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The Lagrangian-Hamiltonian Formalism for Higher Order Field Theories

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Abstract. We generalize the lagrangian-hamiltonian formalism of Skinner and Rusk to higher order field theories on fiber bundles. As a byproduct we solve the long standing problem of defining, in a coordinate free manner, a hamiltonian formalism for higher order lagrangian field theories, which does only depend on the action functional and, therefore, unlike previously proposed formalisms, is free from any relevant ambiguity.

Introduction

First order lagrangian mechanics can be generalized to higher order lagrangian field theory. Moreover, the latter has got a very elegant geometric (and homological) formulation (see, for instance, [1]) on which there is general consensus. On the other side, it seems that the generalization of hamiltonian mechanics of lagrangian systems to higher order field theory presents some more problems. They have been proposed several answers (see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein) to the question: is there any reasonable, higher order, field theoretic analogue of hamiltonian mechanics? To our opinion, non of them is satisfactorily natural, especially because of the common emergence of ambiguities due to either the arbitrary choice of a coordinate system [2] or the choice of a Legendre transform [7, 8, 10]. Namely, the latter seems not to be uniquely definable, except in the case of first order lagrangian field theories when a satisfactory hamiltonian formulation can be presented in terms of multisymplectic geometry (see, for instance, [11] - see also [12] for a recent review, and references therein).

Nevertheless, it is still desirable to have a hamiltonian formulation of higher order lagrangian field theories enjoying the same nice properties as hamiltonian mechanics, which 1) is natural, i.e., is independent of the choice of any other structure than the action functional, 2) gives rise to first order equations of motion, 3) takes advantage from the (pre-)symplectic geometry of phase space, 4) is a natural starting point for gauge reduction, 5) is a natural starting point for quantization. A special mention deserves the relationship between the Euler-Lagrange equations and the Hamilton equations. The Legendre transform maps injectively solutions of the former to solutions of the latter, but, generically, Hamilton equations are not equivalent to Euler-Lagrange ones [11].

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However, the difference between the two is a pure gauge and, therefore, it is irrelevant from a physical point of view.

In this paper we reach the goal of finding a natural (in the above mentioned sense), geometric, higher order, field theoretic analogue of hamiltonian mechanics of lagrangian systems in two steps: first, we find a higher order, field theoretic analogue of the Skinner and Rusk "mixed lagrangian-hamiltonian" formalism [13, 14, 15] (see also [16]), which is rather straightforward, and, second, we show that the derived theory "projects to a smaller space" which is naturally interpreted as phase space. Local expressions of the field equations on the phase space are nothing but de Donder equations [2] and, therefore, are naturally interpreted as the higher order, field theoretic, coordinate free analogue of Hamilton equations. A central role is played in the paper by multisymplectic geometry in the form of partial differential (PD, in the following) hamiltonian system theory, which has been developed in [17].

The paper is divided into eight sections. The first four sections contain reviews of the main aspects of the geometry underlying the paper. They have been included in order to make the paper as self-consistent as possible. The next four sections contain most of the original results.

The first section summarizes notations and conventions adopted throughout the paper. It also contains references to some differential geometric facts which will be often used in the subsequent sections. Finally, we briefly review, in Section 1, the Skinner-Rusk formalism [14]. Section 2 is a short review of the geometric theory of partial differential equations (PDEs) (see, for instance, [19]). Section 3 outlines properties of the main geometric structure of jet spaces and PDEs, namely, the Cartan distribution, and reviews the geometric formulation of the calculus of variations [1]. Section 4 reviews the theory of PD-hamiltonian systems and their PD-Hamilton equations [17]. Moreover, it contains examples of morphisms of PDEs coming from such theory. These examples are presented here for the first time.

In Section 5 we present the higher order, field theoretic analogue of Skinner-Rusk mixed lagrangian-hamiltonian formalism for mechanics. In Section 5 we also discuss the relationship between field equations in the lagrangian-hamiltonian formalism (now on, ELH equations) and the Euler-Lagrange equations. In Section 6 we discuss some natural transformations of the ELH equations. As a byproduct, we prove that they are independent of the choice of a lagrangian density, in the class of those yielding the same Euler-Lagrange equations, up to isomorphisms. ELH equations are, therefore, as natural as possible. In Section 7 we present our proposal for a hamiltonian, higher order, field theory. Since we don't use any additional structure other than the ELH equations and the order of a lagrangian density, we judge our theory satisfactorily natural. Moreover, the associated field equations (HDW equations) are first order and, more specifically, of the PD-Hamilton kind. In Section 8 we study the relationship between the HDW equations and the Euler-Lagrange equations. As a byproduct, we derive a new (and, in our opinion, satisfactorily natural) definition of Legendre transform for higher order, lagrangian field theories. It is a non-local morphism of the Euler-Lagrange equations into the HDW equations.

1. Notations, Conventions and the Skinner-Rusk Formalism

In this section we collect notations and conventions about some general constructions in differential geometry that will be used in the following.

Let N be a smooth manifold. If $L \subset N$ is a submanifold, we denote by $i_L : L \hookrightarrow N$ the inclusion. We denote by $C^\infty(N)$ the $\mathbb{R}$–algebra of smooth, $\mathbb{R}$–valued functions on N. We will always understand a vector field X on N as a derivation $X : C^\infty(N) \longrightarrow C^\infty(N)$. We denote by $\mathrm{D}(N)$ the $C^\infty(N)$–module of vector fields over N, by $\Lambda(M) = \bigoplus_k \Lambda^k(N)$ the graded $\mathbb{R}$–algebra of differential forms over N and by $d : \Lambda(N) \longrightarrow \Lambda(N)$ the de Rham differential. If $F : N_1 \longrightarrow N$ is a smooth map of manifolds, we denote by $F^* : \Lambda(N) \longrightarrow \Lambda(N_1)$ its pull–back. We will understand everywhere the wedge product $\wedge$ of differential forms, i.e., for $\omega, \omega_1 \in \Lambda(N)$, instead of writing $\omega \wedge \omega_1$, we will simply write $\omega\omega_1$.

Let $\alpha : A \longrightarrow N$ be an affine bundle (for instance, a vector bundle) and $F : N_1 \longrightarrow N$ a smooth map of manifolds. Let $\mathscr{A}$ be the affine space of smooth sections of α. For $a \in \mathscr{A}$ and $x \in N$ we put, sometimes, $a_x := a(x)$. The affine bundle on N_1 induced by α via F will be denoted by $F^\circ(\alpha) : F^\circ(A) \longrightarrow N$:

$$\begin{tikzcd} F^\circ(A) \arrow[r] \arrow[d, "F^\circ(\alpha)"'] & A \arrow[d, "\alpha"] \\ N_1 \arrow[r, "F"] & N \end{tikzcd} ,$$

and the space of its section by $F^\circ(\mathscr{A})$. For any section $a \in \mathscr{A}$ there exists a unique section, which we denote by $F^\circ(a) \in F^\circ(\mathscr{A})$, such that the diagram

$$\begin{tikzcd} F^\circ(A) \arrow[r] & A \\ N_1 \arrow[r, "F"] \arrow[u, "F^\circ(a)"] & N \arrow[u, "a"'] \end{tikzcd}$$

commutes. If $F : N_1 \longrightarrow N$ is the embedding of a submanifold, we also write $\bullet\ |_F$ (or, simply, $\bullet\ |_{N_1}$) for $F^\circ(\,\bullet\,)$, and refer to it as the restriction of "$\bullet$" to N_1 (via F), whatever the object "$\bullet$" is (an affine bundle, its total space, its space of sections or a section of it).

Denote by $\mathbb{N}$ the set of natural numbers. We will always understand the sum over repeated upper-lower (multi-)indexes. Our notations about multiindexes are the following. Let $n \in \mathbb{N}$, $\mathbb{I}_n = \{1, \ldots, n\}$ and $\mathbb{M}_n$ be the free abelian monoid generated by $\mathbb{I}_n$. Even if $\mathbb{M}_n$ is abelian, we keep for it the multiplicative notation. Thus if $I = i_1 \cdots i_l, J = j_1 \cdots j_m \in \mathbb{M}_n$ are (equivalence classes of) words, $i_1, \ldots, i_l, j_1, \ldots, j_m \in \mathbb{I}_n$, we denote by $IJ := i_1 \cdots i_l j_1 \cdots j_m$ their composition. If

$I = i_1 \cdots i_l \in \mathbb{M}_n$ is a word, $i_1, \ldots, i_l \in \mathbb{I}_n$, denote by $|I| := l$ its *length*. We denote by O the (equivalence class of the) empty word. An element $I \in \mathbb{M}_n$ is called an *n-multiindex* (or, simply, a *multiindex*). For $I \in \mathbb{M}_n$ and $i \in \mathbb{I}_n$, we denote by $I[i]$ the number of times the "letter" i occurs in the (equivalence class of) word(s) I. In other words $I[i]$ is the greatest positive integer r such that i^r divides I. We stress that our notation about multiindexes is different from more popular ones (see, for instance, [18]).

We conclude this section by briefly reviewing those aspects of the Skinner-Rusk formalism for mechanics [13, 14, 15] that survive in our generalization to higher order field theory.

Let Q be a smooth manifold and $q^1, \ldots, q^m$ coordinates on it, $m = \dim Q$. Let $L \in C^\infty(TQ)$ be a lagrangian function. Consider the induced bundle $\tau_0^\dagger := \tau_Q^\circ(\tau_Q^*) : T^\dagger := \tau_Q^\circ(T^*Q) \longrightarrow TQ$ from the cotangent bundle $\tau_Q^* : T^*Q \longrightarrow Q$ to Q, via the tangent bundle $\tau_Q : TQ \longrightarrow Q$. Let $q_0 : T^\dagger \longrightarrow T^*Q$ be the canonical projection (see Diagram (1))

$$\begin{array}{ccc} T^\dagger & \xrightarrow{q_0} & T^*Q \\ {\scriptstyle \tau_0^\dagger}\downarrow & & \downarrow{\scriptstyle \tau_Q^*} \\ TQ & \xrightarrow{\tau_Q} & Q \end{array} \quad . \tag{1}$$

On $T^\dagger$ there is a canonical function $h \in C^\infty(T^\dagger)$ defined by $h(v,p) := p(v)$, $v \in T_qQ$, $p \in T_q^*Q$, $q \in Q$. Consider also the function $E_L := h - \tau_0^\dagger(L) \in C^\infty(T^\dagger)$. E_L is locally given by $E_L := p_i\dot{q}^i - L$, where $\ldots, q^i, \ldots, \dot{q}^i, \ldots, p_i, \ldots$ are standard coordinates on $T^\dagger$. Finally, put $\omega := q_0^*(\omega_0) \in \Lambda^2(T^\dagger)$, $\omega_0 \in \Lambda^2(T^*Q)$ being the canonical symplectic form on T^*Q, which is locally given by $\omega_0 = dp_i dq^i$. ω is a presymplectic form on $T^\dagger$ whose kernel is made of vector fields over $T^\dagger$ which are vertical with respect to the projection q_0. In the following, denote by $I \subset \mathbb{R}$ a generic open interval. For a curve $\gamma : I \ni t \longmapsto \gamma(t) \in T^\dagger$, consider equations

$$i_{\dot{\gamma}}\omega|_\gamma - (dE_L)|_\gamma = 0, \tag{2}$$

where $\dot{\gamma} \in \gamma^\circ(\mathrm{D}(T^\dagger))$ is the tangent field to γ. Equations (2) read locally

$$\left\{ \begin{array}{l} \frac{d}{dt}q^i = \dot{q}^i \\ p_i = \frac{\partial L}{\partial \dot{q}^i} \\ \frac{d}{dt}p_i = \frac{\partial L}{\partial q^i} \end{array} \right. .$$

In particular, for any solution γ of Equations (2) as above, $\tau_Q \circ \tau_0^\dagger \circ \gamma : I \longrightarrow Q$ is a solution of the Euler-Lagrange equations determined by L. Notice that solutions of Equations (2) can only take values in the submanifold $\mathscr{P} \subset T^\dagger$ defined as

$$\mathscr{P} := \{P \in T^\dagger : \text{there exists } \Xi \in T_P T^\dagger \text{ such that } i_\Xi \omega_P - (dE_L)_P = 0\},$$

and that $\mathscr{P}$ is nothing but the graph of the Legendre transform $FL : TQ \longrightarrow T^*Q$. Finally, consider $\mathscr{P}_0 := q_0(\mathscr{P}) \subset T^*Q$. If $\mathscr{P}_0 \subset T^*Q$ is a submanifold and $q_0 : \mathscr{P} \longrightarrow \mathscr{P}_0$ a submersion with connected fibers, then there exists a (unique) function

$H \in C^\infty(\mathscr{P}_0)$ such that $q_0^*(H) = E_L$. Therefore, for a curve $\sigma : I \ni t \longmapsto \sigma(t) \in \mathscr{P}_0$, we can consider equations

$$i_{\dot\sigma}\omega_0|_\sigma - (dH)|_\sigma = 0, \tag{3}$$

where $\dot\sigma \in \sigma^\circ(\mathrm{D}(T^\dagger))$ is the tangent field to σ. For any solution $\gamma : I \longrightarrow Q$ of the Euler-Lagrange equations, $FL \circ \dot\gamma : I \longrightarrow \mathscr{P}_0$ is a solution of Equations (3). If $\mathscr{P}_0 \subset T^*Q$ is an open submanifold (which, under the above mentioned regularity conditions, happens iff the matrix $\|\partial^2 L/\partial\dot q^i\partial\dot q^j\|_i^j$ has maximum rank, i.e., FL is a local diffeomorphism), then H is a local function on T^*Q and Equations (3) read locally

$$\begin{cases} \frac{d}{dt}q^i = \frac{\partial H}{\partial p_i} \\ \frac{d}{dt}p_i = -\frac{\partial H}{\partial q^i} \end{cases},$$

which are Hamilton equations. In this case, for any solution $\sigma : I \longrightarrow T^*Q$ of Equations (3), $\tau_Q^* \circ \sigma : I \longrightarrow Q$ is a solution of the Euler-Lagrange equations.

2. Geometry of Differential Equations

In this section we recall basic facts about the geometric theory of partial differential equations (PDEs). For more details see [19].

Let $\pi : E \longrightarrow M$ be a fiber bundle, $\dim M = n$, $\dim E = m + n$. For $0 \le l \le k \le \infty$, we denote by $\pi_k : J^k\pi \longrightarrow M$ the bundle of k-jets of local sections of π, and by $\pi_{k,l} : J^k\pi \longrightarrow J^l\pi$ the canonical projection. For any local section $s : U \longrightarrow E$ of π, $U \subset M$ being an open subset, we denote by $j_k s : U \longrightarrow J^k\pi$ its kth jet prolongation. For $x \in U$, put $[s]_x^k := (j_k s)(x)$. Any system of adapted to π coordinates $(\ldots, x^i, \ldots, u^\alpha, \ldots)$ on an open subset U of E gives rise to a system of jet coordinates on $\pi_{k,0}^{-1}(U) \subset J^k\pi$ which we denote by $(\ldots, x^i, \ldots, u^\alpha{}_{|I}, \ldots)$ or simply $(\ldots, x^i, \ldots, u_I^\alpha, \ldots)$ if this does not lead to confusion, $I \in \mathbb{M}_n$, $|I| \le k$, where we put $u_{\mathsf{O}}^\alpha := u^\alpha$, $\alpha = 1, \ldots, m$.

Now, let $k < \infty$, $\tau_0 : T_0 \longrightarrow J^k\pi$ be a vector bundle, and

$$(\ldots, x^i, \ldots, u_I^\alpha, \ldots, v^a, \ldots)$$

adapted to τ_0, local coordinates on T_0. A (possibly non-linear) *differential operator of order $\le k$ 'acting on local sections of π, with values in τ_0'* (in short *'from π to τ_0'*) is a section $\Phi : J^k\pi \longrightarrow T_0$ of τ_0. For any local section $s : U \longrightarrow E$ of π, Φ determines an 'image' section $\Delta_\Phi s := \Phi \circ j_k s : U \longrightarrow T_0$ of the bundle $\underline{\tau}_0 := \pi_k \circ \tau_0 : T_0 \longrightarrow M$.

Let $\pi' : E' \longrightarrow M$ be another fiber bundle and $\varphi : E \longrightarrow E'$ a morphism of bundles. For any local section $s : U \longrightarrow E$ of π, $U \subset M$ an open subset, $\varphi \circ s : U \longrightarrow E'$ is a local section of π'. Therefore, for all $0 \le k \le \infty$, φ_k induces a morphism $j_k\varphi : J^k\pi \longrightarrow J^k\pi'$ of the bundles π_k and π'_k defined by $(j_k\varphi)[s]_x^k := [\varphi \circ s]_x^k$, $x \in U$. Diagram

$$\begin{array}{ccc} J^l\pi & \xrightarrow{j_l\varphi} & J^l\pi' \\ {\scriptstyle \pi_{l,k}}\big\downarrow & & \big\downarrow{\scriptstyle \pi'_{l,k}} \\ J^k\pi & \xrightarrow{j_k\varphi} & J^k\pi' \end{array}$$

commutes for all $0 \leq k \leq l \leq \infty$. $j_k\varphi$ is called the *kth prolongation of* φ.

The above construction generalizes to differential operators as follows. Let Φ be a differential operator of order $\leq k$ from π to $\tau_0 : T_0 \longrightarrow J^k\pi$ and $0 \leq l \leq \infty$. Consider the space $J^l\underline{\tau}_0$ of l-jets of local sections of $\underline{\tau}_0$. In $J^l\underline{\tau}_0$ consider the submanifold $T_0^{(l)}$ made of jets of local sections of the form $\Delta_\Psi s$, where Ψ is any differential operator of order $\leq k$ from π to τ_0 and s is a local section of π. $T_0^{(l)}$ is locally defined by

$$u^\alpha_{I|J} = u^\alpha_{I'|J'}, \text{ whenever } IJ = I'J',$$

$\alpha = 1, \ldots, m$, $|I| = |I'| = k$, $|J| = |J'| = l$. Thus, $(\ldots, x^i, \ldots, u^\alpha_K, \ldots, v^a_J, \ldots)$, $|K| = k + l$, $|J| = l$, are local coordinates on $T_0^{(l)}$, where

$$\ldots, u^\alpha_K := u^\alpha_{K_1|K_2}|_{T_0^{(l)}}, \ldots,$$

K_1, K_2 being any pair of multiindexes such that $K_1K_2 = K$, $|K_1| = k$, $|K_2| = l$. $T_0^{(l)}$ projects canonically onto $J^{k+l}\pi$ and the projection $\tau_0^{(l)} : T_0^{(l)} \longrightarrow J^{k+l}\pi$ is a vector bundle. Moreover, the u^α_K's identify with the corresponding coordinates on $J^{k+l}\pi$ via $\tau_0^{(l)}$. Define the *lth prolongation* $\Phi^{(l)} : J^{k+l}\pi \longrightarrow T_0^{(l)}$ of Φ by putting $\Phi^{(l)}([s]^{k+l}_x) := [\Delta_\Phi s]^l_x \in T_0^{(l)}$, for all local sections s of π and $x \in M$. Then $\Phi^{(l)}$ is a differential operator of order $\leq k + l$ from π to $\tau_0^{(l)}$.

For Φ as above, put $\mathscr{E}_\Phi := \{\theta \in J^k\pi : \Phi(\theta) = 0\}$. $\mathscr{E}_\Phi$ is called the *(system of) PDE(s)* determined by Φ. For $0 \leq l \leq \infty$ put also $\mathscr{E}_\Phi^{(l)} := \mathscr{E}_{\Phi^{(l)}} \subset J^{k+l}\pi$. $\mathscr{E}_\Phi^{(l)}$ is locally determined by equations

$$(D_J\Phi^a)(\ldots, x^i, \ldots, u^\alpha_I, \ldots) = 0, \quad a = 1, \ldots, p,\ |J'| = l, \tag{4}$$

where $\ldots, \Phi^a := \Phi^*(v^a), \ldots$ are local functions on $J^k\pi$, $D_{j_1\cdots j_l} := D_{j_1} \circ \cdots \circ D_{j_l}$, and $D_j := \partial/\partial x^j + u^\alpha_{Ij}\partial/\partial u^\alpha_I$ is the *jth total derivative*, $j, j_1, \ldots, j_l = 1, \ldots, m$. $\mathscr{E}_\Phi^{(l)}$ is called the lth prolongation of the PDE $\mathscr{E}_\Phi$. In the following we put $\partial^I_\alpha := \partial/\partial u^\alpha_I$, $\alpha = 1, \ldots, m$, $I \in \mathbb{M}_n$.

A local section s of π is a *(local) solution of* $\mathscr{E}_\Phi$ iff, by definition, $\operatorname{im} j_k s \subset \mathscr{E}_\Phi$ or, which is the same, $\operatorname{im} j_{k+l}s \subset \mathscr{E}_\Phi^{(l)}$ for some $l \leq \infty$. Notice that the ∞th prolongation of $\mathscr{E}_\Phi$, $\mathscr{E}_\Phi^{(\infty)} \subset J^\infty\pi$, is an inverse limit of the sequence of maps

$$M \xleftarrow{\pi_k} \mathscr{E}_\Phi \longleftarrow \cdots \xleftarrow{\pi_{k+l,k+l-1}} \mathscr{E}_\Phi^{(l)} \xleftarrow{\pi_{k+l+1,k+l}} \mathscr{E}_\Phi^{(l+1)} \longleftarrow \cdots \tag{5}$$

and consists of "formal solutions" of $\mathscr{E}_\Phi$, i.e., possibly non-converging Taylor series fulfilling (4) for every l.

$J^\infty\pi$ is not a finite dimensional smooth manifold. However, it is a *pro-finite dimensional smooth manifold*. For an introduction to the geometry of pro-finite dimensional smooth manifolds see [20] (see also [21], and [22, 23] for different approaches). In the following we will only consider *regular* PDEs, i.e., PDEs $\mathscr{E}_\Phi$ such that $\mathscr{E}_\Phi^{(\infty)} \subset J^\infty\pi$ is a smooth pro-finite dimensional submanifold in $J^\infty\pi$, i.e., $\pi_{\infty,l}(\mathscr{E}_\Phi^{(\infty)}) \subset J^l\pi$ is a smooth submanifold and $\pi_{l+1,l} : \pi_{\infty,l+1}(\mathscr{E}_\Phi^{(\infty)}) \longrightarrow \pi_{\infty,l}(\mathscr{E}_\Phi^{(\infty)})$ is a smooth bundle.

There is a dual concept to the one of a pro-finite dimensional manifold, i.e., the concept of a *filtered smooth manifold* which will be used in the following. We do not give here a complete definition of a filtered manifold, which would take too much space. Rather, we will just outline it. Basically, a filtered smooth manifold is a(n equivalence class of) set(s) $\mathscr{O}$ together with a sequence of embeddings of closed submanifolds

$$\mathscr{O}_0 \xhookrightarrow{i_{0,1}} \mathscr{O}_1 \xhookrightarrow{i_{0,1}} \cdots \hookrightarrow \mathscr{O}_{k-1} \xhookrightarrow{i_{k-1,k}} \mathscr{O}_k \xhookrightarrow{i_{k,k+1}} \cdots \tag{6}$$

and inclusions $i_k : \mathscr{O}_k \hookrightarrow \mathscr{O}$, $k \geq 0$, such that $\mathscr{O}$ (together with the i_k's) is a direct limit of (6). It is associated to the sequence (6) a tower of epimorphisms of algebras

$$C^\infty(\mathscr{O}_0) \longleftarrow \cdots \xleftarrow{i^*_{k-1,k}} C^\infty(\mathscr{O}_k) \xleftarrow{i^*_{k,k+1}} C^\infty(\mathscr{O}_{k+1}) \longleftarrow \cdots. \tag{7}$$

We define $C^\infty(\mathscr{O})$ to be the inverse limit of the tower (7). Every element in $C^\infty(\mathscr{O})$ is naturally a function on $\mathscr{O}$. Thus, we interpret $C^\infty(\mathscr{O})$ as the algebra of *smooth functions on* $\mathscr{O}$. Clearly, there are canonical "restriction homomorphisms" $i_k^* : C^\infty(\mathscr{O}) \longrightarrow C^\infty(\mathscr{O}_k)$, $k \geq 0$. Differential calculus over $\mathscr{O}$ may then be introduced as *differential calculus over* $C^\infty(\mathscr{O})$ [20] respecting the sequence (7). Since the main constructions (smooth maps, vector fields, differential forms, jets and differential operators, etc.) of such calculus and their properties do not look very different from the analogous ones in finite-dimensional differential geometry we will not insist on this. Just as an instance, we report here the definition of a differential form ω on $\mathscr{O}$: it is just a sequence of differential forms $\omega_k \in \Lambda(\mathscr{O}_k)$, $k \geq 0$, such that $i^*_{k-1,k}(\omega_k) = \omega_{k-1}$ for all k.

Finally, notice that, allowing for the $\mathscr{O}_k$'s in (6) to be pro-finite dimensional manifolds we obtain a more general object then both a pro-finite dimensional or a filtered manifold which we will refer to generically as *infinite dimensional smooth manifold* or even just *smooth manifold* if this does not lead to confusion. Our main example of such a kind of infinite dimensional manifold will be presented in the beginning of Section 5.

3. The Cartan Distribution and the Lagrangian Formalism

Let $\pi : E \longrightarrow M$ and Φ be as in the previous section. In the following we will simply write J^∞ for $J^\infty\pi$ and $\mathscr{E}$ for $\mathscr{E}_\Phi^{(\infty)}$. $\mathscr{E}$ will be referred to simply as a PDE (imposed on sections of π) if this does not lead to confusion. Notice that for $\Phi = 0$, $\mathscr{E} = \mathscr{E}_\Phi^{(\infty)} = J^\infty$.

Recall that J^∞ is canonically endowed with the Cartan distribution

$$\mathscr{C} : J^\infty \ni \theta \longmapsto \mathscr{C}_\theta \subset T_\theta J^\infty$$

which is locally spanned by total derivatives, D_i, $i = 1, \ldots, n$. $\mathscr{C}$ is a flat connection in π_∞ which we call the *Cartan connection*. Moreover, it restricts to $\mathscr{E}$ in the sense that $\mathscr{C}_\theta \subset T_\theta\mathscr{E}$ for any $\theta \in \mathscr{E}$. Therefore, the (infinite prolongation of) any PDE is naturally endowed with an involutive distribution whose n-dimensional integral submanifolds are of the form $j_\infty s$, with $s : U \longrightarrow E$ a (local) solution of $\mathscr{E}_\Phi$, $U \subset M$ an open subset. In the following we will identify the space of n-dimensional integral submanifolds of $\mathscr{C}$ and the space of local solutions of $\mathscr{E}_\Phi$.

Let $\pi' : E' \longrightarrow M$ be another bundle and $\mathscr{E}' \subset J^\infty \pi'$ (the infinite prolongation of) a PDE imposed on sections of π'. A smooth map $F : \mathscr{E}' \longrightarrow \mathscr{E}$ is called a *morphism of PDEs* iff it respects the Cartan distributions, i.e., $(d_{\theta'} F)(\mathscr{C}_{\theta'}) \subset \mathscr{C}_{F(\theta')}$ for any $\theta' \in \mathscr{E}'$. The idea of *non-local variables* in the theory of PDEs can be formalized geometrically by special morphisms of PDEs called *coverings* [24] (see also [25]). A *covering* is a morphism $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ of PDEs which is a surjective and submersive. A covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ clearly sends local solution of $\widehat{\mathscr{E}}$ to local solutions of $\mathscr{E}$. If there exists a covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ of PDEs we also say that the PDE $\widehat{\mathscr{E}}$ *covers the PDE* $\mathscr{E}$ *(via* ψ*).* Fiber coordinates in the total space $\widehat{\mathscr{E}}$ of a covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ are naturally interpreted as non-local variables on $\mathscr{E}$. Also notice that given a solution s of the PDE $\mathscr{E}$, a covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ determines a whole family of solutions of $\widehat{\mathscr{E}}$ "projecting onto s via ψ", so that ψ may be interpreted, to some extent, as a fibration over the space of solutions of $\mathscr{E}$.

Many relevant constructions in the theory of PDEs (including Lax pairs, Bäcklund transformations, etc.) are duly formalized in geometrical terms by using coverings. As an instance which will be relevant in the following we report the following

Definition 1. *Let $\mathscr{E}$, $\mathscr{E}'$ be PDEs.* A Bäcklund transformation *between $\mathscr{E}$ and $\mathscr{E}'$ is a diagram*

$$
\begin{tikzcd}
& \widehat{\mathscr{E}} \arrow[dl, "\psi"'] \arrow[dr, "\psi'"] & \\
\mathscr{E} & & \mathscr{E}'
\end{tikzcd}
, \tag{8}
$$

where both ψ and ψ' are coverings.

According to the above interpretation of a covering, a Bäcklund transformation (8) is naturally interpreted as a non-local transformation of the PDE $\mathscr{E}$ into the PDE $\mathscr{E}'$ (and vice versa). Given a solution s of $\mathscr{E}$, the Bäcklund transformation (8) allows one, generically, to obtain a whole family of solutions of $\mathscr{E}'$ by first lifting to $\widehat{\mathscr{E}}$ via ψ (see above) and then projecting to $\mathscr{E}'$ via ψ'.

The Cartan distribution and the fibered structure $\pi_\infty : J^\infty \longrightarrow M$ of J^∞ determine a splitting of the tangent bundle $TJ^\infty \longrightarrow J^\infty$ into the Cartan or horizontal part $\mathscr{C}$ and the vertical (with respect to π_∞) part. Accordingly, $\Lambda^1(J^\infty)$ splits into a direct sum

$$\Lambda^1(J^\infty) = \mathscr{C}\Lambda^1 \oplus \overline{\Lambda}^1, \tag{9}$$

where $\mathscr{C}\Lambda^1 \subset \Lambda^1(J^\infty)$ is locally generated by Cartan forms $\ldots, du^\alpha_I - u^\alpha_{Ii} dx^i, \ldots$, while $\overline{\Lambda}^1$ is canonically isomorphic to $C^\infty(J^\infty) \otimes_{C^\infty(M)} \Lambda^1(M)$ and it is locally generated by forms $\ldots, dx^i, \ldots$. In view of splitting (9), $\Lambda(J^\infty)$ factorizes as $\Lambda(J^\infty) \simeq \mathscr{C}^\bullet\Lambda \otimes \overline{\Lambda}$ (here and in what follows tensor products will be always over $C^\infty(J^\infty)$ if not otherwise specified), where $\mathscr{C}^\bullet\Lambda := \bigoplus_p \mathscr{C}^p\Lambda^p$ and $\mathscr{C}^p\Lambda^p \subset \Lambda(J^\infty)$ is the $C^\infty(J^\infty)$-submodule generated by elements in the form $\omega_1 \cdots \omega_p$, $\omega_1, \ldots, \omega_p \in \mathscr{C}\Lambda^1$, $p \geq 0$, moreover, $\overline{\Lambda} := \bigoplus_q \overline{\Lambda}^q$ and $\overline{\Lambda}^q$ is the $C^\infty(J^\infty)$-submodule generated by elements in the form $\overline{\sigma}_1 \cdots \overline{\sigma}_q$, $\overline{\sigma}_1, \ldots, \overline{\sigma}_q \in \overline{\Lambda}^1$, $q \geq 0$. In particular, there are projections $\mathfrak{p}_{p,q} : \Lambda(J^\infty) \longrightarrow \mathscr{C}^p\Lambda^p \otimes \overline{\Lambda}^q$ for any $p, q \geq 0$. Correspondingly, the de Rham complex of J^∞, $(\Lambda(J^\infty), d)$, splits in a bi-complex $(\mathscr{C}^\bullet\Lambda \otimes \overline{\Lambda}, \overline{d}, d^V)$, defined by

$$\overline{d}(\omega \otimes \overline{\sigma}) := (\mathfrak{p}_{p,q+1} \circ d)(\omega \wedge \overline{\sigma}) \quad \text{and} \quad d^V(\omega \otimes \overline{\sigma}) := (\mathfrak{p}_{p+1,q} \circ d)(\omega \wedge \overline{\sigma}),$$

where $\omega \in \mathscr{C}^p\Lambda^p$ and $\overline{\sigma} \in \overline{\Lambda}^q$, $p, q \geq 0$, called the *variational bi-complex*, which allows a cohomological formulation of the calculus of variations [1, 19, 18]. In the second part of this section we briefly review it. $\overline{d}$ and d^V are called the *horizontal* and the *vertical de Rham differential*, respectively.

In the following we will understand isomorphism $\Lambda(J^\infty) \simeq \mathscr{C}^\bullet\Lambda \otimes \overline{\Lambda}$. The complex

$$0 \longrightarrow C^\infty(J^\infty) \xrightarrow{\overline{d}} \overline{\Lambda}^1 \xrightarrow{\overline{d}} \cdots \longrightarrow \overline{\Lambda}^q \xrightarrow{\overline{d}} \overline{\Lambda}^{q+1} \xrightarrow{\overline{d}} \cdots$$

is called the *horizontal de Rham complex*. An element $\mathscr{L} \in \overline{\Lambda}^n$ is naturally interpreted as a *lagrangian density* and its cohomology class $[\mathscr{L}] \in \overline{H}^n := H^n(\overline{\Lambda}, \overline{d})$ as an *action functional* on sections of π. The associated Euler-Lagrange equations can then be obtained as follows.

Consider the complex

$$0 \longrightarrow \mathscr{C}\Lambda^1 \xrightarrow{\overline{d}} \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^1 \xrightarrow{\overline{d}} \cdots \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q \xrightarrow{\overline{d}} \cdots, \tag{10}$$

and the $C^\infty(J^\infty)$-submodule $\varkappa^\dagger \subset \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$ generated by elements in $\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n \cap \Lambda^{n+1}(J^1\pi)$. $\varkappa^\dagger$ is locally spanned by elements $(du^\alpha - u^\alpha_i dx^i) \otimes d^n x$, where we put $d^n x := dx^1 \cdots dx^n$.

Theorem 1. [1] *Complex (10) is acyclic in the qth term, for $q \neq n$. Moreover, for any $\omega \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$ there exists a unique element $\boldsymbol{E}_\omega \in \varkappa^\dagger \subset \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q$ such that $\boldsymbol{E}_\omega - \omega = \overline{d}\vartheta$ for some $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ and the correspondence $H^n(\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}, \overline{d}) \ni [\omega] \longmapsto \boldsymbol{E}_\omega \in \varkappa^\dagger$ is a vector space isomorphism. In particular, for $\omega = d^V\mathscr{L}$, $\mathscr{L} \in \overline{\Lambda}^n$ being a lagrangian density locally given by $\mathscr{L} = L d^n x$, L a local function on $C^\infty(J^\infty)$, $\boldsymbol{E}(\mathscr{L}) := \boldsymbol{E}_\omega$ is locally given by $\boldsymbol{E}(\mathscr{L}) = \frac{\delta L}{\delta u^\alpha}(du^\alpha - u^\alpha_i dx^i) \otimes d^n x$ where $\frac{\delta L}{\delta u^\alpha} := (-)^{|I|} D_I \partial^I_\alpha L$ are the Euler-Lagrange derivatives of L.*

In view of the above theorem, $\boldsymbol{E}(\mathscr{L})$ does not depend on the choice of $\mathscr{L}$ in a cohomology class $[\mathscr{L}] \in \overline{H}^n$ and it is naturally interpreted as the left hand side of the Euler-Lagrange (EL) equations determined by $\mathscr{L}$. In the following we will denote by $\mathscr{E}_{EL} \subset J^\infty$ the (infinite prolongation of the) EL equations determined by a lagrangian density. Any $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ such that

$$\boldsymbol{E}(\mathscr{L}) - d^V\mathscr{L} = \overline{d}\vartheta \tag{11}$$

will be called a *Legendre form* [10]. Equation (11) may be interpreted as the *first variation formula* for the lagrangian density $\mathscr{L}$. In this respect, the existence of a global Legendre form was first discussed in [26].

Remark 1. *Notice that, if $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ is a Legendre form for a lagrangian density $\mathscr{L} \in \overline{\Lambda}^n$, then $\vartheta + d^V\varrho$ is a Legendre form for the cohomologous lagrangian density $\mathscr{L} + \overline{d}\varrho$, $\varrho \in \overline{\Lambda}^{n-1}$, which determines the same EL equations as $\mathscr{L}$. Moreover, any two Legendre forms ϑ, ϑ' for the same lagrangian density differ by a $\overline{d}$-closed, and, therefore, $\overline{d}$-exact form, i.e., $\vartheta - \vartheta' = \overline{d}\lambda$, for some $\lambda \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-2}$.*

Remark 2. *Finally, notice that complex (10) restricts to* holonomic sections *$j_\infty s$ of π_∞, s being a local sections of π, in the sense that for any such s, there is a (unique) complex*

$$0 \longrightarrow \mathscr{C}\Lambda^1|_j \xrightarrow{\overline{d}|_j} \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^1|_j \xrightarrow{\overline{d}|_j} \cdots \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q|_j \xrightarrow{\overline{d}|_j} \cdots, \tag{12}$$

where $j := j_\infty s$, such that the restriction map $\mathscr{C}\Lambda^1 \otimes \overline{\Lambda} \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}|_j = \mathscr{C}\Lambda^1|_j \otimes_{C^\infty(M)} \Lambda(M)$ is a morphism of complexes. Moreover, complex (12) is acyclic in the qth term and the correspondence defined by $H^n(\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n|_j, \overline{d}|_j) \ni [\omega|_j] \longmapsto \boldsymbol{E}_\omega|_j \in \varkappa^\dagger|_j$, $\omega \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$, is a vector space isomorphism.

4. Partial Differential Hamiltonian Systems

In [17] we defined a partial differential (PD in the following) analogue of the concept of hamiltonian system on an abstract symplectic manifold which we called a *PD-hamiltonian system*. In this section we briefly review those definitions and results in [17] which we will need in the following.

Let $\alpha : P \longrightarrow M$ be a fiber bundle, $A := C^\infty(P)$, $x^1, \ldots, x^n$ coordinates on M, $\dim M = n$, and $y^1, \ldots, y^m$ fiber coordinates on P, $\dim P = n + m$. Denote by $C(P, \alpha)$ the space of (Ehresmann) connections in α. $C(P, \alpha)$ identifies canonically with the space of sections of the first jet bundle $\alpha_{1,0} : J^1\alpha \longrightarrow P$ and in the following we will understand such identification. In particular, for $\nabla \in C(P, \alpha)$, we put $\ldots, \nabla^a_i := \nabla^*(y^a_i), \ldots$, $\ldots, y^a_i, \ldots$ being jet coordinates in $J^1\alpha$.

Denote by $\Lambda_1 = \bigoplus_k \Lambda_1^k \subset \Lambda(P)$ the differential (graded) ideal in $\Lambda(P)$ made of differential forms on P vanishing when pulled-back to fibers of α, by $\Lambda_p = \bigoplus_k \Lambda_p^k$ its p-th exterior power, $p \geq 0$, and by $V\Lambda(P, \alpha) = \bigoplus_k V\Lambda^k(P, \alpha)$ the quotient differential algebra $\Lambda(P)/\Lambda_1$, $d^V : V\Lambda(P, \alpha) \longrightarrow V\Lambda(P, \alpha)$ being its (quotient) differential.

Remark 3. *For instance, if $\alpha = \pi_\infty : P = J^\infty \longrightarrow M$, then, using the Cartan connection $\mathscr{C} \in C(J^\infty, \pi_\infty)$, one can canonically identify $V\Lambda^1(J^\infty, \pi_\infty)$ with $\mathscr{C}\Lambda^1$ and d^V with the vertical de Rham differential. More generally, for any $k \geq 0$, $V\Lambda^1(J^k, \pi_k) \otimes_{C^\infty(J^k\pi)} C^\infty(J^{k+1}\pi)$ identifies canonically with the $C^\infty(J^{k+1}\pi)$-module $\mathscr{C}\Lambda^1 \cap \Lambda(J^{k+1}\pi)$ of $(k+1)$th order* Cartan forms.

For any $k \geq 0$, denote by $'\Omega^{k+1}(P,\alpha)$ the $C^\infty(P)$-module of affine maps $C(P,\alpha) \longrightarrow V\Lambda^k(P,\alpha) \otimes_A \Lambda^n_n$. The linear part of an element in $'\Omega^{k+1}(P,\alpha)$ can be naturally understood as an element in $'\underline{\Omega}^{k+1}(P,\alpha) := V\Lambda^1(P,\alpha) \otimes_A V\Lambda^k(P,\alpha) \otimes_A \Lambda^{n-1}_{n-1}$. Denote by $\underline{\Omega}^{k+1}(P,\alpha)$ the subspace $V\Lambda^{k+1}(P,\alpha) \otimes_A \Lambda^{n-1}_{n-1}$ in $'\underline{\Omega}^{k+1}(P,\alpha)$ and by $\Omega^{k+1}(P,\alpha)$ the subspace in $'\Omega^{k+1}(P,\alpha)$ made of elements whose linear parts lie in $\underline{\Omega}^{k+1}(P,\alpha)$. Finally, put $\underline{\Omega}^0(P,\alpha) := \Omega^0(P,\alpha) := \Lambda^{n-1}_{n-1}$. Clearly, there are canonical projections $\Omega^{k+1}(P,\alpha) \longrightarrow \underline{\Omega}^{k+1}(P,\alpha)$, $k \geq 0$.

Theorem 2. *There are canonical isomorphisms of A-modules $\iota_k : \Lambda^{k+n-1}_{n-1} \longrightarrow \Omega^k(P,\alpha)$, and $\underline{\iota}_k : \Lambda^{k+n-1}_{n-1}/\Lambda^{k+n-1}_n \longrightarrow \underline{\Omega}^k(P,\alpha)$ such that diagrams*

$$\begin{array}{ccc} \Lambda^{k+n-1}_{n-1} & \longrightarrow & \Lambda^{k+n-1}_{n-1}/\Lambda^{k+n-1}_n \\ {\scriptstyle \iota_k}\downarrow & & \downarrow{\scriptstyle \underline{\iota}_k} \\ \Omega^k(P,\alpha) & \longrightarrow & \underline{\Omega}^k(P,\alpha) \end{array} \quad , \tag{13}$$

commute, $k \geq 0$.

In particular, isomorphism ι_k is defined by putting $\iota_k(\omega)(\nabla) := \mathfrak{p}^{k-1,n}_\nabla(\omega) \in V\Lambda^{k-1}(P,\alpha) \otimes_A \Lambda^{n-1}_{n-1}$, $\omega \in \Lambda^{k+n-1}_{n-1}$, $\nabla \in C(P,\alpha)$,

$$\mathfrak{p}^{k-1,n}_\nabla : \Lambda(P) \longrightarrow V\Lambda^{k-1}(P,\alpha) \otimes_A \Lambda^{n-1}_{n-1}$$

being the canonical projection determined by the connection ∇, $k \geq 0$. Notice that we can "transfer structures" from the Λ^{k+n-1}_{n-1}'s to the $\Omega^k(P,\alpha)$'s via the ι_k's. For instance, it is well defined a complex

$$\Omega^0(P,\alpha) \xrightarrow{\delta} \Omega^1(P,\alpha) \xrightarrow{\delta} \cdots \longrightarrow \Omega^k(P,\alpha) \xrightarrow{\delta} \Omega^{k+1}(P,\alpha) \xrightarrow{\delta} \cdots$$

by putting $\delta\omega := (\iota_{k+1} \circ d \circ \iota_k^{-1})(\omega) \in \Omega^{k+1}(P,\alpha)$, $\omega \in \Omega^k(P,\alpha)$. In the following we will understand the isomorphisms ι_k's and put $i_\nabla\omega := \omega(\nabla)$ for any $\omega \in \Omega^{k+1}(P,\alpha)$, $\nabla \in C(P,\alpha)$, $k \geq 0$. Notice that the action of ω on ∇ as above is actually point-wise and, therefore, can be restricted to maps. Namely, if $F : P_1 \longrightarrow P$ is a smooth map, $\square \in F^\circ(C(P,\alpha))$, then it is well defined an element $i_\square F^\circ(\omega) \in F^\circ(V\Lambda^{k-1}(P,\alpha) \otimes_A \Lambda^{n-1}_{n-1})$.

Definition 2. *A* PD-hamiltonian system *on the fiber bundle $\alpha : P \longrightarrow M$ is a δ-closed element $\omega \in \Omega^2(P,\alpha)$. The first order PDEs*

$$i_{j_1\sigma}\omega|_\sigma = 0$$

on (local) sections σ of α are called the PD-Hamilton equations determined by ω. *Geometrically, they correspond to the submanifold*

$$\mathscr{E}^{(0)}_\omega := \{\theta \in J^1\alpha : i_\theta\omega_p = 0,\ p = \alpha_{1,0}(\theta)\} \subset J^1\alpha.$$

Let $\omega \in \Omega^2(P,\alpha)$ be a PD-hamiltonian system on the bundle $\alpha : P \longrightarrow M$ and consider the subset $P_1 := \alpha_{1,0}(\mathscr{E}_\omega^{(0)}) \subset P$. In the following we will assume $P_1 \subset P$ to be a submanifold and $\alpha_1 := \alpha|_{P_1} : P_1 \longrightarrow M$ to be a subbundle of α. α_1 is called *the first constraint subbundle of* ω.

As an example, consider the following canonical constructions. Let $\pi : E \longrightarrow M$ be a fiber bundle and $\ldots, u^\alpha, \ldots$ fiber coordinates on E. $\Omega^1(E,\pi)$ (resp. $\underline{\Omega}^1(E,\pi)$) is the $C^\infty(E)$-module of sections of a vector bundle $\mu_0\pi : \mathscr{M}\pi \longrightarrow E$ (resp. $\tau_0^\dagger\pi : J^\dagger\pi \longrightarrow E$), called the *multimomentum bundle of* π (resp. the *reduced multimomentum bundle of* π). Recall that there is a distinguished element Θ_π in $\Omega^1(\mathscr{M}\pi, \mu\pi)$ (resp. $\underline{\Theta}_\pi \in \underline{\Omega}^1(J^\dagger\pi, \tau^\dagger\pi)$), where $\mu\pi := \pi \circ \mu_0\pi$ (resp. $\tau^\dagger\pi := \pi \circ \tau_0^\dagger\pi$), the tautological one [27], which in standard coordinates $\ldots, x^i, \ldots, u^\alpha, \ldots, p_\alpha^i, \ldots, p$ on $\mathscr{M}\pi$ (resp. $\ldots, x^i, \ldots, u^\alpha, \ldots, p_\alpha^i, \ldots$ on $J^\dagger\alpha$) is given by

$$\Theta_\pi = p_\alpha^i du^\alpha d^{n-1}x_i - pd^n x \quad (\text{resp.} \underline{\Theta}_\pi = p_\alpha^i d^V u^\alpha \otimes d^{n-1}x_i).$$

where $d^{n-1}x_i := i_{\partial/\partial x^i} d^n x$. $\delta\Theta_\pi \in \Omega^2(\mathscr{M}\pi, \mu\pi)$ is then a PD-hamiltonian system on $\mu\pi$ locally given by

$$\delta\Theta_\pi = dp_\alpha^i dq^\alpha d^{n-1}x_i - dpd^n x.$$

Notice that the corresponding PD-Hamilton equations $\mathscr{E}_{\delta\Theta_\pi}^{(0)}$ are empty and, in this sense, $\delta\Theta_\pi$ is a trivial PD-hamiltonian system. Nevertheless at least two (generically nontrivial) PD-hamiltonian systems are canonically determined by a first order lagrangian density in π, $\mathscr{L} \in \overline{\Lambda}^n \cap \Lambda(J^1\pi)$, one on $\tau^\dagger\pi$ and one on $\pi_1 \circ \underline{F}\mathscr{L}^\circ(\tau_0^\dagger\pi) : \underline{F}\mathscr{L}^\circ(J^\dagger\pi) \longrightarrow M$, $\underline{F}\mathscr{L} : J^1\pi \longrightarrow J^\dagger\pi$ being the (reduced) Legendre transform (see, for instance, [12] and [16]). In the next sections we show that a similar result occurs for a lagrangian density of any order.

Example 1. *A PD-hamiltonian system is canonically determined on a fiber bundle $\alpha : P \longrightarrow M$ as above, by the following data: a connection $\nabla \in C(P,\alpha)$ in α and a differential form $\mathscr{L} \in \Lambda_n^n$. Let $\ldots, q^A, \ldots$ be fiber coordinates in P and $\ldots, x^i, \ldots, q^A, \ldots, p_A^i, \ldots, p$ (resp. $\ldots, x^i, \ldots, q^A, \ldots, p_A^i, \ldots$) standard coordinates in $\mathscr{M}\alpha$ (resp. $J^\dagger\alpha$). Let $\mathscr{L}$ be locally given by $\mathscr{L} = Ld^n x$, $\mathscr{L}$ a local function on P. Denote by $\lambda\alpha : \Lambda(P,\alpha) \longrightarrow P$ the kernel bundle of the projection $\mathscr{M}\alpha \longrightarrow J^\dagger\alpha$. Obviously, ∇ induces a splitting of the exact sequence of vector bundles*

$$\begin{tikzcd} 0 \arrow[r] & \Lambda\alpha \arrow[r] & \mathscr{M}\alpha \arrow[r] & J^\dagger\alpha \arrow[r] \arrow[l, bend left, "\Sigma_\nabla"] & 0 \end{tikzcd},$$

which in local standard coordinates reads $\Sigma_\nabla^*(p) = p_A^i \nabla_i^A$. Put $\Theta_\nabla := \Sigma_\nabla^*(\Theta_\alpha) \in \Omega^1(J^\dagger\alpha, \tau^\dagger\alpha)$. In local standard coordinates, $\Theta_\nabla = p_A^i dq^A d^{n-1}x_i - p_A^i \nabla_i^A d^n x$. Put also,

$$\Theta_{\mathscr{L},\nabla} := \Theta_\nabla + (\tau_0^\dagger\pi)^*(\mathscr{L}) \in \Omega^1(J^\dagger\alpha, \tau^\dagger\alpha).$$

Locally, $\Theta_{\mathscr{L},\nabla} = p^i_A dq^A d^{n-1}x_i - E_{\mathscr{L},\nabla} d^n x$, *where* $E_{\mathscr{L},\nabla} := p^i_A \nabla^A_i - L$. *Finally, consider* $\omega_{\mathscr{L},\nabla} := \delta\Theta_{\mathscr{L},\nabla} \in \Omega^2(J^\dagger\alpha, \tau^\dagger\alpha)$. *Locally,*

$$\omega_{\mathscr{L},\nabla} = dp^i_A dq^A d^{n-1}x_i - dE_{\mathscr{L},\nabla} d^n x.$$

$\omega_{\mathscr{L},\nabla}$ *is the* PD-hamiltonian system on $\tau^\dagger\alpha$ determined by ∇ and $\mathscr{L}$. *The associated PD-Hamilton equations read locally*

$$\left\{ \begin{array}{l} p^i_{A},_i = \partial_A L - p^i_B \partial_A \nabla^B_i \\ q^A,_i = \nabla^A_i \end{array} \right. ,$$

where with "$\bullet$ $,_i$" we denoted the partial derivative of "$\bullet$" with respect to the ith independent variable x^i, $i = 1, \ldots, n$.

We conclude this section by discussing two examples of morphisms of PDEs coming from the theory of PD-hamiltonian systems.

Example 2. *Let* $\alpha : P \longrightarrow M$ *be a fiber bundle as above,* $\omega \in \Omega^2(P, \alpha)$ *a PD-hamiltonian system on it,* $\alpha' : P' \longrightarrow M$ *another fiber bundle,* $\beta : P' \longrightarrow P$ *a surjective, submersive, fiber bundle morphism, and* $\omega' := \beta^*(\omega) \in \Omega^2(P', \alpha')$. *Then* ω' *is a PD-hamiltonian system on* α'. *Denote by* $\mathscr{E} \subset J^\infty\alpha$ *(resp.* $\mathscr{E}' \subset J^\infty\alpha'$*) the* ∞*th prolongation of the PD-Hamilton equations determined by* ω *(resp.* ω'*). We want to compare* $\mathscr{E}$ *and* $\mathscr{E}'$. *In order to do this, notice, preliminarily, that* $J^\infty\alpha'$ *covers* $J^\infty\alpha$ *via* $j_\infty\beta : J^\infty\alpha' \longrightarrow J^\infty\alpha$. *Moreover, it can be easily checked that a local section* σ' *of* α' *is a solution of* $\mathscr{E}'$ *iff the section* $\beta \circ \sigma'$ *of* α *is a solution of* $\mathscr{E}$. *We now prove the formal version of this fact.*

Proposition 3. $(j_\infty\beta)(\mathscr{E}') \subset \mathscr{E}$ *and* $j_\infty\beta : \mathscr{E}' \longrightarrow \mathscr{E}$ *is a covering.*

Proof. Consider $j_1\beta : J^1\alpha' \longrightarrow J^1\alpha$. It is easy to check that $\mathscr{E}^{(0)}_{\omega'} := (j_1\beta)^{-1}(\mathscr{E}^{(0)}_\omega) \subset J^1\alpha'$. Similarly, $\mathscr{E}' := (j_\infty\beta)^{-1}(\mathscr{E}) \subset J^\infty\alpha'$. In particular, $j_\infty\beta : \mathscr{E}' \longrightarrow \mathscr{E}$ is the "restriction" of $j_\infty\beta : J^\infty\alpha' \longrightarrow J^\infty\alpha$ to $\mathscr{E} \subset J^\infty\alpha$ and, therefore, is a covering. □

Example 3. *Let* $\alpha : P \longrightarrow M$, $\omega \in \Omega^2(P, \alpha)$ *and* $\mathscr{E} \subset J^\infty\alpha$ *be as in the above example, and* $\alpha_1 : P_1 \longrightarrow M$ *the first constraint subbundle of* ω. *Assume that* $P_1 \subset P$ *is a submanifold and* α_1 *is a subbundle, and put* $\omega_1 := i^*_{P_1}(\omega) \in \Omega^2(P_1, \alpha_1)$. *Then* ω_1 *and it is a PD-hamiltonian system on* α_1. *Denote by* $\mathscr{E}_1 \subset J^\infty\alpha_1$ *the* ∞*th prolongation of the PD-Hamilton equations determined by* ω_1. *We want to compare* $\mathscr{E}$ *and* $\mathscr{E}_1$. *In order to do this, notice, preliminarily, that* $J^\infty\alpha_1$ *may be understood as a submanifold in* $J^\infty\alpha$ *via* $j_\infty i_{P_1} : J^\infty\alpha_1 \hookrightarrow J^\infty\alpha$. *Moreover, it can be easily checked that any solution of* $\mathscr{E}$ *is also a solution of* $\mathscr{E}_1$ *(while the vice-versa is generically untrue). We now prove the formal version of this fact.*

Proposition 4. $\mathscr{E} \subset \mathscr{E}_1$.

Proof. Recall that the projection $\alpha_{1,0} : J^1\alpha \longrightarrow P$ sends $\mathscr{E}^{(0)}_\omega$ to P_1. As a consequence, $\mathscr{E} \subset J^\infty\alpha_1$. Moreover, by definition of ∞th prolongation of a PDE, it is easy to check that

$$
\begin{aligned}
\mathscr{E} &= \mathscr{E} \cap J^\infty \alpha_1 \\
&= \{\theta = [\sigma]_x^\infty \in J^\infty \alpha : \mathrm{im}\,\sigma \subset P_1 \text{ and } [i_{j_1\sigma}\omega|_\sigma]_x^\infty = 0,\ x \in M\} \\
&= \{\theta = [\sigma]_x^\infty \in J^\infty \alpha_1 :\ [i_{j_1\sigma}\omega|_\sigma]_x^\infty = 0,\ x \in M\} \\
&\subset \{\theta = [\sigma]_x^\infty \in J^\infty \alpha_1 :\ [i_{j_1\sigma}\omega_1|_\sigma]_x^\infty = 0,\ x \in M\} \\
&= \mathscr{E}_1.
\end{aligned}
$$

□

5. Lagrangian-Hamiltonian Formalism

We show in this section that the Skinner-Rusk mixed lagrangian-hamiltonian formalism for first order mechanics [13, 14, 15] (see Section 1) is straightforwardly generalized to higher order lagrangian field theories.

First of all, let us present our main example of a filtered manifold. Let $\pi : E \longrightarrow M$ be a fiber bundle as above. Consider the infinite jet bundle $\pi_\infty : J^\infty \longrightarrow M$ for which $\Lambda_q^q = \overline{\Lambda}^q$, $q \geq 0$. Moreover, the $C^\infty(J^\infty)$-module $\underline{\Omega}^1(J^\infty, \pi_\infty) = \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ is canonically filtered by vector subspaces $W_k := \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1} \cap \Lambda(J^{k+1}\pi)$, $k \geq 0$. Denote by $\underline{\Omega}_k^1 \subset \underline{\Omega}^1(J^\infty, \pi_\infty)$ the $C^\infty(J^\infty)$-submodule generated by W_k, $k \geq 0$. Then, for all k, $\underline{\Omega}_k^1$ is canonically isomorphic to $C^\infty(J^\infty) \otimes_{C^\infty(J^{k+1})} W_k$ and

$$\underline{\Omega}_0^1 \subset \underline{\Omega}_1^1 \subset \cdots \subset \underline{\Omega}_k^1 \subset \underline{\Omega}_{k+1}^1 \subset \cdots \subset \underline{\Omega}^1(J^\infty, \pi_\infty), \tag{14}$$

is a sequence of $C^\infty(J^\infty)$-submodules. Notice that, for any k, $\underline{\Omega}_k^1$ is the module of sections of a finite-dimensional vector bundle $\tau_{0,k}^\dagger : J_k^\dagger \longrightarrow J^\infty$. Moreover, the inclusions (14) determine inclusions

$$J_0^\dagger \subset J_1^\dagger \subset \cdots \subset J_k^\dagger \subset J_{k+1}^\dagger \subset \cdots$$

of vector bundles. $J^\dagger := \bigcup_k J_k^\dagger$ is then an infinite dimensional (filtered) manifold and the canonical projection $\tau_0^\dagger : J^\dagger \longrightarrow J^\infty$ an *infinite dimensional vector bundle over* J^∞ whose module of sections identifies naturally with $\underline{\Omega}^1(J^\infty, \pi_\infty)$. We conclude that $\tau_0^\dagger : J^\dagger \longrightarrow J^\infty$ is naturally interpreted as the reduced multimomentum bundle of π_∞. Denote by $\ldots, x^i, \ldots, u_I^\alpha, \ldots, p_\alpha^{I.i}, \ldots$ standard coordinates in $J^\dagger$. Notice that any (local) element $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1} = \underline{\Omega}^1(J^\infty, \pi_\infty)$, in particular a (local) Legendre form, is naturally interpreted as a section $\vartheta : U' \longrightarrow J^\dagger$, $U' \subset J^\infty$ an open subset. Put then $\ldots, \vartheta_\alpha^{I.i} := \vartheta^*(p_\alpha^{I.i}), \ldots$ which are local functions on J^∞ such that $\vartheta = \vartheta_\alpha^{I.i}(du_I^\alpha - u_{Ii}^\alpha dx^i) \otimes d^{n-1}x_i$. It follows that, locally,

$$\overline{d}\vartheta = -(D_i\vartheta_\alpha^{I.i} + \delta_{Ji}^I \vartheta_\alpha^{J.i})(du_I^\alpha - u_{Ii}^\alpha dx^i) \otimes d^n x \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n,$$

where $\delta_K^I = 0$ if $I \neq K$, while $\delta_K^I = 1$ if $I = K$, $I, K \in \mathbb{M}_n$. In the following we will also consider the bundle structures $\tau_k^\dagger := \pi_\infty \circ \tau_{0,k}^\dagger : J_k^\dagger \longrightarrow M$, $k \geq 0$, and $\tau^\dagger := \pi_\infty \circ \tau_0^\dagger : J^\dagger \longrightarrow M$.

Now, in Example 1, put $\alpha = \pi_\infty : P = J^\infty \longrightarrow M$ and $\nabla = \mathscr{C}$, the Cartan connection in π_∞. $\mathscr{L} \in \Lambda^n_n = \overline{\Lambda}^n$ is then a lagrangian density in π. Put $\Sigma := \Sigma_{\mathscr{C}}$, $\Theta_{\mathscr{L}} := \Theta_{\mathscr{L},\mathscr{C}}$ and $\omega_{\mathscr{L}} := \omega_{\mathscr{L},\mathscr{C}}$. $\omega_{\mathscr{L}}$ is a PD-hamiltonian system in $\tau^\dagger : J^\dagger \longrightarrow M$ canonically determined by $\mathscr{L}$. Locally,

$$\omega_{\mathscr{L}} = dp^{I.i}_\alpha du^\alpha_I d^{n-1}x_i - dE_{\mathscr{L}} d^n x,$$

where $E_{\mathscr{L}} := p^{I.i}_\alpha u^\alpha_{Ii} - L$. Let $\sigma : U \longrightarrow J^\dagger$ be a local section of $\tau^\dagger$, $U \subset M$ an open subset, and $j := \tau^\dagger_0 \circ \sigma : U \longrightarrow M$. Put $\ldots, \sigma^\alpha_I := \sigma^*(u^\alpha_I) = j^*(u^\alpha_I), \ldots, \sigma^{I.i}_\alpha := \sigma^*(p^{I.i}_\alpha), \ldots$ which are local functions on M. Then, locally,

$$i_{\dot\sigma}\omega_{\mathscr{L}}|_\sigma = [(-\sigma^{I.i}_\alpha{}_{,i} - \delta^I_{Ji}\sigma^{J.i}_\alpha + \partial^I_\alpha L \circ j) d^V u^I_\alpha|_\sigma + (\sigma^\alpha_I{}_{,i} - \sigma^\alpha_{Ii}) d^V p^{I.i}_\alpha|_\sigma] d^n x,$$

and the PD-Hamilton equations determined by $\omega_{\mathscr{L}}$ read locally

$$\left\{ \begin{array}{l} p^{I.i}_\alpha{}_{,i} = \partial^I_\alpha L - \delta^I_{Ji}\, p^{J.i}_\alpha \\ u^\alpha_I{}_{,i} = u^\alpha_{Ii} \end{array} \right. .$$

We call such equations the *Euler-Lagrange-Hamilton (ELH) equations determined by the lagrangian density $\mathscr{L}$*. Notice that they are first order PDEs (with an infinite number of dependent variables). Denote by $\mathscr{E}_{ELH} \subset J^\infty \tau^\dagger$ their infinite prolongation. In the following theorem we characterize solutions of $\mathscr{E}_{ELH}$. As a byproduct, we derive the relationship between the ELH equations and the EL equations.

Theorem 5. *A local section $\sigma : U \longrightarrow J^\dagger$ of $\tau^\dagger$, $U \subset M$ an open subset, is a solution of the ELH equations determined by the lagrangian density $\mathscr{L}$ iff it is locally of the form $\sigma = \vartheta \circ j^\infty s$ where 1) $s : U \longrightarrow E$ is a solution of the EL equations $\mathscr{E}_{EL}$ and 2) $\vartheta : U' \longrightarrow J^\dagger$ is a Legendre form for $\mathscr{L}$, $U' \subset J^\infty$ an open subset.*

Proof. Let $\sigma : U \longrightarrow J^\dagger$ be a local section of $\tau^\dagger$, $U \subset M$ an open subset. First of all, let σ be in the form $\sigma = \vartheta \circ j$ where 1) $j : U \longrightarrow J^\infty$ is a local section of π_∞ and 2) $\vartheta : U' \longrightarrow J^\dagger$ is a local section of $\tau^\dagger_0 : J^\dagger \longrightarrow J^\infty$, $U' \subset J^\infty$ an open subset. Then,

$$\sigma^{I.i}_\alpha{}_{,i} = D_i \vartheta^{I.i}_\alpha \circ j.$$

Therefore, locally,

$$\begin{aligned} i_{\dot\sigma}\omega_{\mathscr{L}}|_\sigma &= [[(-D_i\vartheta^{I.i}_\alpha - \delta^I_{Ji}\vartheta^{J.i}_\alpha + \partial^I_\alpha L) \circ j] d^V u^I_\alpha|_j + (j^\alpha_I{}_{,i} - j^\alpha_{Ii}) d^V p^{I.i}_\alpha|_\sigma] \otimes d^n x \\ &= (\overline{d}\vartheta + d^V \mathscr{L})|_j + (j^\alpha_I{}_{,i} - j^\alpha_{Ii}) d^V p^{I.i}_\alpha|_\sigma \otimes d^n x, \end{aligned}$$

where $\ldots, j^\alpha_I := j^*(u^\alpha_I), \ldots$ and they are local functions on M. Thus, if ϑ is a Legendre form and $j = j_\infty s$ for some local solution $s : U \longrightarrow E$ of the EL equations then, in particular, $j^\alpha_I{}_{,i} = j^\alpha_{Ii}$, $I \in \mathbb{M}_n$, $\alpha = 1, \ldots, m$, $i = 1, \ldots, n$, and

$$i_{\dot\sigma}\omega_{\mathscr{L}}|_\sigma = (\overline{d}\vartheta + d^V \mathscr{L})|_j + (j^\alpha_I{}_{,i} - j^\alpha_{Ii}) d^V p^{I.i}_\alpha|_\sigma \otimes d^n x = \boldsymbol{E}(\mathscr{L})|_j = 0.$$

On the other hand, let $\sigma : U \longrightarrow J^\dagger$ be a local section of $\tau^\dagger$ and $j := \tau^\dagger_0 \circ \sigma : U \longrightarrow J^\infty$. Locally, there always exists a section $\vartheta : U' \longrightarrow J^\dagger$ of $\tau^\dagger_0$, such that $\sigma = \vartheta \circ j$. Notice, preliminarily, that ϑ is not uniquely determined by σ except for its restriction to $\operatorname{im} j$. If σ is a solution of the ELH equations then, locally,

$$0 = i_{\dot{\sigma}}\omega_{\mathscr{L}}|_\sigma = (\overline{d}\vartheta + d^V\mathscr{L})|_j + (j^\alpha_I{}_{,i} - j^\alpha_{Ii})d^V p^{I.i}_\alpha|_\sigma \otimes d^n x.$$

Since $(d^V p^{I.i}_\alpha)|_\sigma \otimes d^n x$ and $(\overline{d}\vartheta + d^V\mathscr{L})|_j$ are linearly independent, it follows that

$$\begin{cases} (\overline{d}\vartheta + d^V\mathscr{L})|_j = 0 \\ j^\alpha_I{}_{,i} = j^\alpha_{Ii}. \end{cases}.$$

In particular, $j = j_\infty s$, where s is the local section of π defined as $s = \pi_{\infty,0} \circ j$.

Now, let ϑ_0 be a Legendre form for $\mathscr{L}$. Then $d^V\mathscr{L} = \boldsymbol{E}(\mathscr{L}) - \overline{d}\vartheta_0$ and, therefore, $(\overline{d}\vartheta - \overline{d}\vartheta_0 + \boldsymbol{E}(\mathscr{L}))|_j = 0$. Recall that $\overline{d}$ restricts to $j = j_\infty s$ (Remark 2). Thus,

$$\overline{d}|_j(\vartheta - \vartheta_0)|_j = \boldsymbol{E}(\mathscr{L})|_j.$$

In particular, $\boldsymbol{E}(\mathscr{L})|_j$ is $\overline{d}|_j$-exact. In view of Remark 2, this is only possible if $\boldsymbol{E}(\mathscr{L})|_j = 0$, i.e., s is a solution of the EL equations. We conclude that

$$\overline{d}|_j(\vartheta - \vartheta_0)|_j = 0,$$

i.e., $(\vartheta - \vartheta_0)|_j$ is $\overline{d}|_j$-closed. Again in view of Remark 2, this shows that, locally,

$$(\vartheta - \vartheta_0)|_j = \overline{d}|_j\nu|_j = \overline{d}\nu|_j$$

for some $\nu \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-2}$. In particular, we can put $\vartheta = \vartheta_0 + \overline{d}\nu$ and, therefore, ϑ is a Legendre form for $\mathscr{L}$. $\square$

We now prove a *formal version* of the above theorem. Put $p := \tau^\dagger_{\infty,0} \circ \tau^\dagger_0 : J^\infty\tau^\dagger \longrightarrow J^\infty$.

Theorem 6. *$p(\mathscr{E}_{ELH}) \subset \mathscr{E}_{EL}$ and $p : \mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ is a covering of PDEs.*

Proof. In $J^\infty\tau^\dagger$ consider the submanifold $\mathscr{E}_L$ made of ∞th jets of (local) sections $\sigma : U \longrightarrow J^\dagger$, $U \subset M$ an open subset, in the form $\sigma = \vartheta \circ j_\infty s$, where $s : U \longrightarrow E$ is a local section of π, and $\vartheta : U' \longrightarrow J^\dagger$ is a local Legendre form, $U' \subset J^\infty$ an open subset. It can be easily checked that $\mathscr{E}_L$ is locally defined by

$$\begin{cases} p^{I.i}_\alpha{}_{|Ki} + \delta^I_{Ji}\, p^{J.i}_\alpha{}_{|K} = D_K(\partial^I_\alpha L) - \delta^I_{\mathsf{O}} D_K \frac{\delta L}{\delta u^\alpha} \\ u^\alpha_{I|K} = u^\alpha_{IK} \end{cases}. \tag{15}$$

Clearly, the Cartan distribution restricts to $\mathscr{E}_L$ and, therefore, $\mathscr{E}_L$ can be interpreted as a PDE. Moreover, it is easily seen from (15) that $\mathscr{E}_L$ covers J^∞ via p. Denote by

$$D'_j = \partial_j + u^\alpha_{I|Jj}\frac{\partial}{\partial u^\alpha_{I|J}} + p^{I.i}_\alpha{}_{|Jj}\frac{\partial}{\partial p^{I.i}_\alpha{}_{|J}}$$

the jth total derivative on $J^\infty\tau^\dagger$, $j = 1, \ldots, n$. $\mathscr{E}_{ELH}$ is locally defined by

$$\begin{cases} p^{I.i}_\alpha{}_{|Ki} = D'_K(\partial^I_\alpha L) - \delta^I_{Ji}\, p^{J.i}_\alpha{}_{|K} \\ u^\alpha_{I|Ki} = u^\alpha_{Ii|K} \end{cases}, \tag{16}$$

which is clearly equivalent to

$$\left\{ \begin{array}{l} p_{\alpha}^{I.i}{}_{|Ki} = D_K(\partial_\alpha^I L) - \delta_{Ji}^I \, p_\alpha^{J.i}{}_{|K} \\ u_{I\,|K}^\alpha = u_{IK}^\alpha \end{array} \right. .$$

Moreover, on $\mathscr{E}_{ELH}$

$$(-)^{|I|} p_\alpha^{I.i}{}_{|KIi} = D_K \tfrac{\delta L}{\delta u^\alpha} - (-)^{|I|} \delta_{Ji}^I \, p_\alpha^{J.i}{}_{|KI} = D_K \tfrac{\delta L}{\delta u^\alpha} + (-)^{|I|} p_\alpha^{I.i}{}_{|KIi},$$

and, therefore, $D_K \frac{\delta L}{\delta u^\alpha} = 0$, $K \in \mathbb{M}_n$, $\alpha = 1, \ldots, m$. It then follows from (15), that $\mathscr{E}_{ELH} = \mathscr{E}_L \cap p^{-1}(\mathscr{E}_{EL})$. In particular, $p : \mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ is the "restriction" of $p : \mathscr{E}_L \longrightarrow J^\infty$ to $\mathscr{E}_{EL} \subset J^\infty$ and, therefore, is a covering. □

6. Natural Transformations of Euler-Lagrange-Hamilton Equations

Properties of Legendre forms discussed in Remark 1 correspond to specific properties of the ELH equations which we discuss in this section.

First of all, notice that the ELH equations are canonically associated to a lagrangian density. But, how do the ELH equations change when changing the lagrangian density into a $\overline{d}$-cohomology class? In particular, does an action functional uniquely determine a system of ELH equations or not? In order to answer these questions consider $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$. ϑ determines an automorphism $\Psi_\vartheta : J^\dagger \longrightarrow J^\dagger$ of the fiber bundle $\tau_0^\dagger$ via

$$\Psi_\vartheta(P) := P - \vartheta_\theta, \quad P \in J^\dagger, \, \theta = \tau^\dagger(P) \in J^\infty.$$

In particular, $\tau_0^\dagger \circ \Psi_\vartheta = \tau_0^\dagger$. Clearly, $\Psi_\vartheta^{-1} = \Psi_{-\vartheta}$.

Lemma 7. *$\Psi_\vartheta^*(\omega_{\mathscr{L}}) = \omega_{\mathscr{L}} - \tau_0^{\dagger *}(d\vartheta)$.*

Proof. Compute,

$$\begin{aligned} \Psi_\vartheta^*(\omega_{\mathscr{L}}) &= \Psi_\vartheta^*(\delta\Theta_{\mathscr{L}}) \\ &= d\Psi_\vartheta^*(\Theta_{\mathscr{L}}) \\ &= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + (\Psi_\vartheta^* \circ \tau_0^{\dagger *})(\mathscr{L})] \\ &= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + (\tau_0^\dagger \circ \Psi_\vartheta)^*(\mathscr{L})] \\ &= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + \tau_0^{\dagger *}(\mathscr{L})]. \end{aligned}$$

Now, since, locally, $\ldots, \Psi_\vartheta^*(p_\alpha^{I.i}) = p_\alpha^{I.i} - \vartheta_\alpha^{I.i}, \ldots$, we have

$$\begin{aligned} (\Psi_\vartheta^* \circ \Sigma^*)(p_\alpha^{I.i}) &= p_\alpha^{I.i} - \vartheta_\alpha^{I.i}, \\ (\Psi_\vartheta^* \circ \Sigma^*)(p) &= (p_\alpha^{I.i} - \vartheta_\alpha^{I.i}) u_{Ii}^\alpha. \end{aligned}$$

Thus,

$$\begin{aligned} (\Psi_\vartheta^* \circ \Sigma^*)(\Theta) &= (p_\alpha^{I.i} - \vartheta_\alpha^{I.i}) du_I^\alpha d^{n-1}x_i - (p_\alpha^{I.i} - \vartheta_\alpha^{I.i}) u_{Ii}^\alpha d^n x \\ &= \Sigma^*(\Theta) - \tau_0^{\dagger *}(\vartheta). \end{aligned}$$

We conclude that

$$\begin{aligned}\Psi_\vartheta^*(\omega_{\mathscr{L}}) &= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + \tau_0^{\dagger *}(\mathscr{L})] \\ &= d[\Sigma^*(\Theta) - \tau_0^{\dagger *}(\vartheta) + \tau_0^{\dagger *}(\mathscr{L})] \\ &= \omega_{\mathscr{L}} - \tau_0^{\dagger *}(d\vartheta).\end{aligned}$$

□

Theorem 8. *Let $\mathscr{L}' = \mathscr{L} + \overline{d}\varrho$, $\varrho \in \overline{\Lambda}^{n-1}$, be another lagrangian density (thus, $\mathscr{L}'$ determines the same EL equations as $\mathscr{L}$). Then $\Psi_{d^V\varrho}^*(\omega_{\mathscr{L}}) = \omega_{\mathscr{L}'}$.*

Proof. Notice, preliminarily, that

$$\begin{aligned}\tau_0^{\dagger *}(dd^V\varrho) &= \tau_0^{\dagger *}(\overline{d}d^V\varrho) \\ &= -\tau_0^{\dagger *}(d^V\overline{d}\varrho) \\ &= -\tau_0^{\dagger *}(d\overline{d}\varrho) \\ &= -d\tau_0^{\dagger *}(\overline{d}\varrho).\end{aligned}$$

Therefore, in view of the above lemma,

$$\begin{aligned}\Psi_{d^V\varrho}^*(\omega_{\mathscr{L}}) &= \omega_{\mathscr{L}} - \tau_0^{\dagger *}(dd^V\varrho) \\ &= d[\Sigma^*(\Theta) + \tau_0^{\dagger *}(\mathscr{L})] + d\tau_0^{\dagger *}(\overline{d}\varrho) \\ &= d[\Sigma^*(\Theta) + \tau_0^{\dagger *}(\mathscr{L} + \overline{d}\varrho)] \\ &= \delta\Theta_{\mathscr{L}'} \\ &= \omega_{\mathscr{L}'}.\end{aligned}$$

□

Corollary 9. *An action $[\mathscr{L}] \in \overline{H}^n$, $\mathscr{L} \in \overline{\Lambda}^n$, uniquely determines a system of ELH equations, modulo isomorphisms of PD-hamiltonian systems.*

Therefore the ELH equations are basically determined by the sole action functional and not a specific lagrangian density.

Theorem 10. *Let $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ be $\overline{d}$-closed, hence $\overline{d}$-exact. Then, for every lagrangian density $\mathscr{L} \in \overline{\Lambda}^n$, Ψ_ϑ is a symmetry of the ELH equations determined by $\mathscr{L}$ in the sense that $j_\infty\Psi_\vartheta : J^\infty\tau^\dagger \longrightarrow J^\infty\tau^\dagger$ preserves $\mathscr{E}_{ELH}$.*

Proof. By definition of infinite prolongations of a PDE and infinite prolongation of a morphism of bundles, it is enough to prove that $j_1\Psi_\vartheta : J^1\tau^\dagger \longrightarrow J^1\tau^\dagger$ preserves $\mathscr{E}_{ELH}^{(0)} := \mathscr{E}_{\omega_{\mathscr{L}}}^{(0)} \subset J^1\tau^\dagger$. Notice, preliminarily, that, in view of the proof of Theorem 5, we have

$$(j_1\tau_0^\dagger)(\mathscr{E}_{ELH}^{(0)}) \subset \operatorname{im}\mathscr{C} \subset J^1\pi_\infty.$$

Now, let $c \in \mathscr{E}^{(0)}_{ELH}$, $P := \tau^{\dagger}_{1,0}(c)$ and $\xi \in T_P J^{\dagger}$ be a tangent vector, vertical with respect to $\tau^{\dagger}$. Consider also $c' := (j_1\Psi_{-\vartheta})(c)$, $P' := \Psi_{-\vartheta}(P) = \tau^{\dagger}_{1,0}(c')$ and $\xi' := d\Psi_{-\vartheta}(\xi)$. In particular, $\xi' \in T_{P'}J^{\dagger}$ is vertical with respect to $\tau^{\dagger}$ as well. Let us prove that $c' \in \mathscr{E}^{(0)}_{ELH}$. In view of Lemma 7,

$$\Psi^*_{\vartheta}(\omega_{\mathscr{L}}) = \omega_{\mathscr{L}} - \tau^{\dagger}_0{}^*(d\vartheta) = \omega_{\mathscr{L}} - \tau^{\dagger}_0{}^*(d^V\vartheta),$$

so that $\omega_{\mathscr{L}} = \Psi^*_{\vartheta}(\omega_{\mathscr{L}}) + \tau^{\dagger}_0{}^*(d^V\vartheta)$. Compute

$$i_{\xi'}i_{c'}(\omega_{\mathscr{L}})_{P'} = i_{\xi'}i_{c'}[\Psi^*_{\vartheta}(\omega_{\mathscr{L}})_{P'} + \tau^{\dagger}_0{}^*(d^V\vartheta)_{P'}] = i_{\xi}i_c(\omega_{\mathscr{L}})_P + i_{\xi''}i_{\mathscr{C}_\theta}(d^V\vartheta)_\theta = 0,$$

where $\theta = \tau^{\dagger}_0(P) \in J^{\infty}$ and $\xi'' = d\tau^{\dagger}_0(\xi) \in T_\theta J^{\infty}$ is a tangent vector, vertical with respect to π_∞. It follows from the arbitrariness of ξ', that $i_{c'}(\omega_{\mathscr{L}})_{P'} = 0$. □

7. Hamiltonian Formalism

In this section we present our proposal of an hamiltonian formalism for higher order lagrangian field theories. Such proposal is free from ambiguities in that it only depends on the choice of a lagrangian density and its order. Moreover, cohomologous lagrangian of the same order determine equivalent "hamiltonian theories".

First of all, we define a "finite dimensional version" of the ELH equations. In order to do this, notice that, in view of Remark 3, W_k is canonically isomorphic to the $C^{\infty}(J^{k+1}\pi)$-module of section of the induced bundle $\mathscr{C}^{\dagger}\pi_{k+1} := \pi^{\circ}_{k+1,k}(\tau^{\dagger}_0\pi_k) : \mathscr{C}^{\dagger}J^{k+1} := \pi^{\circ}_{k+1,k}(J^{\dagger}\pi_k) \longrightarrow J^{k+1}\pi$ from the vector bundle $\tau^{\dagger}\pi_k : J^{\dagger}\pi_k \longrightarrow J^k\pi$ via $\pi_{k+1,k} : J^{k+1}\pi \longrightarrow J^k\pi$, $k \geq 0$. We conclude that $\tau^{\dagger}_{0,k} : J^{\dagger}_k \longrightarrow J^{\infty}$ is canonically isomorphic to the pull-back bundle $\pi^{\circ}_{\infty,k}(\tau^{\dagger}_0\pi_k) : \pi^{\circ}_{\infty,k}(J^{\dagger}\pi_k) \longrightarrow J^{\infty}$, $k \geq 0$. Denote by $i_k : J^{\dagger}_k \hookrightarrow J^{\dagger}$ the inclusion and by $q'_{0,k} : J^{\dagger}_k \longrightarrow \mathscr{C}^{\dagger}J^{k+1}$ and $q_{0,k} : \mathscr{C}^{\dagger}J^{k+1} \longrightarrow J^{\dagger}\pi_k$ the canonical projections, $k \geq 0$. Notice that the $p^{I.i}_{\alpha}$'s, $|I| \leq k$, identify with the pull-back via $q'_{0,k}$ (resp. $q_{0,k}$) of the corresponding natural coordinates on $\mathscr{C}^{\dagger}J^{k+1}$ (resp. $J^{\dagger}\pi_k$) which we still denote by $\ldots, p^{I.i}_{\alpha}, \ldots$, $k \geq 0$.

Now, let $\mathscr{L} \in \overline{\Lambda}^n$ be a lagrangian density of the order $l+1$, i.e., $\mathscr{L} \in \overline{\Lambda}^n \cap \Lambda(J^{l+1})$. Put $\omega'_l := i^*_l(\omega_{\mathscr{L}}) \in \Omega^2(J^{\dagger}_l, \tau^{\dagger}_l)$. ω'_l is a PD-hamiltonian system on $\tau^{\dagger}_l$, and it is locally given by

$$\omega'_l = \sum_{|I|\leq l} dp^{I.i}_{\alpha} du^{\alpha}_I d^{n-1}x_i - dE_l d^n x,$$

where $E_l := i^*_l(E_{\mathscr{L}}) = \sum_{|I|\leq l} p^{I.i}_{\alpha} u^{\alpha}_{Ii} - L$. Notice that $\omega'_l = q'_{0,l}{}^*(\omega_l)$ for a (unique) PD-hamiltonian system $\omega_l \in \Omega^2(\mathscr{C}^{\dagger}J^{l+1}, q_{l+1})$ on the bundle $q_{l+1} := \pi_{l+1} \circ \mathscr{C}^{\dagger}\pi_{l+1} : \mathscr{C}^{\dagger}J^{l+1} \longrightarrow M$, locally given by the same formula as ω'_l. ω_l is a constrained PD-hamiltonian system, i.e., its first constraint bundle $q : \mathscr{P} \longrightarrow M$ is a proper subbundle of q_{l+1}. Let us compute it. Let $P \in \mathscr{C}^{\dagger}J^{l+1}$ and $\theta := \mathscr{C}^{\dagger}\pi_{l+1}(P) \in J^{l+1}$. Then $P \in \mathscr{P}$ iff there exists $c \in J^1 q_{l+1}$ such that $i_c(\omega_l)_P = 0$, i.e., iff there exist real numbers

$\ldots, c^{a}_{J,j}, \ldots, c^{I.i}_{\alpha}{}_{,j}, \ldots$, $|I| \leq l$, such that

$$\begin{cases} c^{I.i}_{\alpha}{}_{,i} = (\partial^{I}_{\alpha} L)(\theta) - \delta^{I}_{Ji}\, P^{J.i}_{\alpha}, & |I| \leq l+1 \\ c^{\alpha}_{J,i} = P^{\alpha}_{Ji}, & |J| \leq l+1 \end{cases}$$

where we put $c^{I.i}_{\alpha}{}_{,i} = 0$ for $|I| = l+1$, and $\ldots, P^{\alpha}_{Ji} := p^{\alpha}_{Ji}(P), \ldots$, $|J| \leq l+1$, $\alpha = 1, \ldots, m$. Thus, for $|I| = l+1$, P should be a solution of the system

$$\partial^{I}_{\alpha} L - \delta^{I}_{Ji}\, p^{J.i}_{\alpha} = 0, \quad |I| = l+1. \tag{17}$$

Equations (17) define $\mathscr{P}$ locally.

Assumption 1. *We will always assume $\mathscr{P}$ to be a submanifold in $\mathscr{C}^{\dagger} J^{l+1}$ and $q = q_{l+1}|_{\mathscr{P}} : \mathscr{P} \longrightarrow M$ a smooth subbundle of q_l. Similarly, we assume $\mathscr{P}_0 := q_{0,l}(\mathscr{P})$ to be a submanifold in $J^{\dagger}\pi_l$ and $\tau^{\dagger}\pi_l|_{\mathscr{P}_0} : \mathscr{P}_0 \longrightarrow M$ to be a smooth subbundle of $\tau^{\dagger}\pi_l$. Finally, we assume $q_{0,l}|_{\mathscr{P}} : \mathscr{P} \longrightarrow \mathscr{P}_0$ to be a smooth bundle with connected fibers.*

Notice that, as usual, all the above regularity conditions are true if we restrict all the involved maps to suitable open subsets.

The following commutative diagram summarizes the above described picture:

$$\begin{tikzcd}
J^{\dagger} \arrow[d, "\tau_0^{\dagger}"'] & J_l^{\dagger} \arrow[l, hook', "i_l"'] \arrow[dl, "\tau^{\dagger}_{0,l}"'] \arrow[d] & \\
J^{\infty} \arrow[d, "\pi_{\infty,l+1}"'] & \mathscr{C}^{\dagger} J^{l+1} \arrow[dl, "\mathscr{C}^{\dagger}\pi_{l+1}"'] \arrow[d, "q_{0,l}"] & \mathscr{P} \arrow[l, hook', "i_{\mathscr{P}}"'] \arrow[d] \\
J^{l+1} \arrow[d, "\pi_{l+1,l}"'] & J^{\dagger}\pi_l \arrow[dl, "\tau_0^{\dagger}\pi_l"'] \arrow[ddl, "\tau^{\dagger}\pi_l"] & \mathscr{P}_0 \arrow[l, hook', "i_{\mathscr{P}_0}"'] \arrow[ddll] \\
J^{l} \arrow[d, "\pi_l"'] & & \\
M & &
\end{tikzcd}\quad .$$

Theorem 11. *Under the regularity Assumption 1, there exists a unique PD-hamiltonian system ω_0 in $\tau^{\dagger}\pi_l|_{\mathscr{P}_0} : \mathscr{P}_0 \longrightarrow M$, such that $i^{*}_{\mathscr{P}}(\omega_l) = q_{0,l}|^{*}_{\mathscr{P}}(\omega_0)$.*

Proof. Since $q_{0,l}|_{\mathscr{P}} : \mathscr{P} \longrightarrow \mathscr{P}_0$ has connected fibers, it is enough to prove that $i_{\overline{Y}} i^{*}_{\mathscr{P}}(\omega_l) = L_{\overline{Y}} i^{*}_{\mathscr{P}}(\omega_l) = 0$ for all vector fields $\overline{Y} \in \mathrm{D}(\mathscr{P})$ vertical with respect to

$q_{0,l}$. Let $Y \in \mathrm{D}(\mathscr{C}^\dagger J^{l+1})$ be a vector field on $\mathscr{C}^\dagger J^{l+1}$, vertical with respect to $q_{0,l}$, and $\overline{Y} := Y|_{\mathscr{P}}$ its restriction to $\mathscr{P}$. Then $\overline{Y}$ is locally of the form

$$\overline{Y} = \sum_{|K|=l+1} Y^\beta_K \partial^K_\beta|_{\mathscr{P}},$$

for some $\ldots, Y^\beta_K, \ldots$ local functions on $\mathscr{P}$. Now $\overline{Y} \in \mathrm{D}(\mathscr{P})$ iff, locally,

$$\sum_{|I|=l+1} Y^\beta_K \partial^K_\beta \partial^I_\alpha L|_{\mathscr{P}} = 0.$$

Compute

$$\overline{Y}(E_l|_{\mathscr{P}}) = \sum_{|K|=l+1} Y^\beta_K \partial^K_\beta E_l|_{\mathscr{P}} = \sum_{|I|=l+1} Y^\alpha_I (\delta^I_{Ji} p^{J.i}_\alpha - \partial^I_\alpha L)|_{\mathscr{P}} = 0.$$

Therefore

$$i_{\overline{Y}} i^*_{\mathscr{P}}(\omega_l) = -\overline{Y}(E_l|_{\mathscr{P}}) d^n x = 0.$$

Similarly,

$$L_{\overline{Y}} i^*_{\mathscr{P}}(\omega_l) = -d\overline{Y}(E_l|_{\mathscr{P}}) d^n x = 0.$$

It follows from the arbitrariness of $\overline{Y}$ that $i^*_{\mathscr{P}}(\omega_l) = q_{0,l}|^*_{\mathscr{P}}(\omega_0)$, where, locally

$$\omega_0 = \sum_{|I|\leq l} i^*_{\mathscr{P}_0}(dp^{I.i}_\alpha du^\alpha_I) d^{n-1}x_i - dH d^n x,$$

and H is the local function on $\mathscr{P}_0$ uniquely defined by putting $q_{0,l}|^*_{\mathscr{P}}(H) = E_l|_{\mathscr{P}}$. □

Definition 3. ω_0 *is called the* PD-hamiltonian system determined by the $(l+1)$th order lagrangian density $\mathscr{L}$ *and the corresponding PD-Hamilton equations are the* Hamilton-de Donder-Weyl (HDW) equations determined by $\mathscr{L}$.

Definition 4. *A lagrangian density of order $l+1$ $\mathscr{L}$ is* regular at the order $l+1$ *if $\mathscr{P}_0$ is an open submanifold of $J^\dagger\pi_l$.*

Clearly, under the regularity assumptions 1, the lagrangian density of order $l+1$, $\mathscr{L}$, is *regular at the order* $l+1$ iff the matrix

$$\left\| (\partial^K_\beta \partial^I_\alpha L)(\theta) \right\|^{(\alpha,I)}_{(\beta,K)}, \quad |I|, |K| = l+1,$$

where the pairs (α, I) and (β, K) are understood as single indexes, is of maximal rank at every point $\theta \in J^\infty$. In this case, ω_0 is a PD-hamiltonian system on an open subbundle of $\tau^\dagger\pi_l$, and it is locally given by

$$\omega_0 = \sum_{|I|\leq l} dp^{I.i}_\alpha du^\alpha_I d^{n-1}x_i - dH d^n x,$$

where, now, H is a local function on $J^\dagger\pi_l$. Then, as expected, the HDW equations read locally

$$\begin{cases} p_\alpha^{I.i}{}_{,i} = -\frac{\partial H}{\partial u_I^\alpha} \\ u_I^\alpha{}_{,i} = \frac{\partial H}{\partial p_\alpha^{I.i}} \end{cases} .$$

Notice that the HDW equations are canonically associated to a lagrangian density and its order and no additional structure is required to define them. Moreover, in view of Theorem 8, two lagrangian densities of the same order determining the same system of EL equations, determine equivalent HDW equations. Finally, to write down the HDW equations there is no need of any distinguished Legendre transform. Actually, the emergence of ambiguities in all proposed Hamiltonian formalism for higher order field theories in literature seems to rely on the common attempt to first define a higher order analogue of the Legendre transform and, only thereafter, to define the "hamiltonian theory". In the next section we present our own point of view about the Legendre transform in higher order lagrangian field theories.

8. The Legendre Transform

Keeping the same notations as in the previous section, denote by ${}^l\mathscr{E}_{ELH} \subset J^\infty q_{l+1}$ the infinite prolongation of the PD-Hamilton equations of ω_l and by $p' : \mathscr{C}^\dagger\pi_{l+1} \longrightarrow E$ the natural projection.

Proposition 12. *$(j_\infty p')({}^l\mathscr{E}_{ELH}) \subset \mathscr{E}_{EL}$ and $j_\infty p' : {}^l\mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ is a covering.*

Proof. The proof is the finite dimensional version of the proof of Theorem 6 and will be omitted. □

Denote also by $\mathscr{E}_H^{\mathscr{P}} \subset J^\infty q \subset J^\infty q_l$ the infinite prolongation of the PD-Hamilton equations of $i_{\mathscr{P}}^*(\omega_l)$ and $\mathscr{E}_H \subset J^\infty \tau^\dagger\pi_l|_{\mathscr{P}_0}$ the infinite prolongation of the HDW equations.

Proposition 13. *$(j_\infty q_{0,l})(\mathscr{E}_H^{\mathscr{P}}) \subset \mathscr{E}_H$ and $j_\infty q_{0,l} : \mathscr{E}_H^{\mathscr{P}} \longrightarrow \mathscr{E}_H$ is a covering.*

Proof. It immediately follows from Theorem 11 and Proposition 3. □

Notice that, in view of Propositions 4, 12 and 13, there is a diagram of morphisms of PDEs,

$$\begin{array}{ccc} {}^l\mathscr{E}_{ELH} & \hookrightarrow & \mathscr{E}_H^{\mathscr{P}} \\ {\scriptstyle j_\infty p'}\downarrow & & \downarrow{\scriptstyle j_\infty q_{0,l}} \\ \mathscr{E}_{EL} & & \mathscr{E}_H \end{array} , \tag{18}$$

whose vertical arrows are coverings. Therefore the inclusion ${}^l\mathscr{E}_{ELH} \subset \mathscr{E}_H^{\mathscr{P}}$ may be understood as a non local morphism of $\mathscr{E}_{EL}$ into $\mathscr{E}_H$. We interpret such morphism as Legendre transform according to the following

Definition 5. *We call diagram (18) the* Legendre transform determined by the lagrangian density $\mathscr{L}$.

Any Legendre form of order l, $\vartheta : J^\infty \longrightarrow J_l^\dagger \longrightarrow \mathscr{C}^\dagger\pi_{l+1}$, determines a section $j_\infty\vartheta|_{\mathscr{E}_{EL}} : \mathscr{E}_{EL} \longrightarrow {}^l\mathscr{E}_{ELH}$ of the covering $j_\infty p' : {}^l\mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ and, therefore, via composition with $j_\infty q_{0,l}$, a concrete map $\mathscr{E}_{EL} \longrightarrow \mathscr{E}_H$. Nevertheless, among these maps, there is no distinguished one.

We now prove that, if $\mathscr{L}$ is regular at the order $l+1$, then $\mathscr{E}_H$ itself covers $\mathscr{E}_{EL}$. This result should be interpreted as the higher order analogue of the theorem stating the equivalence of EL equations and HDW equations for first order theories with regular lagrangian (see, for instance, [11]). Let us first prove the following

Lemma 14. *If $\mathscr{L}$ is regular at the order $l+1$, then ${}^l\mathscr{E}_{ELH} = \mathscr{E}_H^{\mathscr{P}}$.*

Proof. The proof is in local coordinates. Let $\sigma : U \longrightarrow \mathscr{C}^\dagger\pi_{l+1}$ be a local section of q_{l+1}, $U \subset M$ an open subset. Suppose $\operatorname{im}\sigma \subset \mathscr{P}$. Then, locally,

$$\partial_\alpha^I L \circ \sigma - \delta_{Jj}^I \sigma_\alpha^{J.j} = 0, \quad |I| = l+1.$$

Now, $i_{\dot\sigma}\omega_l|_\sigma$ is locally given by

$$\begin{aligned} &i_{\dot\sigma}\omega_l|_\sigma \\ &= \Big[\textstyle\sum_{|I|\leq l+1}(-\sigma_\alpha^{I.i},_i - \delta_{Jj}^I\sigma_\sigma^{J.j} + \partial_\alpha^I L\circ\sigma)du_I^\alpha + \sum_{|I|\leq l}(\sigma_I^\alpha,_i - \sigma_{Ii}^\alpha)dp_\alpha^{I.i}\Big]|_\sigma d^n x. \end{aligned}$$

As already outlined, the annihilator of $\mathrm{D}(\mathscr{P})$ in $\Lambda^1(\mathscr{C}^\dagger\pi_{l+1})|_{\mathscr{P}}$ is locally spanned by 1-forms

$$\lambda_\alpha^I := d(\partial_\alpha^I L - \delta_{Jj}^I p_\alpha^{J.j})|_{\mathscr{P}} = \Big(\textstyle\sum_{|K|\leq l+1}\partial_\beta^K\partial_\beta^I L\, du_K^\beta - \delta_{Jj}^I dp_\alpha^{Jj}\Big)|_{\mathscr{P}}, \quad |I| = l+1.$$

Therefore, $i_{\dot\sigma}i_{\mathscr{P}}^*(\omega_l)|_\sigma = 0$ iff, locally,

$$i_{\dot\sigma}\omega_l|_\sigma = \sum_{|I|=l+1} f_I^a\lambda_\alpha^I|_\sigma, \tag{19}$$

for some local functions $\ldots, f_I^\alpha, \ldots$ on $\operatorname{im}\sigma$. Equations (19) read

$$\begin{array}{c} \sum_{|I|\leq l+1}(-\sigma_\alpha^{I.i},_i + \partial_\alpha^I L\circ\sigma - \delta_{Jj}^I\sigma_\sigma^{J.j} - \sum_{|K|=l+1} f_K^\beta\partial_\beta^K\partial_\alpha^I L\circ\sigma)du_I^\alpha|_\sigma \\ +\sum_{|I|<l}(\sigma_{I,i}^\alpha - \sigma_{Ii}^\alpha)dp_\alpha^{I.i}|_\sigma + \sum_{|I|=l}(\sigma_I^\alpha,_i - \sigma_{Ii}^\alpha + \frac{I[i]+i}{l+1}f_{Ii}^\alpha)dp_\alpha^{I.i}|_\sigma = 0 \end{array}.$$

Since the forms $\ldots, du_I^\alpha|_\sigma, \ldots, dp_\alpha^{I.i}|_\sigma, \ldots$ are linearly independent, $i_{\dot\sigma}i_{\mathscr{P}}^*(\omega_l)|_\sigma = 0$ iff, locally,

$$\left\{\begin{array}{ll} -\sigma_\alpha^{I.i},_i + \partial_\alpha^I L\circ\sigma - \delta_{Jj}^I\sigma_\sigma^{J.j} - \sum_{|K|=l+1} f_K^\beta\partial_\beta^K\partial_\alpha^I L\circ\sigma = 0, & |I|\leq l+1 \\ \sigma_{I,i}^\alpha - \sigma_{Ii}^\alpha = 0, & |I|<l \\ \sigma_I^\alpha,_i - \sigma_{Ii}^\alpha + \frac{I[i]+i}{l+1}f_{Ii}^\alpha = 0, & |I| = l \end{array}\right., \tag{20}$$

for some $\ldots, f_I^\alpha, \ldots$. It follows from the third of Equations (20) that

$$f_{Ii}^\alpha = -\tfrac{l+1}{I[i]+1}(\sigma_I^\alpha,_i - \sigma_{Ii}^\alpha), \quad |I| = l. \tag{21}$$

Moreover, since $\operatorname{im}\sigma \subset \mathscr{P}$, the first equation, for $|I| = l+1$, gives

$$0 = \sum_{|K|=l+1} f^{\beta}_{K}\partial^{K}_{\beta}\partial^{I}_{\alpha}L\circ\sigma = \sum_{|J|=l} \tfrac{J[j]+1}{l+1} f^{\beta}_{Jj}\partial^{Jj}_{\beta}\partial^{I}_{\alpha}L\circ\sigma = -\sum_{|J|=l}(\sigma^{\beta}_{J}{}_{,j} - \sigma^{\beta}_{Jj})\partial^{Jj}_{\beta}\partial^{I}_{\alpha}L\circ\sigma,$$

and, in view of the regularity of $\mathscr{L}$ and Equations (21),

$$\sigma^{\alpha}_{I}{}_{,i} - \sigma^{\alpha}_{Ii} = f^{\alpha}_{Ii} = 0, \quad |I| = l.$$

Substituting again into (20), we finally find that the PD-Hamilton equations $i_{\dot\sigma} i^{*}_{\mathscr{P}}(\omega_l)|_{\sigma} = 0$ are locally equivalent to equations

$$\begin{cases} p^{I.i}_{\alpha}{}_{,i} = \partial^{I}_{\alpha}L - \delta^{I}_{Jj}p^{J.j}_{\sigma}, & |I| \leq l+1 \\ u^{\alpha}_{I}{}_{,i} = u^{\alpha}_{Ii}, & |I| \leq l \end{cases},$$

which are the PD-Hamilton equations of ω_l. □

Now, notice that, in view of the above lemma, if $\mathscr{L}$ is regular at the order $l+1$, the Legendre transform (18) reduces to a Bäcklund transformation

$$\begin{array}{ccc} {}^{l}\mathscr{E}_{ELH} & = & \mathscr{E}^{\mathscr{P}}_{H} \\ {\scriptstyle j_\infty p'}\downarrow & & \downarrow{\scriptstyle j_\infty q_{0,l}} \\ \mathscr{E}_{EL} & & \mathscr{E}_{H} \end{array}.$$

Finally, $J^{\dagger}\pi_l$ maps to E via $\pi_{l,0}\circ\tau^{\dagger}_{0}\pi_l$ and such map is a morphisms of bundles (over M). Therefore, it induces a morphism $J^{\infty}\tau^{\dagger}_{0}\pi_l \longrightarrow J^{\infty}$ and, by restriction, a morphism of PDEs $\kappa : \mathscr{E}_H \longrightarrow J^{\infty}$, locally defined as $\kappa^{*}(u^{\alpha}_{K}) = u^{\alpha}_{\mathsf{O}|K}$, $|K| \geq 0$. It is easy to show that diagram

$$\begin{array}{ccccc} {}^{l}\mathscr{E}_{ELH} & & = & & \mathscr{E}^{\mathscr{P}}_{H} \\ {\scriptstyle j_\infty p'}\downarrow & & & & \downarrow{\scriptstyle j_\infty q_{0,l}} \\ \mathscr{E}_{EL} & \hookrightarrow & J^{\infty} & \xleftarrow{\kappa} & \mathscr{E}_{H} \end{array},$$

commutes, so that $\kappa(\mathscr{E}_H) \subset \mathscr{E}_{EL}$ and $\kappa : \mathscr{E}_H \longrightarrow \mathscr{E}_{EL}$ is a covering. We have thus proved the following

Theorem 15. *If $\mathscr{L}$ is regular at the order $l+1$, then $\mathscr{E}_H$ covers $\mathscr{E}_{EL}$.*

Conclusions

In this paper, using the geometric theory of PDEs, we solved the long standing problem of finding a reasonably natural, higher order, field theoretic analogue of hamiltonian mechanics of lagrangian systems. By naturality we mean dependence on no other structure than the action functional. we achieved our goal in two steps. First we found a higher order, field theoretic analogue of the Skinner-Rusk mixed lagrangian-hamiltonian formalism [13, 14, 15] and, second, we shew that such theory projects naturally to a PD-hamiltonian system on a smaller space. The obtained hamiltonian field equations enjoy the following nice properties: 1) they are first order, 2) there is a canonical, non-local embedding of the Euler-Lagrange equations into them, and 3) for regular lagrangian theories, they cover the Euler-Lagrange equations. Moreover, for regular lagrangian theories, the coordinate expressions of the obtained field equations are nothing but the de Donder higher order field equations. This proves that our theory is truly the coordinate-free formulation of de Donder one [2].

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