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Cohomology background in geometry of PDE

by

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Abstract. Using techniques of Frölicher – Nijenhuis brackets, we associate to any formally integrable equation $\mathcal{E}$ a cohomology theory ($\mathcal{C}$-complex) $H^*_{\mathcal{C}}(\mathcal{E})$ related to deformations of the equation structure on the infinite prolongation $\mathcal{E}^\infty$. A subgroup in $H^1_{\mathcal{C}}(\mathcal{E})$ is identified with recursion operators acting on the Lie algebra $\operatorname{sym}\mathcal{E}$ of symmetries. On the other hand, another subgroup of $H^*_{\mathcal{C}}(\mathcal{E})$ can be understood as the algebra of supersymmetries of the "superization" of the equation $\mathcal{E}$. This pass to superequations makes it possible to obtain a well-defined action of recursion operators in nonlocal setting. Relations to Poisson structures on $\mathcal{E}^\infty$ are briefly discussed.

This text is based on my lecture at the Moscow Conference, as well as on series of papers [Kr1, Kr2, Kr3, KrKe1] published in recent years. Unlike Jim Stasheff ([Sta]), I was able to use only one letter of my native language here.

In [Vin1] A. Vinogradov showed that Lagrangian formalism with constraints, as well as the theory of conservation laws, is, in essence, a cohomological theory expressed in terms of the *$\mathcal{C}$-spectral sequence*. My main purpose here is to show that the theory of symmetries for partial differential equations (including the theory of recursion operators for these symmetries, [Olv]) is of cohomological nature as well. The corresponding cohomologies are determined by the *Frölicher – Nijenhuis bracket* due to flatness of the *Cartan connection* and are called *$\mathcal{C}$-cohomologies*. Moreover, the term E_1 of the $\mathcal{C}$-spectral sequence and $\mathcal{C}$-cohomologies are in a sense mutually dual. Using this duality, one can introduce the concept of *Poisson structure* on a differential equation as a cochain map of a special type, [Kr3, Kr4]. On the other hand, both cohomology theories can be considered as *horizontal cohomologies* with appropriate coefficients [Kr4, Ver] which gives a unified way to estimate them [Ver].

Section 1 is a brief introduction to the theory of Frölicher – Nijenhuis brackets in a general algebraic framework. In Section 2, I construct the complex associated to a flat connection in a commutative algebra and introduce the corresponding cohomologies. Section 3 contains specialization of these construction in the case of the algebra of smooth functions on the infinite prolongation of a formally integrable equation endowed with the Cartan connection. In Section 4, it is shown that to obtain a self-contained, theory one needs to extend the initial setting with graded variables naturally associated to any differential equation and to take into consideration nonlocal picture. Section 5 contains a brief outline of some problems related to the main topic.

Contents

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1. Bracket formalism in commutative algebras

Let $\mathbb{F}$ be a field of characteristic zero and A be a commutative unitary $\mathbb{F}$-algebra. We shall use below notations and definitions from differential calculus over commutative algebras (see for details [Vin2, KLV, Kr5]). Recall the basics:

- $D(P)$ is the module of P-valued derivations $A \to P$, where P is an A-module;
- $D_i(P)$ is the module of P-valued derivations skew-symmetric i-derivations. In particular, $D_1(P) = D(P)$;
- $\Lambda^i(A)$ is the module of differential i-forms of the algebra A;
- $\mathrm{d} : \Lambda^i(A) \to \Lambda^{i+1}(A)$ is the de Rham differential.

If A is the algebra of smooth functions on a smooth manifold M, we use the notation $D(M)$ and $\Lambda^i(M)$ for $D(A)$ and $\Lambda^i(A)$ respectively. The modules $\Lambda^i(A)$ are representative objects for the functors $D_i : P \Rightarrow D_i(P)$, i.e., $D_i(P) = \hom_A(\Lambda^i(A), P)$. The isomorphism $D(P) = \hom_A(\Lambda^1(A), P)$ can be expressed in more exact terms: for any derivation $X : A \to P$ there exists a uniquely defined homomorphism $f_X : \Lambda^1(A) \to P$ satisfying $X = f_X \circ \mathrm{d}$. Denote by $\langle Z, \omega\rangle \in P$ the value of the derivation $Z \in D_i(P)$ at $\omega \in \Lambda^i(A)$.

Both $\Lambda^*(A) = \oplus_{i\geqslant 0}\Lambda^i(A)$ and $D_*(A) = \oplus_{i\geqslant 0}D_i(A)$ are endowed with the structures of commutative superalgebras with respect to the wedge product operation $\wedge : \Lambda^i(A)\otimes\Lambda^j(A) \to \Lambda^{i+j}(A)$, $D_i(A)\otimes D_j(A) \to D_{i+j}(A)$, the de Rham differential $\mathrm{d} : \Lambda^*(A) \to \Lambda^*(A)$ becoming a derivation. Note also that $D_*(P) = \oplus_{i\geqslant 0}D_i(P)$ is a $D_*(A)$-module.

Using the paring $\langle\cdot,\cdot\rangle$ and the wedge product, we define the *interior product* (or *contraction*) $\mathrm{i}_X\,\omega \in \Lambda^{j-i}(A)$ of $X \in D_i(A)$ and $\omega \in \Lambda^j(A)$, $i \leqslant j$, by setting

$$\langle Y, \mathrm{i}_X\,\omega\rangle = \langle Y \wedge X, \omega\rangle \tag{1.1}$$

where Y is an arbitrary element of $D_{j-i}(A)$. We formally set $\mathrm{i}_X\,\omega = 0$ for $i > j$. We now define the *Lie derivative* of ω along X as

$$\mathrm{L}_X\,\omega = (\mathrm{i}_X \circ \mathrm{d} - (-1)^X\,\mathrm{d} \circ \mathrm{i}_X)\omega = [\mathrm{i}_X, \mathrm{d}]\omega \tag{1.2}$$

where (as everywhere below) $(-1)^a$ substitutes $(-1)^{\deg a}$ for any element a of a "superobject" (i.e., algebra or module), while $[\cdot,\cdot]$ denotes the supercommutator.

Consider now the graded module $D(\Lambda^*(A))$ of $\Lambda^*(A)$-valued derivations $A \to \Lambda^*(A)$ corresponding to form-valued vector fields (or, which is the same, vector-valued differential forms) on a smooth manifold. Note that the graded structure in $D(\Lambda^*(A))$ is determined by the splitting $D(\Lambda^*(A)) = \oplus_{i\geqslant 0}D(\Lambda^i(A))$ and thus elements of grading i are derivations X such that $\operatorname{im} X \subset \Lambda^i(A)$. We shall need three algebraic structures associated to $D(\Lambda^*(A))$. First note that $D(\Lambda^*(A))$ is a graded $\Lambda^*(A)$-module: for any $X \in D(\Lambda^*(A)), \omega \in \Lambda^*(A)$ and $a \in A$ we set $(\omega \wedge X)a = \omega \wedge X(a)$. Second, we can define interior product $\mathrm{i}_X\,\omega \in \Lambda^{i+j-1}(A)$ of $X \in D(\Lambda^i(A))$ and $\omega \in \Lambda^j(A)$ using the same definition (1.1). Finally, we can contract elements of $D(\Lambda^*(A))$ with each other by setting

$$(\mathrm{i}_X\,Y)a = \mathrm{i}_X(Ya),\ \ X, Y \in D(\Lambda^*(A)), a \in A.$$

Two properties of contractions are essential in the sequel.

Proposition 1.1. *Let* $X, Y \in D(\Lambda^*(A))$ *and* $\omega, \theta \in \Lambda^*(A)$. *Then*

$$\mathrm{i}_X(\omega \wedge \theta) = \mathrm{i}_X(\omega) \wedge \theta + (-1)^{\omega(X-1)}\omega \wedge \mathrm{i}_X(\theta), \tag{1.3}$$

$$\mathrm{i}_X(\omega \wedge Y) = \mathrm{i}_X(\omega) \wedge Y + (-1)^{\omega(X-1)}\omega \wedge \mathrm{i}_X(Y), \tag{1.4}$$

$$[\mathrm{i}_X, \mathrm{i}_Y] = \mathrm{i}_{[\![X,Y]\!]^{\mathrm{rn}}}, \tag{1.5}$$

where

$$[\![X, Y]\!]^{\mathrm{rn}} = \mathrm{i}_X(Y) - (-1)^{(X-1)(Y-1)}\,\mathrm{i}_Y(X). \tag{1.6}$$

Proof. Equalities (1.3) and (1.4) are proved by direct use of definitions. To prove (1.5), it suffices to use expression (1.6). □

Definition 1.1. The element $[\![X, Y]\!]^{\mathrm{rn}}$ defined by (1.6) is called the *Richardson – Nijenhuis bracket* of elements X and Y.

Directly from Proposition 1.1 we obtain the following

Proposition 1.2. *For any* $X, Y, Z \in D(\Lambda^*(A))$ *and* $\omega \in \Lambda^*(A)$ *one has*

$$[\![X, Y]\!]^{\mathrm{rn}} + (-1)^{(X+1)(Y+1)}[\![Y, X]\!]^{\mathrm{rn}} = 0, \tag{1.7}$$

$$\oint (-1)^{(Y+1)(X+Z)}[\![[\![X, Y]\!]^{\mathrm{rn}}, Z]\!]^{\mathrm{rn}} = 0, \tag{1.8}$$

$$[\![X, \omega \wedge Y]\!]^{\mathrm{rn}} = \mathrm{i}_X(\omega) \wedge Y + (-1)^{(X+1)\omega}\omega \wedge [\![X, Y]\!]^{\mathrm{rn}}. \tag{1.9}$$

Here and below the symbol $\oint$ denotes the sum of cyclic permutations.

Note that Proposition 1.2 means that $D(\Lambda^*(A))^{\downarrow}$ is a Gerstenhaber algebra with respect to the Richardson –Nijenhuis bracket [K-S]. Here the superscript $\downarrow$ denotes the shift of grading by 1.

Similar to (1.2) define the Lie derivative of $\omega \in \Lambda^*(A)$ along $X \in D(\Lambda^*(A))$ by

$$\mathrm{L}_X\,\omega = (\mathrm{i}_X \circ \mathrm{d} + (-1)^X\,\mathrm{d} \circ \mathrm{i}_X)\omega = [\mathrm{i}_X, \mathrm{d}]\omega \tag{1.10}$$

(change of sign is due to the fact that $\deg(\mathrm{i}_X) = \deg(X) - 1$). From the properties of i_X and d we obtain

Proposition 1.3. *For any* $X \in D(\Lambda^*(A))$ *and* $\omega, \theta \in \Lambda^*(A)$ *one has*

$$\mathrm{L}_X(\omega \wedge \theta) = \mathrm{L}_X(\omega) \wedge \theta + (-1)^{X\omega}\omega \wedge \mathrm{L}_X(\theta), \tag{1.11}$$

$$\mathrm{L}_{\omega \wedge X} = \omega \wedge \mathrm{L}_X + (-1)^{\omega + X}\,\mathrm{d}(\omega) \wedge \mathrm{i}_X, \tag{1.12}$$

$$[\mathrm{L}_X, \mathrm{d}] = 0. \tag{1.13}$$

Our main concern now is to analyze the commutator $[\mathrm{L}_X, \mathrm{L}_Y]$ of two Lie derivatives. It may be done efficiently for the following class of algebras.

Definition 1.2. We say that a commutative algebra A is *smooth*, if $\Lambda^1(A)$ is a projective module of finite type.

Proposition 1.4. *Let* A *be a smooth algebra. Then for any elements* $X, Y \in D(\Lambda^*(A))$ *there exists a uniquely determined element* $[\![X, Y]\!]^{\mathrm{fn}}$ *such that*

$$[L_X, L_Y] = L_{[\![X,Y]\!]^{\mathrm{fn}}}. \tag{1.14}$$

Proof. To prove existence, note that for smooth algebras one has

$$D_i(P) = \hom_A(\Lambda^i(A), P) = P \otimes_A \hom_A(\Lambda^i(A), A) = P \otimes_A D_i(A)$$

for any A-module P and integer $i \geqslant 0$. Using this identification represent elements $X, Y \in D(\Lambda^*(A))$ in the form

$$X = \omega \otimes X', Y = \theta \otimes Y', \ \omega, \theta \in \Lambda^*(A), X', Y' \in D(A).$$

Then it is easily checked that the element

$$\begin{aligned} Z &= \omega\wedge\theta\otimes[X',Y'] + \omega\wedge \mathrm{L}_{X'}\theta\otimes Y + (-1)^{\omega}\,\mathrm{d}\,\omega\wedge \mathrm{i}_{X'}\theta\otimes Y' \\ &\quad - (-1)^{\omega\theta}\theta\wedge \mathrm{L}_{Y'}\omega\otimes X' - (-1)^{(\omega+1)\theta}\,\mathrm{d}\,\theta\wedge\mathrm{i}_{Y'}\omega\otimes X' \\ &= \omega\wedge\theta\otimes[X',Y'] + \mathrm{L}_X\theta\otimes Y' - (-1)^{\omega\theta}\,\mathrm{L}_Y\omega\otimes X' \end{aligned} \tag{1.15}$$

satisfies (1.14).

Uniqueness follows from the fact that $\mathrm{L}_X(a) = X(a)$ for $a\in A$. □

Definition 1.3. The element $[\![X,Y]\!]^{\mathrm{fn}}$ defined by formula (1.14) is called the *Frölicher – Nijenhuis bracket* of elements X and Y.

The basic properties of this bracket are summarized in the following

Proposition 1.5. *Let A be a smooth algebra, $X,Y,Z\in D(\Lambda^*(A))$ and $\omega\in\Lambda^*(A)$. Then*

$$[\![X,Y]\!]^{\mathrm{fn}} + (-1)^{XY}[\![Y,X]\!]^{\mathrm{fn}} = 0, \tag{1.16}$$

$$\oint (-1)^{Y(X+Z)}[\![X,[\![Y,Z]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}} = 0, \tag{1.17}$$

$$\mathrm{i}_{[\![X,Y]\!]^{\mathrm{fn}}} = [\mathrm{L}_X,\mathrm{i}_Y] + (-1)^{X(Y+1)}\,\mathrm{L}_{\mathrm{i}_Y X}, \tag{1.18}$$

$$\begin{aligned} \mathrm{i}_Z\,[\![X,Y]\!]^{\mathrm{fn}} &= [\![\mathrm{i}_Z X,Y]\!]^{\mathrm{fn}} + (-1)^{X(Z+1)}[\![X,\mathrm{i}_Z Y]\!]^{\mathrm{fn}} \\ &\quad + (-1)^X\,\mathrm{i}_{[\![Z,X]\!]^{\mathrm{fn}}}Y - (-1)^{(X+1)Y}\,\mathrm{i}_{[\![Z,Y]\!]^{\mathrm{fn}}}X, \end{aligned} \tag{1.19}$$

$$[\![X,\omega\wedge Y]\!]^{\mathrm{fn}} = \mathrm{L}_X\omega\wedge Y - (-1)^{(X+1)(Y+\omega)}\,\mathrm{d}\,\omega\wedge\mathrm{i}_Y X + (-1)^{X\omega}\omega\wedge[\![X,Y]\!]^{\mathrm{fn}}. \tag{1.20}$$

Note that the first two equalities in the previous proposition mean that the module $D(\Lambda^*(A))$ is a Lie superalgebra with respect to the Frölicher – Nijenhuis bracket.

Remark 1.1. The above exposed algebraic scheme obtains a geometrical realization, if one takes $C^\infty(M)$ for the algebra A, M being a smooth finite-dimensional manifold.[1] The algebra $A = C^\infty(M)$ is smooth in this case. However, in geometrical theory of differential equations we have to work with infinite-dimensional manifolds of the form $N = \varprojlim_{\{\pi_{k+1,k}\}} N_k$, where $\pi_{k+1,k} : N_{k+1}\to N_k$ are surjections of finite-dimensional smooth manifolds. The corresponding algebraic object is a filtered algebra $A = \bigcup_{k\in\mathbb{Z}} A_k,\ A_i\subset A_{k+1}$, where all A_k are subalgebras in A. A self-contained differential calculus over A is constructed, if one considers the category of all filtered A-modules with filtered homomorphisms for morphisms between them. Then all functors of differential calculus in this category become filtered, as well as their representative objects.

In particular, the A-modules $\Lambda^i(A)$ are filtered by A_k-modules $\Lambda^i(A_k)$. We say that the algebra A is *finitely smooth*, if $\Lambda^1(A_k)$ is a projective A_k-module of finite type for any $k\in\mathbb{Z}$. For finitely smooth algebras, elements of $D(P)$ may be represented as formal infinite sums $\sum_k p_k\otimes X_k$, such that any finite sum $S_n = \sum_{k\leqslant n} p_k\otimes X_k$ is a derivation $A_n\to P_{n+s}$ for some fixed $s\in\mathbb{Z}$. Any derivation X is completely determined by the system $\{S_n\}$ and Proposition 1.5 obviously remains valid.

[1] To obtain exact correspondence between algebraic and geometrical pictures, one needs to restrict oneself with the subcategory of all A-modules consisting of the so-called *geometrical* modules. See the details in [KLV].

2. Connections and cohomologies

We now introduce the second object of our interest. Let A be an $\mathbb{F}$-algebra and B be an algebra over A. We shall assume that the corresponding homomorphism $\varphi : A \to B$ is an embedding. Let P be a B-module; then it is an A-module as well and we can consider the B-module $D(A, P)$ of P-valued derivations $A \to P$.

Definition 2.1. Let $\nabla^\bullet : D(A, \bullet) \Rightarrow D(\bullet)$ be a natural transformations of functors $D(A, \bullet) : A \Rightarrow D(A, P), D(\bullet) : P \Rightarrow D(\bullet)$ in the category of B-modules, i.e., a system of homomorphisms $\nabla^P : D(A, P) \to D(P)$ such that the diagram

$$\begin{array}{ccc} D(A,P) & \xrightarrow{\nabla^P} & D(P) \\ {\scriptstyle D(A,f)}\downarrow & & \downarrow{\scriptstyle D(f)} \\ D(A,Q) & \xrightarrow{\nabla^Q} & D(Q) \end{array}$$

is commutative for any B-homomorphism $f : P \to Q$. We say that $\nabla^\bullet$ is a *connection* in the triad (A, B, φ), if $\nabla^P(X)\,|_A = X$ for any $X \in D(A, P)$.

Here and below we use the notation $Y\,|_A = Y \circ \varphi$ for any $Y \in D(P)$.

Remark 2.1. When $A = C^\infty(M), B = C^\infty(E), \varphi = \pi^*$, where M, E are smooth manifolds and $\pi : E \to M$ is a smooth fiber bundle, Definition 2.1 reduces to the ordinary definition of a connection in the bundle π. In fact, if we have a connection $\nabla^\bullet$ in the sense of Definition 2.1, then the correspondence

$$D(A) \hookrightarrow D(A, B) \xrightarrow{\nabla^B} D(B)$$

allows one to lift any vector field on M up to a π-protectible field on E. Conversely, if ∇ is such a correspondence, then we can construct a natural transformation $\nabla^\bullet$ of the functors $D(A, \bullet)$ and $D(\bullet)$ due to the fact that for smooth manifolds one has $D(A, P) = P \otimes_A D(A)$ and $D(P) = P \otimes_B D(P)$, P being an arbitrary B-module. We use the notation $\nabla = \nabla^B$ in the sequel.

Definition 2.2. Let $\nabla^\bullet$ be a connection in (A, B, φ) and $X, Y \in D(A, B)$ be two derivations. The *curvature form* of the connection $\nabla^\bullet$ on the pair X, Y is defined by

$$R_\nabla(X, Y) = [\nabla(X), \nabla(Y)] - \nabla(\nabla(X) \circ Y - \nabla(Y) \circ X). \tag{2.1}$$

Note that (2.1) makes sense, since $\nabla(X) \circ Y - \nabla(Y) \circ X$ is a B-valued derivation of A.

Consider now the de Rham differential $\mathrm{d} = \mathrm{d}_B : B \to \Lambda^1(B)$. Then the composition $\mathrm{d}_B \circ \varphi : A \to B$ is a derivation. Consequently, we may consider the derivation $\nabla(\mathrm{d}_B \circ \varphi) \in D(\Lambda^1(B))$.

Definition 2.3. The element $U_\nabla \in D(\Lambda^1(B))$ defined by

$$U_\nabla = \nabla(\mathrm{d}_B \circ \varphi) - \mathrm{d}_B \tag{2.2}$$

is called the *connection form* of ∇.

Directly from definition we obtain the following

Lemma 2.1. *The equality*

$$\mathrm{i}_X(U_\nabla) = X - \nabla(X\,|_A) \tag{2.3}$$

holds for any $X \in D(B)$.

Using this result, we now prove

Proposition 2.2. *If B is a smooth algebra, then*

$$\mathrm{i}_Y\,\mathrm{i}_X\,[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2R_\nabla(X\,|_A\,, Y\,|_A\,) \tag{2.4}$$

for any $X, Y \in D(B)$.

Proof. First note that $\deg U_\nabla = 1$. Then using (1.19) and (1.16) we obtain

$$\begin{aligned}\mathrm{i}_X\,[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} &= [\![\mathrm{i}_X\,U_\nabla, U_\nabla]\!]^{\mathrm{fn}} + [\![U_\nabla, \mathrm{i}_X\,U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X,U_\nabla]\!]^{\mathrm{fn}}}\,U_\nabla - \mathrm{i}_{[\![X,U_\nabla]\!]^{\mathrm{fn}}}\,U_\nabla\\ &= 2\big([\![\mathrm{i}_X\,U_\nabla, U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X,U_\nabla]\!]^{\mathrm{fn}}}\,U_\nabla\big).\end{aligned}$$

Applying i_Y to the last expression and using (1.17) and (1.19), we get now

$$\mathrm{i}_Y\,\mathrm{i}_X\,[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2\big([\![\mathrm{i}_X\,U_\nabla, \mathrm{i}_Y\,U_\nabla]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![X,Y]\!]^{\mathrm{fn}}}\,U_\nabla\big).$$

But $[\![V, W]\!]^{\mathrm{fn}} = [V, W]$ for any $V, W \in D(\Lambda^0(A)) = D(A)$. Hence, by (2.3), we have

$$\mathrm{i}_Y\,\mathrm{i}_X\,[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 2\big([X - \nabla(X\,|_A\,), Y - \nabla(Y\,|_A\,)] - ([X, Y] - \nabla([X, Y]\,|_A\,))\big).$$

It only remains to note now that $\nabla(X\,|_A\,)\,|_A\, = X\,|_A$ and $[X, Y]\,|_A\, = X \circ Y\,|_A\, - Y \circ X\,|_A\,$. □

Definition 2.4. A connection ∇ in (A, B, φ) is called *flat*, if $R_\nabla = 0$.

Thus for flat connections we have

$$[\![U_\nabla, U_\nabla]\!]^{\mathrm{fn}} = 0. \tag{2.5}$$

Let $U \in D(\Lambda^1(B))$ be an element satisfying (2.5). Then from the graded Jacobi identity (1.17) we obtain $2[\![U, [\![U, X]\!]^{\mathrm{fn}}]\!]^{\mathrm{fn}} = [\![[\![U, U]\!]^{\mathrm{fn}}, X]\!]^{\mathrm{fn}} = 0$ for any $X \in D(\Lambda^*(A))$. Consequently, the operator $\partial_U = [\![U, \cdot]\!]^{\mathrm{fn}} : D(\Lambda^i(B)) \to D(\Lambda^{i+1}(B))$ defined by $\partial_U(X) = [\![U, X]\!]^{\mathrm{fn}}$ satisfies $\partial_U \circ \partial_U = 0$.

Consider now the case $U = U_\nabla$, where ∇ is a flat connection.

Definition 2.5. An element $X \in D(\Lambda^*(B))$ is called *vertical*, if $X(a) = 0$ for any $a \in A$. Denote the B-submodule of such elements by $D^v(\Lambda^*(B))$.

Lemma 2.3. *Let ∇ be a connection in (A, B, φ). Then*

(i) *an element $X \in D(\Lambda^*(B))$ is vertical if and only if $\mathrm{i}_X\,U_\nabla = X$;*
(ii) *the connection form U_∇ is vertical, $U_\nabla \in D^v(\Lambda^1(B))$;*
(iii) *the map ∂_{U_∇} preserves verticality, $\partial_{U_\nabla}(D^v(\Lambda^i(B))) \subset D^v(\Lambda^{i+1}(B))$.*

Proof. To prove (i), use Lemma 2.5: from (2.3) it follows that $\mathrm{i}_X\,U_\nabla = X$ if and only if $\nabla(X\,|_A\,) = 0$. But $\nabla(X\,|_A\,)\,|_A\, = X\,|_A\,$. The second statements follows from the same lemma and from first one:

$$\mathrm{i}_{U_\nabla} = U_\nabla - \nabla(U_\nabla\,|_A\,) = U_\nabla - \nabla\big((U_\nabla - \nabla(U_\nabla\,|_A\,)\,|_A\,\big) = U_\nabla.$$

Finally, (iii) is a consequence of (1.19). □

Definition 2.6. Denote the restriction $\partial_{U_\nabla}\,\big|_{D^v(\Lambda^*(A))}$ by ∂_∇ and call the complex

$$0 \to D^v(A) \xrightarrow{\partial_\nabla} D^v(\Lambda^1(A)) \to \ldots \to D^v(\Lambda^i(A)) \xrightarrow{\partial_\nabla} D^v(\Lambda^{i+1}(A)) \to \ldots \tag{2.6}$$

the *U-complex* of the triple (A, B, φ). The corresponding cohomology groups are denoted by $H^i_\nabla(B; A, \varphi)$, $H^*_\nabla(B; A, \varphi) = \bigoplus_{i\geqslant 0} H^i_\nabla(B; A, \varphi)$ and are called the *U-cohomologies* of the triple (A, B, φ).

Introduce the notation

$$\mathrm{d}^v_\nabla = \mathrm{L}_{U_\nabla} : \Lambda^i(B) \to \Lambda^{i+1}(B). \tag{2.7}$$

Proposition 2.4. *Let ∇ be a flat connection in the triple (A, B, φ) and B be a smooth (or finitely smooth) algebra. Then for any $X, Y \in D^v(\Lambda^*(A))$ and $\omega \in \Lambda^*(A)$ one has*

$$\partial_\nabla [\![X, Y]\!]^{\mathrm{fn}} = [\![\partial_\nabla X, Y]\!]^{\mathrm{fn}} + (-1)^X [\![X, \partial_\nabla Y]\!]^{\mathrm{fn}}, \tag{2.8}$$

$$[\mathrm{i}_X, \partial_\nabla] = (-1)^X \,\mathrm{i}_{\partial_\nabla X}, \tag{2.9}$$

$$\partial_\nabla(\omega \wedge X) = (\mathrm{d}^v_\nabla - \mathrm{d})(\omega) \wedge X + (-1)^\omega \omega \wedge \partial_\nabla X, \tag{2.10}$$

$$[\mathrm{d}^v_\nabla, \mathrm{i}_X] = \mathrm{i}_{\partial_\nabla X} + (-1)^X \,\mathrm{L}_X\,. \tag{2.11}$$

Proof. Equality (2.8) is a direct consequence of (1.17). Equality (2.9) follows from (1.19). Equality (2.10) follows from (1.20) and (2.3). Finally, (2.11) is obtained from (1.18). □

Corollary 2.5. *The cohomology module $H^*_\nabla(B; A, \varphi)$ inherits the graded Lie algebra structure with respect to $[\![\cdot, \cdot]\!]^{\mathrm{fn}}$, as well as the contraction operation.*

Proof. Note that $D^v(\Lambda^*(A))$ is closed with respect to the Frölicher–Nijenhuis bracket: to prove this fact, it suffices to apply (1.19). Then the first statement follows from (2.8). The second one is a consequence of (2.9). □

Remark 2.2. We preserve the same notations for the inherited structures. Note, in particular, that $H^0_\nabla(B; A, \varphi)$ is a Lie algebra with respect to the Frölicher–Nijenhuis bracket (which reduces to the ordinary Lie bracket in this case). Moreover, $H^1_\nabla(B; A, \varphi)$ is an associative algebra with respect to the inherited contraction, while the action

$$\mathcal{R}_\Omega : X \mapsto \mathrm{i}_X \Omega, \;\; X \in H^0_\nabla(B; A, \varphi), \Omega \in H^1_\nabla(B; A, \varphi)$$

is a representation of this algebra as endomorphisms of $H^0_\nabla(B; A, \varphi)$.

Consider now the map $\mathrm{d}^v_\nabla : \Lambda^*(B) \to \Lambda^*(B)$ defined by (2.7) and define $\mathrm{d}^h_\nabla = \mathrm{d}_B - \mathrm{d}^v_\nabla$.

Proposition 2.6. *Let B be a (finitely) smooth algebra and ∇ be a smooth connection in the triple $(B; A, \varphi)$. Then*

(i) *The pair $(\mathrm{d}^h_\nabla, \mathrm{d}^v_\nabla)$ forms a bicomplex, i.e.*

$$\mathrm{d}^v_\nabla \circ \mathrm{d}^v_\nabla = 0, \;\; \mathrm{d}^h_\nabla \circ \mathrm{d}^h_\nabla = 0, \;\; \mathrm{d}^h_\nabla \circ \mathrm{d}^v_\nabla + \mathrm{d}^v_\nabla \circ \mathrm{d}^h_\nabla = 0. \tag{2.12}$$

(ii) *The differential d^h_∇ possesses the following properties*

$$[\mathrm{d}^h_\nabla, \mathrm{i}_X] = -\,\mathrm{i}_{\partial_\nabla X}, \tag{2.13}$$

$$\partial_\nabla(\omega \wedge X) = -\,\mathrm{d}^h_\nabla(\omega) \wedge X + (-1)^\omega \omega \wedge \partial_\nabla X, \tag{2.14}$$

where $\omega \in \Lambda^*(B), X \in D^v(\Lambda^*(B))$.

Proof. (i) Since $\deg \mathrm{d}^v_\nabla = 1$, we have

$$2\,\mathrm{d}^v_\nabla \circ \mathrm{d}^v_\nabla = [\mathrm{d}^v_\nabla, \mathrm{d}^v_\nabla] = [\mathrm{L}_{U_\nabla}, \mathrm{L}_{U_\nabla}] = \mathrm{L}_{[\![U_\nabla, U_n a]\!]^{\mathrm{fn}}} = 0.$$

Since $\mathrm{d}^v_\nabla = \mathrm{L}_{U_\nabla}$, the identity $[\mathrm{d}_B, \mathrm{d}^v_\nabla] = 0$ holds (see (1.13)), and it finishes the proof of the first part.

(ii) To prove (2.13), note that $[\mathrm{d}^h_\nabla, \mathrm{i}_X] = [\mathrm{d}_B - \mathrm{d}^h_\nabla, \mathrm{i}_X] = (-1)^X \,\mathrm{L}_X - [\mathrm{d}^v_\nabla, \mathrm{i}_X]$, and (2.13) holds due to (2.11). Finally, (2.14) is just the other form of (2.10). □

Definition 2.7. Let ∇ be a connection in $(A, B\varphi)$.

(i) The bicomplex $(B, \mathrm{d}^h_\nabla, \mathrm{d}^v_\nabla)$ is called the *variational bicomplex* associated to the connection ∇.

(ii) Corresponding spectral sequence is called ∇-*spectral sequence* of the triple $(A, B\varphi)$.

Obviously, the ∇-spectral sequence converges to the de Rham cohomology of B.

To finish this section, note the following. Since the module $\Lambda^1(B)$ is generated by the image of the operator $\mathrm{d}_B : B \to \Lambda^1(B)$ while the graded algebra $\Lambda^*(B)$ is generated by $\Lambda^1(B)$, we have the direct sum decomposition

$$\Lambda^*(B) = \bigoplus_{i\geqslant 0} \bigoplus_{p+q=i} \Lambda_v^p(B) \otimes \Lambda_h^q(B),$$

where

$$\Lambda_v^p(B) = \underbrace{\Lambda_v^1(B) \wedge \ldots \wedge \Lambda_v^1(B)}_{p \text{ times}}, \ \Lambda_h^q(B) = \underbrace{\Lambda_h^1(B) \wedge \ldots \wedge \Lambda_h^1(B)}_{q \text{ times}},$$

while $\Lambda_v^1(B) \subset \Lambda^1(B), \Lambda_h^1(B) \subset \Lambda^1(B)$ are spanned in $\Lambda^1(B)$ by the images of d_∇^v and d_∇^h respectively. Obviously,

$$\mathrm{d}_\nabla^h \left(\Lambda_v^p(B) \otimes \Lambda_h^q(B)\right) \subset \Lambda_v^p(B) \otimes \Lambda_h^{q+1}(B),$$
$$\mathrm{d}_\nabla^v \left(\Lambda_v^p(B) \otimes \Lambda_h^q(B)\right) \subset \Lambda_v^{p+1}(B) \otimes \Lambda_h^q(B).$$

Denote by $D^{p,q}(B)$ the module $D^v(\Lambda_v^p(B) \otimes \Lambda_h^q(B))$. Then, obviously, $D^v(B) = \bigoplus_{i\geqslant 0} \bigoplus_{p+q=i} D^{p,q}(B)$, while from (2.13) and (2.14) we obtain

$$\partial_\nabla \big(D^{p,q}(B)\big) \subset D^{p,q+1}(B).$$

Consequently, the module $H_\nabla^*(B; A, \varphi)$ is split as

$$H_\nabla^*(B; A, \varphi) = \bigoplus_{i\geqslant 0} \bigoplus_{p+q=i} H_\nabla^{p,q}(B; A, \varphi) \tag{2.15}$$

with obvious meaning of the notation $H_\nabla^{p,q}(B; A, \varphi)$.

3. Nonlinear differential equations: $\mathcal{C}$-cohomologies, deformations and recursion operators

Now we apply the above exposed algebraic results to the case of infinitely prolonged differential equation. In what follows, we use the introductory material of A. Vinogradov's paper published in this volume.

Let $\pi : E \to M$ be a locally trivial vector[2] bundle, $\dim M = n, \dim \pi = m$, and $\mathcal{E} \subset J^k(\pi)$ be a k-th order differential equation in the bundle π. Consider the prolongations $\mathcal{E}^l \subset J^{k+l}(\pi)$ and the maps $\pi_{k+l+1,k+l} : \mathcal{E}^{l+1} \to \mathcal{E}^l$.

Definition 3.1. Equation $\mathcal{E} \subset J^k(\pi)$ is called *formally integrable* if all prolongations $\mathcal{E}^l$ are smooth manifolds and the maps $\pi_{k+l+1,k+l} : \mathcal{E}^{l+1} \to \mathcal{E}^l$ are surjections and affine fiber bundles.

In what follows, we assume that all equations under consideration are formally integrable.

Let $\mathcal{E}^\infty \subset J^\infty(\pi)$ be the infinite prolongation of $\mathcal{E}$ with the natural projections $\pi_{\infty,k+l} : \mathcal{E}^\infty \to \mathcal{E}^l, \pi_\infty : \mathcal{E}^\infty \to M$. We use the following notations below:

- $\mathcal{F}(\mathcal{E})$ denotes the filtered algebra of smooth (local) functions on $\mathcal{E}^\infty$, i.e., $\mathcal{F}(\mathcal{E}) = \bigcup_{l\geqslant 0} \mathcal{F}_l(\mathcal{E})$, where $\mathcal{F}_l(\mathcal{E}) = C^\infty(\mathcal{E}^l)$,
- $\mathcal{F}(\mathcal{E}, \xi) = \bigcup_{l\geqslant 0} \mathcal{F}_l(\mathcal{E}, \xi)$ is the filtered $\mathcal{F}(\mathcal{E})$-module of sections of the pull-back $\pi_\infty^*\xi$, where $\xi : F \to M$ is a vector bundle,
- for any filtered $\mathcal{F}(\mathcal{E})$-module P, $D_i(P)$ denotes the module of P-valued *filtered* i-derivations of $\mathcal{F}(\mathcal{E})$,

[2] In fact, all necessary construction can be defined in the case of an arbitrary locally trivial bundle, but it adds to technical difficulties only and does not affect the general conceptual scheme.

- in particular, $D(\mathcal{E}) = D_1(\mathcal{E})$ is identified with the module of vector fields on $\mathcal{E}^\infty$. We also denote by $D^v(P)$ the submodule of π_∞-vertical derivations,
- $\Lambda^*(\mathcal{E}) = \bigoplus_{i\geqslant 0} \Lambda^i(\mathcal{E})$ is the graded filtered algebra of local differential forms on $\mathcal{E}^\infty$, i.e., $\Lambda^i(\mathcal{E}) = \bigcup_{l\geqslant 0} \Lambda^i(\mathcal{E}^l)$.

For the "empty" equation $\mathcal{E}^\infty = J^\infty(\pi)$ we write $\mathcal{F}(\pi), D(\pi), \Lambda^i(\pi)$, etc.

Let $\xi : F \to M, \xi' : F' \to M$ be two vector bundles and $\Delta : \Gamma(\xi) \to \Gamma(\xi')$ be a $C^\infty(M)$-linear differential operator taking sections of ξ to sections of ξ'. Let f be a (local) section of π and $j_\infty(f) \in \mathcal{J}^\infty(\pi)$ be its infinite jet. Set

$$j_\infty(f)^*\big(\mathcal{C}\Delta(\varphi)\big) = \Delta\big(j_\infty(f)^*\varphi\big), \tag{3.1}$$

where $\varphi \in \mathcal{F}(\pi, \xi)$.

Proposition 3.1 (see [KLV])**.** *Equality* (3.1) *uniquely defines an $\mathcal{F}(\pi)$-linear differential operator $\mathcal{C}\Delta : \mathcal{F}(\pi, \xi) \to \mathcal{F}(\pi, \xi')$ possessing the following properties:*

(i) *The correspondence $\Delta \mapsto \mathcal{C}\Delta$ is $C^\infty(M)$-linear.*

(ii) *If $\xi'' : F'' \to M$ is another vector bundle and $\Delta' : \Gamma(\xi') \to \Gamma(\xi'')$ is a linear differential operator, then*

$$\mathcal{C}(\Delta' \circ \Delta) = \mathcal{C}(\Delta') \circ \mathcal{C}(\Delta). \tag{3.2}$$

(iii) *If $\epsilon : \mathcal{E}^\infty \hookrightarrow J^\infty(\pi)$ is an infinite prolongation of a formally integrable equation, then any operator of the form $\mathcal{C}\Delta$ admits restriction onto $\mathcal{E}^\infty$, i.e., there exists an operator $\widetilde{\mathcal{C}\Delta}$ such that the diagram*

$$\begin{array}{ccc}
\mathcal{F}(\pi,\xi) & \xrightarrow{\mathcal{C}\Delta} & \mathcal{F}(\pi,\xi') \\
{\scriptstyle \epsilon^*}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon^*} \\
\mathcal{F}(\mathcal{E},\xi) & \xrightarrow{\widetilde{\mathcal{C}\Delta}} & \mathcal{F}(\mathcal{E},\xi')
\end{array}$$

is commutative.

Note also that (3.1) means that operators $\mathcal{C}\Delta$ admit restrictions onto graphs of infinite jets.

Definition 3.2. Operators of the form $\widetilde{\mathcal{C}\Delta} : \mathcal{F}(\mathcal{E}, \xi) \to \mathcal{F}(\mathcal{E}, \xi')$ are called $\mathcal{C}$-*differential* (or *total*) *operators.* The $\mathcal{F}(\mathcal{E})$-module of such operators is denoted by $\mathcal{C}\,\mathrm{Diff}(\xi, \xi')$.

In particular, if $X \in D(M)$ is a vector field on the manifold M, then $\mathcal{C}X$ is a vector field on $J^\infty(\pi)$ admitting restriction onto $\mathcal{E}^\infty$. Such fields are called *total derivatives.* From the definition and Proposition 3.1 it follows that the correspondence $X \mapsto \mathcal{C}X$ is a connection in the bundle $\pi_\infty : \mathcal{E}^\infty \to M$.

Definition 3.3. The correspondence $X \mapsto \mathcal{C}X$ is called the *Cartan connection* on $\mathcal{E}$.

Proposition 3.2. *The Cartan connection is flat for any formally integrable equation.*

Proof. It is a direct consequence of (3.2). □

Thus, to any formally integrable equation $\mathcal{E} \subset J^k(\pi)$ we can associate the complex $(D(\Lambda^i(\mathcal{E})), \partial_{\mathcal{C}})$ and the cohomology theory determined by the Cartan connection. We denote the corresponding cohomology modules by $H^*_{\mathcal{C}}(\mathcal{E}) = \bigoplus_{i\geqslant 0} H^i_{\mathcal{C}}(\mathcal{E})$. In the case of "empty" equation, we use the notation $H^*_{\mathcal{C}}(\pi) = \bigoplus_{i\geqslant 0} H^i_{\mathcal{C}}(\pi)$.

Definition 3.4. Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\mathcal{C}$ be the Cartan connection in the bundle $\pi_\infty : \mathcal{E}^\infty \to M$. Then

(i) The connection form $U_{\mathcal{C}} = U_{\mathcal{C}}(\mathcal{E}) \in D^v(\Lambda^1(\mathcal{E}))$ is called the *structural element* of the equation $\mathcal{E}$.
(ii) The modules $H^i_{\mathcal{C}}(\mathcal{E})$ are called *$\mathcal{C}$-cohomologies* of $\mathcal{E}$.

The following result contains an interpretation of the first two modules of $\mathcal{C}$-cohomologies.

Theorem 3.3. *For any formally integrable equation $\mathcal{E} \subset J^k(\pi)$, one has*
(i) *The module $H^0_{\mathcal{C}}(\mathcal{E})$ as a Lie algebra is isomorphic to the algebra* $\operatorname{sym}\mathcal{E}$ *of higher symmetries of the equation $\mathcal{E}$.*
(ii) *The module $H^1_{\mathcal{C}}(\mathcal{E})$ coincides with the equivalence classes of nontrivial vertical deformations of the structural element.*

Proof. To prove (i), take a vertical vector field $Y \in D^v(\mathcal{E})$ and an arbitrary field $Z \in D(\mathcal{E})$. Then, due to (1.19), one has

$$\begin{aligned}\mathrm{i}_Z\,\partial_{\mathcal{C}}Y &= \mathrm{i}_Z\,[\![U_{\mathcal{C}}, Y]\!]^{\mathrm{fn}} = [\mathrm{i}_Z\,U_{\mathcal{C}}, Y] - \mathrm{i}_{[Z,Y]}\,U_{\mathcal{C}}\\ &= [Z^v, Y] - [Z^v + (Z - Z^v), Y]^v = [Z - Z^v, Y]^v,\end{aligned}$$

where $Z^v = \mathrm{i}_Z\,U_{\mathcal{C}}$. Hence, $\partial_{\mathcal{C}}Y = 0$ if and only if $[Z - Z^v, Y]^v = 0$ for any $Z \in D(\mathcal{E})$. But the last equality holds if and only if $[\mathcal{C}X, Y] = 0$ for any $X \in D(M)$ which means that

$$\ker\big(\partial_{\mathcal{C}} : D^v(\mathcal{E}) \to D^v(\Lambda^1(\mathcal{E}))\big) = \operatorname{sym}\mathcal{E}.$$

Consider the second statement. Let $U(\varepsilon)$ be a deformation of the structural element satisfying $U(\varepsilon) \in D^v(\Lambda^1(\mathcal{E})), [\![U(\varepsilon), U(\varepsilon)]\!]^{\mathrm{fn}} = 0$ and $U(0) = U_{\mathcal{C}}$. Then $U(\varepsilon) = U_{\mathcal{C}} + U_1\varepsilon + O(\varepsilon^2)$. Consequently,

$$[\![U(\varepsilon), U(\varepsilon)]\!]^{\mathrm{fn}} = [\![U_{\mathcal{C}}, U_{\mathcal{C}}]\!]^{\mathrm{fn}} + 2[\![U_{\mathcal{C}}, U_1]\!]^{\mathrm{fn}}\varepsilon + O(\varepsilon^2) = 0,$$

from where it follows that $[\![U_{\mathcal{C}}, U_1]\!]^{\mathrm{fn}} = \partial_{\mathcal{C}}U_1 = 0$. Hence the linear part of the deformation $U(\varepsilon)$ determines an element of $H^1_{\mathcal{C}}(\mathcal{E})$ and vice versa. On the other hand, let $A : \mathcal{E}^\infty \to \mathcal{E}^\infty$ be a diffeomorphism[3] of $\mathcal{E}^\infty$. Define the action A^* of A on the elements $\Omega \in D(\Lambda^*(\mathcal{E}))$ in such a way that the diagram

$$\begin{array}{ccc} \Lambda^*(\mathcal{E}) & \xrightarrow{\mathrm{L}_\Omega} & \Lambda^*(\mathcal{E}) \\ {\scriptstyle A^*}\big\downarrow & & \big\downarrow{\scriptstyle A^*} \\ \Lambda^*(\mathcal{E}) & \xrightarrow{\mathrm{L}_\Omega} & \Lambda^*(\mathcal{E}) \end{array}$$

is commutative. Then, if A_t is a one-parameter group of diffeomorphisms, then, obviously,

$$\left.\frac{d}{dt}\right|_{t=0} A_{t,*}(\mathrm{L}_\Omega) = \left.\frac{d}{dt}\right|_{t=0} A_t^* \circ \mathrm{L}_\Omega \circ (A_t^*)^{-1} = [\mathrm{L}_X, \mathrm{L}_\Omega] = \mathrm{L}_{[\![X,\Omega]\!]^{\mathrm{fn}}}\,.$$

Hence, infinitesimal action is given by the Frölicher – Nijenhuis bracket. Taking $\Omega = U_{\mathcal{C}}$ and $X \in D^v(\mathcal{E})$, we see that $\operatorname{im}\partial_{\mathcal{C}}$ consists of infinitesimal deformation arising due to infinitesimal action of diffeomorphisms on the structural element. Such deformations are natural to be called trivial. ☐

Remark 3.1. From the general theory [Ger], we obtain that the module $H^2_{\mathcal{C}}(\mathcal{E})$ consists of obstructions to prolongation of infinitesimal deformations up to formal ones. In the case under consideration, elements $H^2_{\mathcal{C}}(\mathcal{E})$ have another nice interpretation discussing later (see Remark 3.14).

[3]Since $\mathcal{E}^\infty$ is, in general, infinite-dimensional, vector fields on $\mathcal{E}^\infty$ do not usually possess one-parameter groups of diffeomorphisms. Thus the arguments below are of a heuristic nature

We shall now compute the modules $H^p_{\mathcal{C}}(\pi)$, $p \geqslant 0$. To this, recall the splitting $\Lambda^i(\mathcal{E}) = \bigoplus_{p+q=i} \Lambda^p_v(\mathcal{E}) \otimes \Lambda^q_h(\mathcal{E})$ (see Section 2).

Theorem 3.4. *One has* $H^p_{\mathcal{C}}(\pi) = \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \Lambda^p_v(\mathcal{E})$ *for any* $p \geqslant 0$.

Proof. Define a filtration in $D^v(\Lambda^*(\pi))$ by setting

$$F^l D^v(\Lambda^p(\pi)) = \{X \in D^v(\Lambda^p(\pi)) \mid X\big|_{\mathcal{F}_{l-p-1}} = 0\}.$$

Evidently,

$$F^l D^v(\Lambda^p(\pi)) \subset F^{l+1} D^v(\Lambda^p(\pi)),\ \partial_\pi\big(F^l D^v(\Lambda^p(\pi))\big) \subset F^l D^v(\Lambda^{p+1}(\pi)).$$

Thus we obtain the spectral sequence associated to this filtration. To compute the term E_0, choose local coordinates $x_1, \dots, x_n, u^1, \dots, u^m$ in the bundle π and consider the corresponding special coordinates u^j_σ, $j = 1, \dots, m, \sigma = (\sigma_1, \dots, \sigma_n)$ in $J^\infty(\pi)$. In these coordinates, the structural element is represented as

$$U_\pi = \sum_{|\sigma|\geqslant 0} \sum_{j=1}^m \left(\mathrm{d}\, u^j_\sigma - \sum_{i=1}^n u_{\sigma+1_i}\, \mathrm{d}\, x_i \right) \otimes \frac{\partial}{\partial u^j_\sigma}, \tag{3.3}$$

where $\sigma + 1_i = (\sigma_1, \dots, \sigma_i + 1, \dots, \sigma_n)$, while for $X = \sum_{\sigma,j} \theta^j_\sigma \otimes \partial/\partial u^j_\sigma$, $\theta \in \Lambda^*(\pi)$, one has

$$\partial_\pi(X) = \sum_{|\sigma|\geqslant 0} \sum_{j=1}^m \sum_{i=1}^n \mathrm{d}\, x_i \wedge \left(\theta^j_{\sigma+1_i} - D_i(\theta^j_\sigma)\right) \otimes \frac{\partial}{\partial u^j_\sigma}, \tag{3.4}$$

where

$$D_i = \mathcal{C}\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma+1_i} \frac{\partial}{\partial u^j_\sigma}$$

is the total derivative along x_i.

Obviously, the term $E_0^{p,-q} = F^p D^v(\Lambda^{p-q}(\pi))/F^{p-1} D^v(\Lambda^{p-q}(\pi))$, $p \geqslant 0, 0 \leqslant q \leqslant p$, identifies with the tensor product $\Lambda^{p-q}(\pi) \otimes_{\mathcal{F}(\pi)} \Gamma(\pi^*_{\infty,q-1}(\pi_{q,q-1}))$, where $\pi_{\infty,q-1} : J^\infty(\pi) \to J^{q-1}(\pi), \pi_{q,q-1} : J^q(\pi) \to J^{q-1}(\pi)$ are natural projections. These modules can be locally represented as $\mathcal{F}(\pi,\pi) \otimes \Lambda^{p-q}(\pi)$-valued homogeneous polynomials of order q, while the differential $\partial_0^{p,-q} : E_0^{p,-q} \to E_0^{p,-q+1}$ acts as the Spencer δ-differential (or , which is the same, as the Koszul differential). Hence, all homology groups are trivial except for the term $E_0^{p,0}$ and one has $\operatorname{coker} \partial_0^{p,0} = \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \Lambda^p_v(\pi)$. Consequently, only the 0-th line survives in the term E_1 and this line is of the form

$$\mathcal{F}(\pi,\pi) \xrightarrow{\partial_1^{0,0}} \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \Lambda^1_v(\pi) \to \dots$$
$$\dots \to \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \Lambda^p_v(\pi) \xrightarrow{\partial_1^{p,0}} \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \Lambda^{p+1}_v(\pi) \to \dots$$

But the image of $\partial_{\mathcal{C}}$ contains at least one horizontal component (see the end of Section 2). Therefore, all differentials $\partial_1^{p,0}$ vanish. □

Let us now establish the correspondence between the last result (describing $\mathcal{C}$-cohomology in terms of $\Lambda^*_v(\pi)$) and representation $H^*_{\mathcal{C}}(\pi)$ as classes of derivations $\mathcal{F}(\pi) \to \Lambda^*(\pi)$. To do this, for any $\omega = (\omega^1, \dots, \omega^m) \in \mathcal{F}(\pi,\pi) \otimes_{\mathcal{F}(\pi)} \Lambda^*_v(\pi)$ set

$$\reflectbox{E}_\omega = \sum_{\sigma,j} D_\sigma(\omega^j) \otimes \frac{\partial}{\partial u^j_\sigma}, \tag{3.5}$$

where $D_\sigma = D_1^{\sigma_1} \circ \dots \circ D_n^{\sigma_n}$ for $\sigma = (\sigma_1, \dots \sigma_n)$.

Definition 3.5. The element $\reflectbox{$\mathrm{E}$}_\omega \in D^v(\Lambda^*(\pi))$ defined by (3.5) is called *evolution superderivation* with the *generating section* $\omega \in \Lambda^*_v(\pi)$.

Proposition 3.5. *Definition of $\ni_\omega$ is independent of coordinate choice.*

Proof. It is easily checked that

$$\ni_\omega\bigl(\mathcal{F}(\pi)\bigr) \subset \Lambda_v^*(\pi),\ \ni_\omega \in \ker \partial_{\mathcal{C}}.$$

But derivations possessing these properties are uniquely determined by their restriction onto $\mathcal{F}_0(\pi)$ (see [Kr1, KLV] for details). □

From this result and from Corollary 2.5 it follows that given two evolution superderivations $\ni_\omega, \ni_\theta$, the elements

$$\text{(i)}\ \ [\![\ni_\omega, \ni_\theta]\!]^{\mathrm{fn}}, \qquad\qquad \text{(ii)}\ \ \mathrm{i}_{\ni_\omega}(\ni_\theta)$$

are evolution superderivations as well.

In the first case, the corresponding generating section is called the *Jacobi superbracket* of elements $\omega = (\omega^1, \dots, \omega^m)$ and $\theta = (\theta^1, \dots, \theta^m)$ and is denoted by $\{\omega, \theta\}$. The components of this bracket are expressed by

$$\{\omega, \theta\}^j = \mathrm{L}_{\ni_\omega}(\theta^j) - (-1)^{\omega\theta}\,\mathrm{L}_{\ni_\theta}(\omega^j),\ j = 1, \dots, m. \tag{3.6}$$

Obviously, the module $\mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \Lambda_v^*(\pi)$ is a graded Lie algebra with respect to the Jacobi superbracket.

In the case (ii), the generating section is $\mathrm{i}_{\ni_\omega}(\theta)$. Note now that any element $\rho \in \Lambda_v^1(\pi)$ is of the form $\rho = \sum_{\sigma,\alpha} a_{\sigma,\alpha}\omega_\sigma^\alpha$, where

$$\omega_\sigma^\alpha = \mathrm{d}_{\mathcal{C}}^v\, u_\sigma^\alpha = \mathrm{d}\, u_\sigma^\alpha - \sum_{i=1}^n u_{\sigma+1_i}^\alpha\, \mathrm{d}\, x_i.$$

Hence, if $\theta \in \mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \Lambda_v^1(\pi)$ and $\theta^j = \sum_{\sigma,\alpha} a_{\sigma,\alpha}^j \omega_\sigma^\alpha$, then

$$\bigl(\mathrm{i}_{\ni_\omega}(\theta)\bigr)^j = \sum_{\sigma,\alpha} a_{\sigma,\alpha}^j D_\sigma(\omega^\alpha). \tag{3.7}$$

In particular, we see that (3.7) establishes an isomorphism between the modules $\mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \Lambda_v^1(\pi)$ and $\mathcal{C}\,\mathrm{Diff}(\pi, \pi)$ and defines the action of $\mathcal{C}$-differential operators on elements of $\Lambda_v^*(\pi)$. This is a really well-defined action because of the fact that $\mathrm{i}_{\mathcal{C}X}\,\omega = 0$ for any $X \in D(M)$ and $\omega \in \Lambda_v^*(\pi)$.

Consider now a differential equation $\mathcal{E} \subset J^k(\pi)$ and assume that it is determined by a differential operator $\Delta \in \mathcal{F}(\pi, \xi)$. Denote by $\ell_{\mathcal{E}}$ the restriction of the operator of universal linearization ℓ_Δ onto $\mathcal{E}^\infty$. Let $\ell_{\mathcal{E}}^{[p]}$ be the extension of $\ell_{\mathcal{E}}$ to $\mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \Lambda_v^p(\mathcal{E})$ which is well defined due to the above said. Then the module $H_{\mathcal{C}}^{p,0}(\mathcal{E})$ is identified with the set of evolution superderivations $\ni_\omega$ whose generating sections $\omega \in \mathcal{F}(\pi, \pi) \otimes_{\mathcal{F}(\pi)} \Lambda_v^p(\mathcal{E})$ satisfy the equation

$$\ell_{\mathcal{E}}^{[p]}(\omega) = 0 \tag{3.8}$$

(see [Ver] for the proof and more details). If $\mathcal{E}$ satisfies the assumptions of the 2-lines theorem (see [Vin1, Ver]), then $H_{\mathcal{C}}^{p,1}(\mathcal{E})$ is identified with the cokernel of $\ell_{\mathcal{E}}^{[p-1]}$ and thus

$$H_{\mathcal{C}}^i(\mathcal{E}) = \ker \ell_{\mathcal{E}}^i \oplus \operatorname{coker} \ell_{\mathcal{E}}^{[i-1]}$$

in this case.

As it was noted in Remark 2.2, $H_{\mathcal{C}}^1(\mathcal{E})$ is an associative algebra with respect to contraction represented in the algebra of endomorphisms of $H_{\mathcal{C}}^0(\mathcal{E})$. It is easily seen that the action of the $H_{\mathcal{C}}^{0,1}(\mathcal{E})$ is trivial while $H_{\mathcal{C}}^{1,0}(\mathcal{E})$ acts on $H_{\mathcal{C}}^0(\mathcal{E}) = \operatorname{sym}\mathcal{E}$ as $\mathcal{C}$-differential operators (see above).

Definition 3.6. Elements of the module $H_{\mathcal{C}}^{1,0}(\mathcal{E})$ are called *recursion operators* for symmetries of the equation $\mathcal{E}$.

We use the notation $\mathcal{R}(\mathcal{E})$ for the algebra of recursion operators.

Remark 3.2. The algebra $\mathcal{R}(\mathcal{E})$ is always nonempty, since it contains the structural element $U_{\mathcal{E}}$ which is the unit of this algebra. "Usually" this is the only solution of (3.8) for $p = 1$. This fact apparently contradicts to practical experience (cf. with well-known recursion operators for the KdV and other integrable systems). The reason is that these operators contain nonlocal terms like D^{-1} or of a more complicated form. An adequate framework to deal with such constructions will be described in the next section.

Remark 3.3. Let $\varphi \in \operatorname{sym}\mathcal{E}$ be a symmetry and $R \in \mathcal{R}(\mathcal{E})$ be a recursion operator. Then we obtain a series of symmetries $\varphi_0 = \varphi, \varphi_1 = R(\varphi), \dots, \varphi_n = R^n(\varphi), \dots$. Using identity (1.19), one can compute the commutators $[\varphi_m, \varphi_n]$ in terms of $[\![\varphi, R]\!]^{\mathrm{fn}} \in H^{1,0}_{\mathcal{C}}(\mathcal{E})$ and $[\![R, R]\!]^{\mathrm{fn}} \in H^{2,0}_{\mathcal{C}}(\mathcal{E})$. In particular, it can be shown that when both $[\![\varphi, R]\!]^{\mathrm{fn}}$ and $[\![R, R]\!]^{\mathrm{fn}}$ vanish, all symmetries φ_n mutually commute (see [Kr2]).

For example, if $\mathcal{E}$ is an evolution equation, $H^{p,0}_{\mathcal{C}}(\mathcal{E}) = 0$ for all $p \geqslant 2$. Hence, if φ is a symmetry and R is a φ-*invariant* recursion operator (i.e., such that $[\![\varphi, R]\!]^{\mathrm{fn}} = 0$), then R generates a commutative series of symmetries. This is exactly the case for the KdV and other evolution integrable equations.

4. Nonlocal and superextensions

Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation and $\mathcal{E}^\infty \subset J^\infty(\pi)$ be its infinite prolongation. Consider the algebra $\Lambda^*_v(\mathcal{E})$ and a vector field $X \in D(M)$. Define $\mathcal{C}_{\mathcal{S}}X : \Lambda^*_v(\mathcal{E}) \to \Lambda^*_v(\mathcal{E})$ by setting

$$\mathcal{C}_{\mathcal{S}}X(\omega) = \mathrm{L}_{\mathcal{C}X}(\omega), \ \omega \in \Lambda^*_v(\mathcal{E}).$$

Then

$$\mathcal{C}_{\mathcal{S}}(fX)(\omega) = \mathrm{L}_{\mathcal{C}fX}(\omega) = f\,\mathrm{L}_{\mathcal{C}X}(\omega) + \mathrm{d}\,f \wedge \mathrm{i}_{\mathcal{C}X}\,\omega = f\,\mathrm{L}_{\mathcal{C}X}(\omega) = f\mathcal{C}_{\mathcal{S}}(X)(\omega)$$

since, by definition, $\mathrm{i}_{\mathcal{C}X}\,\omega = 0$ for all $X \in D(M), \omega \in \Lambda^*_v(\mathcal{E})$. Obviously one has $\mathcal{C}_{\mathcal{S}}X\,\big|_{D(M)} = X$, and we obtain a flat connection in the algebra $\Lambda^*_v(\mathcal{E})$.

Definition 4.1. The pair $\mathcal{S}\mathcal{E} = (\Lambda^*_v(\mathcal{E}), \mathcal{C}_{\mathcal{S}})$ is called the *superization* of the equation $\mathcal{E}$.

Remark 4.1. The object $\mathcal{S}\mathcal{E}$ may be considered as a superdifferential equation. Its even part is the equation $\mathcal{E}$, while the odd one may be described as follows. Let $\mathcal{E}$ be given by $F_\alpha(x_1, \dots, x_n, \dots, u^j_\sigma, \dots) = 0$ in a special coordinate system (x_i, u^j_σ). Denote by v^j the form $\mathrm{d}\,u^i - \sum_i u^j_{1_i}\,\mathrm{d}\,x_i$. Then the odd component of $\mathcal{S}\mathcal{E}$ is of the form

$$\ell_{F_\alpha} v \equiv \sum \frac{\partial F_\alpha}{\partial u^j_\sigma} D_\sigma v^j = 0.$$

Thus, $\mathcal{S}\mathcal{E}$ governs the behavior of the generators in $\Lambda^*_v(\mathcal{E})$.

The construction of $\mathcal{S}\mathcal{E}$ is a particular case of a more general concept.

Definition 4.2. Let $\mathcal{E}$ be a differential equation and $\mathcal{A}$ be a commutative graded $\mathcal{F}(\mathcal{E})$-algebra. Suppose that there exists a flat connection $\mathcal{C}_{\mathcal{A}}$ in $\mathcal{A}$ as in a $C^\infty(M)$-algebra. Then $(\mathcal{A}, \mathcal{C}_{\mathcal{A}})$ is called an *extension* of the equation $\mathcal{E}$, if $\mathcal{C}_{\mathcal{A}}X\,\big|_{\mathcal{F}(\mathcal{E})} = \mathcal{C}X$ for any derivation $X : C^\infty(M) \to \mathcal{A}$. Two extensions, $(\mathcal{A}, \mathcal{C}_{\mathcal{A}})$ and $(\mathcal{B}, \mathcal{C}_{\mathcal{B}})$, are said to be *equivalent*, if there exists an $\mathcal{F}(\mathcal{E})$-isomorphism $f : \mathcal{A} \to \mathcal{B}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle \mathcal{C}_{\mathcal{A}}(X)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{C}_{\mathcal{B}}(f\circ X)} \\ \mathcal{A} & \xrightarrow{f} & \mathcal{B} \end{array}$$

is commutative for any derivation $X : C^\infty(M) \to \mathcal{A}$.

For any extension $(\mathcal{A}, \mathcal{C}_{\mathcal{A}})$, one can repeat all the constructions of Section 2 and to define symmetries, $\mathcal{C}$-cohomologies, etc. Let us describe the symmetries of the superization $\mathcal{SE}$.

Lemma 4.1. *Let $\mathcal{E}$ be a formally integrable equation. Then a derivation $X \in D^v(\Lambda^*(\mathcal{E}))$ lies in $H^{*,0}_{\mathcal{C}}(\mathcal{E})$ if and only if $[\![\mathcal{C}Y, X]\!]^{\mathrm{fn}} = 0$ for any $Y \in D(M)$.*

Proof. Let $X \in D^v(\Lambda^{*,0}(\mathcal{E}))$. Then $\partial_{\mathcal{C}} X \in D^v(\Lambda^{*,1}(\mathcal{E}))$ and consequently $\partial_{\mathcal{C}} X = 0$ if and only if $\mathrm{i}_{\mathcal{C}Y}\, \partial_{\mathcal{C}} X = 0$ for all $Y \in D(M)$. Use (1.19) now:

$$\mathrm{i}_{\mathcal{C}Y}\, \partial_{\mathcal{C}} X = \mathrm{i}_{\mathcal{C}Y}\, [\![U_{\mathcal{C}}, X]\!]^{\mathrm{fn}} = [\![\mathrm{i}_{\mathcal{C}Y}\, U_{\mathcal{C}}, X]\!]^{\mathrm{fn}} - [\![U_{\mathcal{C}}, \mathrm{i}_{\mathcal{C}Y}\, X]\!]^{\mathrm{fn}} - \mathrm{i}_{[\![\mathcal{C}Y, U_{\mathcal{C}}]\!]^{\mathrm{fn}}}\, X - \mathrm{i}_{[\![\mathcal{C}Y, X]\!]^{\mathrm{fn}}}\, U_{\mathcal{C}}.$$

But all terms in right-hand side vanish except for the last one, which equals to $-[\![\mathcal{C}Y, X]\!]^{\mathrm{fn}}$ due to verticality of $[\![\mathcal{C}Y, X]\!]^{\mathrm{fn}}$. □

Theorem 4.2. *For any formally integrable equation one has*

$$\operatorname{sym} \mathcal{SE} = H^{*,0}_{\mathcal{C}}(\mathcal{E}) \oplus H^{*,0}_{\mathcal{C}}(\mathcal{E}).$$

Proof. Let $X : \Lambda^{*,0}(\mathcal{E}) \to \Lambda^{*,0}(\mathcal{E})$ be a vertical derivation. Any such a derivation can be uniquely represented in the form

$$X = \mathrm{L}^{\mathcal{C}}_{X_0} + \mathrm{i}_{X_1},\ X_0, X_1 \in D^v(\Lambda^{*,0}(\mathcal{E})),$$

where $\mathrm{L}^{\mathcal{C}}_{X_0} = [\mathrm{i}_{X_0}, \mathrm{d}^v_{\mathcal{C}}]$. Then, using Proposition 1.5, we obtain

$$\begin{aligned}[][\mathcal{C}_{\mathcal{S}}Y, X] &= [\mathrm{L}_{\mathcal{C}Y}, \mathrm{L}^{\mathcal{C}}_{X_0} + \mathrm{i}_{X_1}] = [\mathrm{L}_{\mathcal{C}Y}, [\mathrm{i}_{X_0}, \mathrm{d}^v_{\mathcal{C}}]] + [\mathrm{L}_{\mathcal{C}Y}, \mathrm{i}_{X_1}] \\ &= [[\mathrm{L}_{\mathcal{C}Y}, \mathrm{i}_{X_0}], \mathrm{d}^v_{\mathcal{C}}] + [\mathrm{i}_{X_0}, [\mathrm{L}_{\mathcal{C}Y}, \mathrm{d}^v_{\mathcal{C}}]] + \mathrm{i}_{[\![\mathcal{C}Y, X_1]\!]^{\mathrm{fn}}} - \mathrm{L}_{\mathrm{i}_{X_1}\mathcal{C}Y}.\end{aligned}$$

where $Y \in D(M)$. But the second and fourth summands vanish while the first one equals to $\mathrm{L}_{[\![\mathcal{C}Y, X_0]\!]^{\mathrm{fn}}}$. Hence $[\![\mathcal{C}Y, X_0]\!]^{\mathrm{fn}} = 0$, $[\![\mathcal{C}Y, X_1]\!]^{\mathrm{fn}} = 0$, which gives the result by Lemma 4.1. □

Thus we see that there are two embeddings of $H^{*,0}_{\mathcal{C}}(\mathcal{E})$ into $\operatorname{sym} \mathcal{SE}$: the first one, $\mathrm{L}^{\mathcal{C}}$, given by the Lie derivative and another one, i, defined by the interior product. Note that if $R \in \mathcal{R}(\mathcal{E})$ is a recursion operator and $\varphi \in \operatorname{sym} \mathcal{E}$ is a symmetry, then $[\mathrm{i}(\varphi), \mathrm{L}^{\mathcal{C}}(R)] = \mathrm{L}^{\mathcal{C}}(R\varphi)$ and thus the action of recursion operators on symmetries is expressed in terms of the Lie bracket in $\operatorname{sym} \mathcal{SE}$. Note also that the algebra $\operatorname{sym} \mathcal{SE}$ is always nonempty: it contains two elements, $\mathrm{L}^{\mathcal{C}}(U_{\mathcal{E}})$ and $\mathrm{i}(U_{\mathcal{E}})$, at least.

Consider a particular and a very important case of extensions. Let $\tau : \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a fiber bundle and $\mathcal{F}_\tau(\mathcal{E})$ denote the function algebra $C^\infty(\tilde{\mathcal{E}})$.

Definition 4.3 (cf. [KrVin])**.** We say that $\tau : \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ is a *covering* over $\mathcal{E}$, if there exists an extension structure $(\mathcal{F}_\tau(\mathcal{E}), \mathcal{C}_\tau)$ in $\mathcal{F}_\tau(\mathcal{E})$. A symmetry of $(\mathcal{F}_\tau(\mathcal{E}), \mathcal{C}_\tau)$ is called a *nonlocal symmetry of the type* τ of the equation $\mathcal{E}$, or τ-symmetry.

More general, let $(\mathcal{A}, \mathcal{C}_{\mathcal{A}})$ be an extension of $\mathcal{E}$ and $(\mathcal{B}, \mathcal{C}_{\mathcal{B}})$ be an extension of $(\mathcal{A}, \mathcal{C}_{\mathcal{A}})$. We say that $(\mathcal{B}, \mathcal{C}_{\mathcal{B}})$ is a *covering* over $(\mathcal{A}, \mathcal{C}_{\mathcal{A}})$, if they are of the same grading.

Geometrically, it means that a flat connection $\mathcal{C}_\tau$ is given in the bundle $\pi_\infty \circ \tau : \tilde{\mathcal{E}} \to M$ satisfying $\mathcal{C}_\tau X(f) = \mathcal{C}X(f)$ for any $f \in \mathcal{F}(\mathcal{E})$.

Example 4.1. Let $\mathcal{E}$ be given by an operator $\Delta : \Gamma(\pi) \to \Gamma(\pi^1)$. The infinite prolongation Δ^∞ determines the map $\varphi_\Delta : J^\infty(\pi) \to J^\infty(\xi^1)$. Then, under unrestrictive regularity conditions, the image $\varphi_\Delta J^\infty(\pi)$ is of the form $\mathcal{E}_1^\infty$ for some equation $\mathcal{E}_1 \subset J^{k_1}(\pi^1)$. This equation $\mathcal{E}_1$ consists of *compatibility conditions* for $\mathcal{E}$ and is generated by the *Nöther identities* of $\mathcal{E}$. If $\mathcal{E}_1$ is a *proper* equation, i.e., $\mathcal{E}_1^\infty$ does not coincide with $J^\infty(\pi^1)$, we can repeat the construction, etc. Thus we obtain the sequence of coverings

$$\mathcal{E}^\infty \subset J^\infty(\pi) \xrightarrow{\varphi_\Delta} \mathcal{E}_1^\infty \subset J^\infty(\pi^1) \xrightarrow{\varphi_{\Delta_1}} \dots \xrightarrow{\varphi_{\Delta_{s-1}}} \mathcal{E}_s^\infty \subset J^\infty(\pi^s) \xrightarrow{\varphi_{\Delta_s}} \dots$$

which may be called the *nonlinear compatibility sequence* of $\mathcal{E}$ (cf. [Ver]) and was discovered by A. Vinogradov some 10 years ago, [Vin3].

Consider on $J^\infty(\pi^1)$ the evolution derivation $\ni_\delta$ whose generating section δ is the *diagonal* in $\mathcal{F}(\pi^1, \pi^1)$ (i.e., $j_\infty(\varphi)^*(\delta) = \varphi$ for any $\varphi \in \Gamma(\pi^1)$). The composition $\ni_\delta^\Delta = \varphi_\Delta^* \circ \ni_\delta$ is a $\mathcal{F}(\pi)$-valued derivation of $\mathcal{F}(\pi^1)$.[4] Then the contraction $\mathrm{i}_{\ni_\delta^\Delta}$ coincides with the Koszul – Tate differential while elements of $\Lambda_v^1(\mathcal{E}_1)$ are identified with anti-fields.

Example 4.2. Let $\mathcal{E}$ be an equation and $(\mathcal{A}, \mathcal{C}_\mathcal{A})$ be its extension. Consider a horizontal 1-form $\omega \in \Lambda_h^1(\mathcal{A})$ and the algebra $\mathcal{A}_\omega = \mathcal{A}(u_\omega)$, where the variable u_ω carries the same grading as the one inherited by ω from $\mathcal{A}$. For any $X \in D(M)$, define $\mathcal{C}_{\mathcal{A}_\omega} u_\omega = \mathrm{i}_{\mathcal{C}X}\, \omega$. Then the pair $(\mathcal{A}_\omega, \mathcal{C}_{\mathcal{A}_\omega})$ is the extension of $\mathcal{E}$ and a covering over $(\mathcal{A}, \mathcal{C}_\mathcal{A})$.

If two horizontal forms, ω and ω', are $\mathrm{d}_\mathcal{C}^h$-homologous (say, $\omega' = \omega + \mathrm{d}_\mathcal{C}^h\, a,\ a \in \mathcal{A}$), then the coverings $(\mathcal{A}_\omega, \mathcal{C}_\omega)$ and $(\mathcal{A}_{\omega'}, \mathcal{C}_{\omega'})$ are equivalent and the corresponding isomorphism is given by $u_\omega \mapsto u_{\omega'} - a$. Thus we established a correspondence between the group $H_h^{0,1}(\mathcal{A})$ and the set of classes of equivalent coverings over $\mathcal{A}$. Note that the $\mathrm{d}_\mathcal{C}^h$-cohomology class of ω is "killed" in $(\mathcal{A}_\omega, \mathcal{C}_\omega)$, since, by definition,

$$\mathrm{i}_{\mathcal{C}X}\, \mathrm{d}_\mathcal{C}^h\, u_\omega = \mathrm{L}_{\mathcal{C}X}\, u_\omega = \mathrm{i}_{\mathcal{C}X}\, \omega$$

for any $X \in D(M)$.

Let now $\omega_1, \dots, \omega_\alpha, \dots$ be an $\mathbb{R}$-basis in $H_h^{0,1}(\mathcal{A})$. Then by a similar construction, we can consider the extension $(\mathcal{K}^1\mathcal{A}, \mathcal{C}^1)$, where $\mathcal{K}^1\mathcal{A} = \mathcal{A}(u_{\omega_1}, \dots, u_{\omega_\alpha}, \dots)$ and the connection $\mathcal{C}_\mathcal{A}^1$ is given as above. Evidently the equivalence class of $\mathcal{K}^1\mathcal{A}$ is independent of basis choice. All elements of $H_h^{0,1}(\mathcal{A})$ will be killed in $\mathcal{K}^1\mathcal{A}$, but new ones may arise and we shall kill them by constructing $\mathcal{K}^2\mathcal{A} = \mathcal{K}^1(\mathcal{K}^1\mathcal{A})$, etc. Thus we obtain the series of extensions

$$(\mathcal{A}, \mathcal{C}_\mathcal{A}) \subset (\mathcal{K}^1\mathcal{A}, \mathcal{C}_\mathcal{A}^1) \subset \dots \subset (\mathcal{K}^s\mathcal{A}, \mathcal{C}_\mathcal{A}^s) \subset (\mathcal{K}^{s+1}\mathcal{A}, \mathcal{C}_\mathcal{A}^{s+1}) \subset \dots$$

whose direct limit is denoted by $(\mathcal{K}^*\mathcal{A}, \mathcal{C}^*)$ and is called the *universal Abelian covering* (or *extension*) of $\mathcal{A}$ (cf. [Kh]). By construction, $H_h^{0,1}(\mathcal{K}^*\mathcal{A}) = 0$.

Let again $\mathcal{E}$ be an equation (or a superequation) and $\tau : \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a covering. The any $\mathcal{C}$-differential operator Δ on $\mathcal{E}^\infty$ can be lifted up to an operator $\tilde{\Delta}$ on $\tilde{\mathcal{E}}$. In particular, we can consider the operator $\tilde{\ell}_\mathcal{E}$ and solutions of the equations

$$\tilde{\ell}_\mathcal{E}\varphi = 0, \quad \varphi \in \Gamma((\pi_\infty \circ \tau)^*, \pi). \tag{4.1}$$

Solutions of (4.1) are called *shadows* of τ-symmetries. We say that a shadow φ is *recoverable*, if there exists a τ-symmetry S such that $S\,\big|_{\mathcal{F}(\mathcal{E})} = \ni_\varphi\,\big|_{\mathcal{F}(\mathcal{E})}$. Of course, not all shadows are recoverable, but the universal Abelian covering is a happy case.

Denote by $\mathfrak{k}^s : \mathfrak{K}_\mathcal{E}^s \to \mathcal{E}^\infty$ the covering corresponding to the extension $\mathcal{K}^s\mathcal{F}(\mathcal{E})$ and by $\mathfrak{k}^* : \mathfrak{K}_\mathcal{E}^* \to \mathcal{E}^\infty$ the covering corresponding to $\mathcal{K}^*\mathcal{F}(\mathcal{E})$

[4] We consider $\mathcal{F}(\pi)$ as an $\mathcal{F}(\pi^1)$-algebra due to φ_Δ^*.

Theorem 4.3 ([Kh])**.** *Any $\mathfrak{k}^*$-shadow is recoverable in the covering $\mathfrak{k}^*$.*

Though the proof in [Kh] was given for nongraded equations, it remains to be literary valid for the super case as well. Note now that by construction any $\mathfrak{k}^s$-shadow is also a $\mathfrak{k}^*$-shadow. Hence we have a tautological consequence of the above theorem.

Corollary 4.4. *Any $\mathfrak{k}^s$-shadow is recoverable in the covering $\mathfrak{k}^*$.*

Using this result, we prove the following theorem stating existence of well-defined action of nonlocal recursion operators in $\mathfrak{k}^*$.

Theorem 4.5 (action of recursion operators in $\mathcal{K}^*$)**.** *Let $\mathcal{E}$ be a formally integrable equation and $\mathcal{SE}$ be its superization. Consider the universal Abelian extension $\mathcal{K}^*\mathcal{SE}$ and a $\mathfrak{k}^*$-shadow ω of grading 1. Then, if $S = S_0$ is a symmetry of $\mathfrak{K}^*_{\mathcal{E}}$, then there exists a series of symmetries in $\mathfrak{K}^*_{\mathcal{E}}$ defined by $S_{i+1} = \mathrm{i}_{S_i}\,\omega$.*

Proof. First, following Theorem 4.3 let us recover the shadow ω up to a symmetry S_ω (of grading 1) in $\mathcal{K}^*\mathcal{SE}$. Let S be a symmetry of $\mathcal{K}^*\mathcal{E}$. Then the restriction $\mathrm{i}_S|_{\mathcal{SE}}$ is a shadow in $\mathcal{K}^*\mathcal{E}$ (it is a consequence of Remark 4.1) and thus can be lifted up to a symmetry S^- of $\mathcal{K}^*\mathcal{SE}$. Since S^- is of grading -1, the commutator $[S^-, S_\omega]$ is of grading 0 and thus can be restricted onto $\mathcal{K}^*\mathcal{E} \subset \mathcal{K}^*\mathcal{SE}$ which gives the result desired. □

Example 4.3. Let $u_t = u_{xxx} + uu_x$ be the KdV equation. Choose the functions $t, x, u_0, \dots, u_k, \dots$, where u_k corresponds to $\partial^k u/\partial x^k$, for the internal coordinates in $\mathcal{E}^\infty$. Denote by ω_k the form $\mathrm{d}^v_{\mathcal{C}}\, u_k$. The form $\omega = u_0\,\mathrm{d}\,x + (u_2 + 1/2u_0^2)\,\mathrm{d}\,t$ is a $\mathrm{d}^h_{\mathcal{C}}$-closed form with nontrivial cohomology class. Let u_{-1} be the new variable in the corresponding covering. Then $\rho = \omega_2 + 2/3u_0\omega_0 + 1/3u_1\omega_{-1}$ is a shadow in $\mathfrak{k}^*$. The corresponding recursion operator is $D_x^2 + 2/3u_0 + 1/3u_1\omega_{-1}$ and coincides with the classical one, [Olv]. Other, less trivial examples one can find in [KrKe2].

5. Concluding remarks

I want to finish this paper with mentioning some topics not discussed above — partially because of lack of space, but mainly because they need to be more clarified.

1. $\mathcal{C}$-cohomologies and Poisson structures. As it was mentioned in the Introduction, the term E_1 of Vinogradov's $\mathcal{C}$-spectral sequence and $\mathcal{C}$-complex introduced here are in a sense mutually dual: while $\mathcal{C}$-cohomologies contain information on (super)symmetries of equation $\mathcal{E}$, the $(n-1)$-st line in $E_1(\mathcal{E})$ consists of (super)conservation laws of the same equation. Moreover, elements of $H^*_{\mathcal{C}}(\mathcal{E})$ act on $E_1(\mathcal{E})$ by the Lie derivative and if a cochain map $\mathcal{P} : E_0(\mathcal{E}) \to D^v(\Lambda^*(\mathcal{E}))$ is given, we can define the bracket on $E_1(\mathcal{E})$,

$$\langle[\omega], [\omega']\rangle_{\mathcal{P}} = [\mathrm{L}_{\mathcal{P}(\omega)}\,\omega'],\ \omega, \omega' \in E_0(\mathcal{E}),$$

where $[\,\cdot\,]$ denotes the cohomology class. In [Kr3, Kr4] it was shown that in the case when $\mathcal{E}$ of evolution equations and when $\mathcal{P}$ is a $\mathcal{C}$-differential operator, $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ is a Poisson bracket, if $\mathcal{P}$ (i) satisfies two conditions (skew-symmetry and the Jacobi identity) deduced in [AsVin] for the case $J^\infty(\pi)$ and (ii) is a solution of the operator equation

$$\ell_{\mathcal{E}} \circ \mathcal{P} + \mathcal{P}^* \circ \ell^*_{\mathcal{E}} = 0,$$

where Δ^* denotes the operator formally adjoint to Δ. It seems that the same is true when $\mathcal{E}$ satisfies assumptions of the 2-lines theorem.

2. Generalized connections and higher differential forms. The definition of a connection used here (see Definition 2.1) may be fruitfully used in

different situation generating corresponding cohomology theories. For example, taking a pair of algebras $A \subset B$, one can consider natural transformations of the functors $\mathrm{Diff}_*(A, \bullet)$ and $\mathrm{Diff}_*(\bullet)$. Then, literary repeating the scheme of Section 2, we shall obtain a differential in the module $\mathrm{Diff}_*^v(\Lambda^{\sigma(*)})$, where $\Lambda^{\sigma(*)}$ denotes the module of *higher differential forms* of the algebra B, [VeVi]. It is almost obvious that its 0-cohomology coincides with the module of *secondary differential operators* (when $B = \mathcal{F}(\mathcal{E})$, see [GVY] for the definition). It would be quite interesting to consider arising invariants in more details.

3. Relation to the Koszul – Tate complex. Finally, I would like to note that Example 4.1 puts the Koszul – Tate complex in a natural geometric framework which opens promising perspectives for further constructions.

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