ is even.

In the case when $n-p-1$ is odd, the operator $\omega_1 \times \omega_2 \mapsto (*\bar{d}\omega_1) \wedge (*\bar{d}\omega_2)$ is skew-symmetric, hence the map $d_1^{1,2(n-p-1)}$ is an isomorphism. Thus, $\dim E_1^{2,2(n-p-1)} = 1$ and $E_1^{0,2(n-p-1)} = 0$.

Continuing this line of reasoning, we obtain the following result.

Theorem 3.7. *For $i = q = 0$ one has $\dim E_1^{0,0} = 1$. If either or both i or q are positive, there are two cases:*

1. *if $n-p-1$ is even then $\dim E_1^{i,q} = \begin{cases} 1 & \text{for } i = l(n-p-1) \text{ and } q = 0, 1, \\ 0 & \text{otherwise;} \end{cases}$*
2. *if $n-p-1$ is odd then $\dim E_1^{i,q} = \begin{cases} 1 & \text{for } i = l(n-p-1) \text{ and } q = l-1, l, \\ 0 & \text{otherwise;} \end{cases}$*

here $1 \leq l < \frac{n-1}{n-p-1}$.

In other words, let $\bar{\mathcal{A}}$ be the exterior algebra generated by two forms: $\omega_1 = *\bar{d}A \in \bar{\Lambda}^{n-p-1}$ and $\omega_2 = \bar{d}_1(\omega_1) = *\bar{d} \in \bar{\Lambda}^{n-p-1} \otimes \mathcal{C}\Lambda^1$; then we see that the space $\bigoplus_{i,q \leq n-2} E_1^{i,q}$ is isomorphic to the subspace of $\bar{\mathcal{A}}$ containing no forms of degree $q > n-2$. In particular, for horizontal cohomology with trivial coefficients this result agrees with that of [8] obtained previously by means of the Koszul-Tate resolution.

Remark 3.8. One can consider the quotient equation under the action of gauge symmetries. The techniques described here allow one to compute the terms $E_1^{i,q}$, $q < n-1$, for the quotient equation. This will be considered elsewhere.

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