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ABSTRACT. We define partial differential (PD in the following), i.e., field theoretic, analogues of hamiltonian systems on abstract symplectic manifolds and study their main properties, namely, PD-Hamilton equations, PD-Noether theorem, PD-Poisson bracket, etc.. Unlike in standard multisymplectic approach to hamiltonian field theory, in our formalism the geometric structure (kinematics) and the dynamical information on the “phase space” appear as just different components of one single geometric object.

INTRODUCTION

First order lagrangian mechanics can be naturally generalized to higher order lagrangian field theory. Moreover, the latter can be presented in a very elegant and precise algebro-geometric fashion [1]. In particular, it is clear what are all the involved geometric structures (higher order jets, Cartan distribution, \mathcal{C} -spectral sequence, etc., [1, 2]). On the other hand it seems to be quite hard to understand what is the most “reasonable, unambiguous, higher order, field theoretic generalization” of hamiltonian mechanics on abstract symplectic manifolds. Actually, there exists a universally accepted generalization of the standard mechanical picture

$$\text{lagrangian mechanics on } TQ \implies \text{hamiltonian mechanics on } T^*Q,$$

Q being a smooth manifold, to the picture

$$\text{lagrangian field theory on } J^1\pi \implies \text{hamiltonian field theory on } \mathcal{M}\pi, \quad (1)$$

π being a fiber bundle, $J^1\pi$ its first jet space and $\mathcal{M}\pi$ its multimomentum space [3] (see also [4] for a recent review). Picture (1) includes, in particular, a generalization of the Legendre transform. Along this path it has been discovered an analogous structure of the symplectic structure on T^*Q , which has been named the *multisymplectic structure of $\mathcal{M}\pi$* (see, for instance, [5]). A whole literature exists about properties of such structure, which are generically referred to as *multisymplectic geometry of $\mathcal{M}\pi$* (see references in [4]). In particular, efforts were made to find multisymplectic analogues of all properties of T^*Q (including, for instance, the Poisson bracket [6, 7, 8, 9, 10]). Now, it is natural to wonder if it is possible to reasonably further generalize in two different directions. The first one is towards a picture

$$\text{lagrangian field theory on } J^\infty\pi \implies \text{higher order hamiltonian field theory}, \quad (2)$$

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$J^\infty\pi$ being the ∞ th jet space of π , including a higher order generalization of the Legendre transform. There is no universally accepted answer about picture (2) (see, for instance, [11, 12, 13, 14, 15, 16, 17, 18] and references therein). Moreover, it seems that all the proposed answers are ambiguous in some aspects. Most often they involve the choice of some extra structure other than the natural ones on $J^\infty\pi$.

The second direction in which to generalize picture (1) can be illustrated as follows. T^*Q is just a very special example of (pre)symplectic manifold. Actually hamiltonian mechanics can be (and should, in some cases [19]) formulated on abstract (pre)symplectic manifolds. Similarly, it is natural to wonder if there exists the concept of abstract multi(pre)symplectic manifolds in such a way that hamiltonian field theory could be reasonably formulated on them. In literature they can be found some proposals of should be abstract multi(pre)symplectic manifolds (see, for instance, [20, 21]). In particular, definitions have been given in such a way to be able to prove multisymplectic analogues of the celebrated Darboux lemma [22, 23]. The recent definitions by Forger and Gomes [23] appear to be the most satisfactory in that they are “minimal” on one side and duly model in an abstract fashion the relevant geometric properties of $\mathcal{M}\pi$ on the other side. In their work Forger and Gomes illustrate, in particular, the role played by fiber bundles in the should be definition of multi(pre)symplectic structure. The next step forward should be to formulate hamiltonian field theory on multisymplectic bundles.

In this paper we present our own proposal about what should it be an abstract, first order, hamiltonian field theory. We call such proposal the *theory of partial differential (PD in the following) hamiltonian systems* so to 1) stress that it is a natural generalization of the theory of hamiltonian systems on abstract symplectic manifolds, 2) distinguish it from the special case of hamiltonian field theory on $\mathcal{M}\pi$. A PD-hamiltonian system encompasses both the kinematics (encoded, in picture (1), by the multisymplectic structure in $\mathcal{M}\pi$) and the dynamics (encoded, in picture (1), by the so called *hamiltonian section* [4]) which appear as just different components of one single geometric object. Namely, the main difference between a PD-hamiltonian system and a multi(pre)symplectic structure (whatever the reader understand for this) is the dynamical content of the former (as opposed to the just kinematical one of the latter).

The paper is divided into 6 sections. In Section 1 we collect our notations and conventions and recall basic differential geometric facts that will be used in the main part of the paper.

In Sections 2 we define what we call *affine forms* on fiber bundles. The introduction of affine forms can be motivated as follows. Trajectories in hamiltonian mechanics are curves, whose 1st derivative at a point is naturally understood as a tangent vector. In their turn, tangent vectors can be inserted into differential forms and, in particular, a symplectic one, and Hamilton equations can be written in terms of such insertion. Trajectories in field theory are sections of a fiber bundle $\alpha : P \longrightarrow M$, whose 1st derivative at a point is naturally understood as a point in $J^1\alpha$. In their turn, points

of $J^1\alpha$ can be inserted into affine forms and, in particular, a PD-hamiltonian system (see Section 4), and PD-Hamilton equations can be written in terms of such insertion. Recall now that the natural projection $J^1\alpha \rightarrow P$ is an affine bundle whose sections are naturally interpreted as (Ehresmann) connections in α . Thus, connections and affine geometry play a prominent role in the theory of PD-hamiltonian systems. The affine geometry is hidden in standard hamiltonian mechanics by an *a priori* choice of the parametrization of the time axis (see [24, 25], and references therein, for the role of affine geometry in theoretical mechanics). Similarly, even if the role of connections in field theory has been often recognized (see, for instance, [26, 27]) their affine geometry is sometimes hidden in hamiltonian field theory on $\mathcal{M}\pi$ by the use of multivectors, or even decomposable ones [28, 29] (which is just a multidimensional analogue of choosing a parameterization of time). Actually, we show in Subsection 2.3 that affine forms can be understood as standard differential forms of a special kind. Nevertheless, we prefer to keep the distinction for foundational reasons.

In Section 3 we discuss standard operations with affine forms. Essentially because of the interpretation of affine forms as standard differential forms we mentioned above, some of this operations (for instance, the insertion of a connection into an affine form [30], or the differential of an affine form) were actually already defined in the literature, or can be understood as standard operations with forms. We stress again that we will keep the distinction. Finally, we discuss relevant affine form cohomologies proving an affine form version of the Poincaré lemma.

In Section 4 we introduce PD-(pre)hamiltonian systems, discuss their geometry and the geometry of the associated PD-Hamilton equations, with some references to the singular, constrained case (see [31, 32, 33, 34] for an account of the constraint algorithm in first order field theory). For completeness, we also relate PD-hamiltonian systems to multi(pre)symplectic structures *à la* Forger [23] and the calculus of variations.

In Section 5 we introduce PD-Noether symmetries and currents of a PD-hamiltonian system. In view of the dynamical content of the latter we are able to prove a Noether theorem (see also [35]). Moreover, there is a natural Lie bracket (named PD-Poisson bracket) among PD-Noether currents. As already mentioned, multisymplectic analogues of the Poisson bracket have been already discussed in the literature [6, 7, 8, 9, 10]. However, we emphasise here the dynamical nature of PD-Poisson bracket (see also [36, 37]). Namely, such bracket is just part of the Peierls bracket [37] among conservation laws of the underlying lagrangian theory and we don't try to extend it to non-conserved currents. Indeed, our opinion is that the existence of a Poisson bracket among non-conserved functions in hamiltonian mechanics is essentially due to the existence of a preferred hamiltonian system on any symplectic manifold N , i.e., the one with 0 hamiltonian, for which every function on N is a conservation law. Finally we discuss the (gauge) reduction of a degenerate (but unconstrained) PD-hamiltonian system.

In Section 6 we propose few examples of PD-hamiltonian systems, including the computation of their PD-Noether symmetries and currents or, in one case, their reduction.

1. NOTATIONS AND CONVENTIONS

In this section we collect notations and conventions about some general constructions that will be used in the following.

Let N be a manifold. We denote by $C^\infty(N)$ the \mathbb{R} -algebra of smooth, \mathbb{R} -valued functions on N . A vector field X over N will be always understood as a derivation $X : C^\infty(N) \rightarrow C^\infty(N)$. We denote by $D(N)$ the $C^\infty(N)$ -module of vector fields over N , by $\Lambda(N) = \bigoplus_k \Lambda^k(N)$ the graded \mathbb{R} -algebra of differential forms over N , by $d_N : \Lambda(N) \rightarrow \Lambda(N)$ (or, simply, $d : \Lambda(N) \rightarrow \Lambda(N)$ if this does not lead to confusion) the de Rham differential and by $H(N) = \bigoplus_k H^k(N)$ the de Rham cohomology. If $F : N_1 \rightarrow N$ is a smooth map of manifolds, we denote by $F^* : \Lambda(N) \rightarrow \Lambda(N_1)$ its pull-back. We will understand everywhere the wedge product \wedge of differential forms, i.e., for $\omega, \omega_1 \in \Lambda(N)$, instead of writing $\omega \wedge \omega_1$, we will simply write $\omega\omega_1$. We assume the reader to be familiar with Frölicher-Nijenhuis calculus on form valued vector fields (insertion $i_Z\omega$ of a form valued vector field Z into a differential form ω , Lie derivative $L_Z\omega$ of a differential form ω along a form valued vector fields Z , Frölicher-Nijenhuis bracket, etc.).

Let $\varpi : W \rightarrow N$ be an affine bundle (or, possibly, a vector bundle) and $F : N_1 \rightarrow N$ a smooth map of manifolds. The affine space of smooth sections of ϖ will be denoted by $\Gamma(\varpi)$. For $x \in N$, we put, sometimes, $\Gamma(\varpi)|_x := \varpi^{-1}(x)$ and, for $\chi \in \Gamma(\varpi)$, we also put $\chi_x := \chi(x)$. The affine bundle on N_1 induced by ϖ via F will be denoted by $\varpi|_F : W|_F \rightarrow N_1$:

$$\begin{array}{ccc} W|_F & \longrightarrow & W \\ \varpi|_F \downarrow & & \downarrow \varpi \\ N_1 & \xrightarrow{F} & N \end{array} .$$

We also denote $\Gamma(\varpi)|_F := \Gamma(\varpi|_F)$. For any section $s \in \Gamma(\varpi)$ there exists a unique section, which abusing the notation we denote by $s|_F \in \Gamma(\varpi|_F)$, such that the diagram

$$\begin{array}{ccc} W|_F & \longrightarrow & W \\ s|_F \uparrow & & \uparrow s \\ N_1 & \xrightarrow{F} & N \end{array}$$

commutes. Elements in $\Gamma(\varpi)|_F$ are called *sections of ϖ along F* . If F is an embedding $\varpi|_F$, $\Gamma(\varpi)|_F$ and $s|_F$ will be referred to as *the restriction to N_1 of ϖ , $\Gamma(\varpi)$ and s* , respectively. If $\varpi_1 : W_1 \rightarrow N$ is an other affine bundle and $A : \Gamma(\varpi) \rightarrow \Gamma(\varpi_1)$ is an affine map then there exists a unique affine map $A|_F : \Gamma(\varpi)|_F \rightarrow \Gamma(\varpi_1)|_F$ such that $A|_F(s|_F) = A(s)|_F$ for all $s \in \Gamma(\varpi)$.

Let $\alpha : P \rightarrow M$ be a fiber bundle. A vector field $X \in D(P)$ is called *α -projectable* iff there exists $\tilde{X} \in D(P)$ such that $X \circ \alpha^* = \alpha^* \circ \tilde{X}$. \tilde{X} is called the *α -projection of X* . α -projectable vector fields form a Lie subalgebra in $D(P)$ denoted by $D_V(P, \alpha)$ (or

simply D_V if this does not lead to confusion). An α -projectable vector field projecting onto the 0 vector field is a α -vertical vector field. α -vertical vector fields form an ideal in D_V denoted by $VD(P, \alpha)$ (or simply VD). Notice that, if α has connected fiber, then D_V is the stabilizer of VD in $D(P)$, i.e., $D_V = \{X \in D(P) \mid [X, VD] \subset VD\}$.

Let $\alpha : P \rightarrow M$ be as above, $\dim M = n$, $\dim P = m + n$. Denote by $\alpha_1 : J^1\alpha \rightarrow M$ the bundle of 1-jets of local sections of α [38, 2], and by $\alpha_{1,0} : J^1\alpha \rightarrow P$ the canonical projection. For any local section $\sigma : U \rightarrow P$ of α , $U \subset M$ being an open subset, we denote by $\dot{\sigma} : U \rightarrow J^1\alpha$ its 1th jet prolongation. Any system of adapted to α coordinates $(\dots, x^i, \dots, y^a, \dots)$ on P , \dots, x^i, \dots being coordinates on M and \dots, y^a, \dots fiber coordinates on P , gives rise to the system of jet coordinates $(\dots, x^i, \dots, u^a, \dots, y_i^a, \dots)$ on $J^1\alpha$, $i = 1, \dots, n$, $a = 1, \dots, m$. Recall that $\alpha_{1,0}$ is an affine bundle and a section $\nabla : P \rightarrow J^1\alpha$ of it is naturally interpreted as a (Ehresmann) connection in α . We assume the reader to be familiar with the geometry of connections (see, for instance, [39, 40]). ∇ is locally represented as

$$\nabla : y_i^a = \nabla_i^a,$$

with $\nabla_i^a = \nabla_i^a(x^1, \dots, x^n, y^1, \dots, y^m)$, $i = 1, \dots, n$, $a = 1, \dots, m$. The space $\Gamma(\alpha_{1,0})$ of all such sections will be also denoted by $C(P, \alpha)$ (or simply C).

Let $\alpha : P \rightarrow M$ be as above, $\alpha' : P' \rightarrow M$ another fiber bundle and $G : P \rightarrow P'$ a bundle morphism (over the identity $\text{id}_M : M \rightarrow M$), i.e., a smooth map such that $\alpha' \circ G = \alpha$. First of all, recall that there exists a unique bundle morphism $j^1G : J^1\alpha \rightarrow J^1\alpha'$ such that $j^1G \circ \dot{\sigma} = (G \circ \sigma)^\cdot$ for all local sections σ of α . j^1G is the *first jet prolongation of G* and diagram

$$\begin{array}{ccc} J^1\alpha & \xrightarrow{j^1G} & J^1\alpha' \\ \alpha_{1,0} \downarrow & & \downarrow \alpha'_{1,0} \\ P & \xrightarrow{G} & P' \\ & \searrow \alpha & \swarrow \alpha' \\ & & M \end{array}$$

commutes. Now, a connection $\nabla \in C(P, \alpha)$ and a connection $\nabla' \in C(P', \alpha')$ are said *G-compatible* iff $\nabla' \circ G = j^1G \circ \nabla$.

Let

$$\cdots \longrightarrow K_{l-1} \xrightarrow{\delta_{l-1}} K_l \xrightarrow{\delta_l} K_{l+1} \xrightarrow{\delta_{l+1}} \cdots$$

be a complex. Put $K := \bigoplus_l K_l$ and $\delta := \bigoplus_l \delta_l$. We denote by $H(K, \delta) := \bigoplus_l H^l(K, \delta)$, the cohomology space of (K, δ) , $H^l(K, \delta) := \ker \delta_l / \text{im } \delta_{l-1}$.

Let A be a commutative \mathbb{R} -algebra, \mathbf{M}, \mathbf{M}_1 be A -modules and \mathbf{A} an affine space modelled over \mathbf{M} . We denote by $\text{Aff}_A(\mathbf{A}, \mathbf{M}_1)$ (resp. $\text{Hom}_A(\mathbf{M}, \mathbf{M}_1)$) the A -module of affine (resp. A -linear) maps $\mathbf{A} \rightarrow \mathbf{M}_1$ (resp. $\mathbf{M} \rightarrow \mathbf{M}_1$). If $\phi \in \text{Aff}_A(\mathbf{A}, \mathbf{M}_1)$, its *linear part* $\underline{\phi}$ is an element in $\text{Hom}_A(\mathbf{M}, \mathbf{M}_1)$.

Let m, r be positive integers and $\dots, A_{a_1 \dots a_r}, \dots$ be elements in a real vector space, $a_1, \dots, a_r = 1, \dots, m$. We denote by $\dots, A_{[a_1 \dots a_r]}, \dots$ their skew-symmetrization, i.e.,

$$A_{[a_1 \dots a_r]} := \frac{1}{s!} \sum_{\sigma \in S_r} \varepsilon(\sigma) A_{a_{\sigma(1)} \dots a_{\sigma(r)}},$$

S_r being the group of permutations of $\{1, \dots, r\}$ and $\varepsilon(\sigma)$ the sign of $\sigma \in S_r$.

We denote by \simeq (resp. \approx) a canonical (resp. non-canonical) isomorphism between algebraic structures and by \equiv an equivalence of notations. For instance, for $\alpha : P \rightarrow M$ as above, $VD \equiv VD(P, M)$. Finally, we understand the sum over upper-lower pairs of repeated indexes.

2. AFFINE FORMS ON FIBER BUNDLES

2.1. Special Forms on Fiber Bundles. Let $\alpha : P \rightarrow M$ be a fiber bundle, $A := C^\infty(P)$, $A_0 := C^\infty(M)$, x^1, \dots, x^n coordinates on M , $\dim M = n$, and y^1, \dots, y^m fiber coordinates on P , $\dim P = n + m$. In the following we will often understand the monomorphism of algebras $\alpha^* : A_0 \rightarrow A$, whose image is made of functions on P which are constant along the fibers of α . D_V (resp. VD) is made of vector fields X locally in the form $X = X^i \partial_i + Y^a \partial_a$ (resp. $X = Y^a \partial_a$) where $X^i = X^i(x^1, \dots, x^n)$, $\partial_i := \partial / \partial x^i$, $i = 1, \dots, n$, $\partial_a = \partial / \partial y^a$, $a = 1, \dots, m$.

Denote by $\Lambda_1(P, \alpha) = \bigoplus_k \Lambda_1^k(P, \alpha)$ (or simply $\Lambda_1 = \bigoplus_k \Lambda_1^k$) the differential (graded) ideal of differential forms on P vanishing when restricted to fibers of α , i.e., $\omega \in \Lambda_1^k$, $k \geq 0$ iff $\omega \in \Lambda^k(P)$ and $i_{\alpha^{-1}(x)}^*(\omega) = 0$ for all $x \in M$, $i_{\alpha^{-1}(x)} : \alpha^{-1}(x) \rightarrow P$ being the embedding of the fiber $\alpha^{-1}(x)$ of α through $x \in M$. Moreover, denote by $\Lambda_p(P, \alpha) = \bigoplus_k \Lambda_p^k(P, \alpha)$ (or simply $\Lambda_p = \bigoplus_k \Lambda_p^k$) the p -th exterior power of Λ_1 . For all k and p , Λ_p^k is made of differential k -forms ω such that $(i_{Y_1} \circ \dots \circ i_{Y_{k-p+1}})\omega = 0$ for every $Y_1, \dots, Y_{k-p+1} \in VD$ or, which is the same, differential k -forms ω locally in the form

$$\omega = \sum_{l \geq 0} \omega_{i_1 \dots i_{p+l} a_1 \dots a_{k-p-l}} dx^{i_1} \dots dx^{i_{p+l}} dy^{a_1} \dots dy^{a_{k-p-l}},$$

$\dots, \omega_{i_1 \dots i_{p+l} a_1 \dots a_{k-p-l}}, \dots$ being local functions on P , $i_1, \dots, i_{p+l} = 1, \dots, n$, $a_1, \dots, a_{k-p-l} = 1, \dots, m$.

Denote by $V\Lambda(P, \alpha) = \bigoplus_k V\Lambda^k(P, \alpha)$ (or simply $V\Lambda = \bigoplus_k V\Lambda^k$) the quotient differential algebra $\Lambda(P)/\Lambda_1$, with $d^V : V\Lambda \rightarrow V\Lambda$ its differential and with $p^V : \Lambda(P) \ni \omega \mapsto \omega^V := \omega + \Lambda_1 \in V\Lambda$ the projection onto the quotient. Notice that d^V is A_0 -linear. An element ρ^V in $V\Lambda^k$ is locally in the form

$$\rho^V = \rho_{a_1 \dots a_k} d^V y^{a_1} \dots d^V y^{a_k},$$

$\dots, \rho_{a_1 \dots a_k}, \dots$ being local functions on P , and $d^V \rho^V$ is locally given by

$$d^V \rho^V = \partial_a \rho_{a_1 \dots a_k} d^V y^a d^V y^{a_1} \dots d^V y^{a_k} = \partial_{[a} \rho_{a_1 \dots a_k]} d^V y^a d^V y^{a_1} \dots d^V y^{a_k}.$$

Clearly, $V\Lambda^1$ is the dual A -module of VD and $V\Lambda$ its exterior algebra. In particular, elements in $V\Lambda$ may be interpreted as multilinear, skew-symmetric forms on VD .

Denote by $\overline{\Lambda}(P, \alpha) = \bigoplus_k \overline{\Lambda}^k(P, \alpha) := \bigoplus_k \Lambda_k^k \subset \Lambda(P)$ (or simply $\overline{\Lambda} = \bigoplus_k \overline{\Lambda}^k$) the sub-algebra generated by Λ_1^1 . An element $\omega \in \overline{\Lambda}^k$ is locally in the form

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \dots dx^{i_k}.$$

Notice that $\overline{\Lambda}$ is naturally isomorphic to $A \otimes_{A_0} \Lambda(M)$ as an A -algebra.

For any p , the quotient (graded) differential module $E_0^{p, \bullet} \equiv E_0^{p, \bullet}(P, \alpha) := \Lambda_p / \Lambda_{p+1}$ ¹ is naturally isomorphic to $V\Lambda \otimes_A \overline{\Lambda}^p$ (or, which is the same, $V\Lambda \otimes_{A_0} \Lambda^p(M)$) via the correspondence

$$E_0^{p, q} \ni \omega + \Lambda_{p+1}^{p+q} \longmapsto \varpi \in V\Lambda^q \otimes_A \overline{\Lambda}^p, \quad (3)$$

well defined by putting

$$\varpi(Y_1, \dots, Y_q) := (i_{Y_q} \circ \dots \circ i_{Y_1})(\omega) \in \overline{\Lambda}^p,$$

$Y_1, \dots, Y_q \in VD$, $E_0^{p, q}$ being the q th homogeneous piece of $E_0^{p, \bullet}$, $q \in \mathbb{Z}$. In particular, $V\Lambda \otimes_A \overline{\Lambda}$ (or, which is the same, $V\Lambda \otimes_{A_0} \Lambda(M)$) is the graded object associated with the filtration $\Lambda(P) \supset \Lambda_1 \supset \dots \supset \Lambda_p \supset \dots$. As we will see in the next subsection, a connection in α allows one to identify the filtration with its graded object.

Let us now focus on the ideals Λ_{n-1} and Λ_n . Put $d^n x := dx^1 \dots dx^n$ and $d^{n-1} x_i := i_{\partial_i} d^n x$, so that $dx^j d^{n-1} x_i = \delta_i^j d^n x$, $i, j = 1, \dots, n$. Then an element $\omega \in \Lambda_{n-1}^{q+n-1}$ (resp. $\omega \in \Lambda_n^{q+n-1}$) is locally in the form

$$\omega = \omega_{a_1 \dots a_q}^i dy^{a_1} \dots dy^{a_q} d^{n-1} x_i + \omega_{a_1 \dots a_{q-1}} dy^{a_1} \dots dy^{a_{q-1}} d^n x$$

(resp.

$$\omega = \omega_{a_1 \dots a_{q-1}} dy^{a_1} \dots dy^{a_{q-1}} d^n x),$$

$\dots, \omega_{a_1 \dots a_q}^i, \dots, \omega_{a_1 \dots a_{q-1}}, \dots$ being local functions on P . In particular Λ_{n-1}^{q+n-1} (resp. Λ_n^{q+n-1}) is the module of sections of an $[n \binom{q}{m} + \binom{q-1}{m}]$ (resp. $\binom{q-1}{m}$)-dimensional vector bundle over P . In few lines we will provide an alternative description of Λ_{n-1} and Λ_n (Theorem 1). In our opinion, such description is more suitable for a better understanding of the role of Λ_{n-1} and Λ_n in first order field theories (see, for instance, [5]).

2.2. Affine Forms. Let $\nabla \in C(P, \alpha)$. Recall, preliminarily, that C is an affine space modelled over the A -module $\overline{\Lambda}^1 \otimes_A VD$, or, which is the same, $\Lambda^1(M) \otimes_{A_0} VD$. ∇ allows one to split the tangent bundle TP to P into its vertical part VP and a horizontal part $H_\nabla P$. denote by $H_\nabla D(P, \alpha) \subset D(P)$ (or simply $H_\nabla D \subset D(P)$) the submodule of ∇ -horizontal vector fields. An element $X \in H_\nabla D$ is locally in the form $X = X^i \nabla_i$, where $\nabla_i := \partial_i + \nabla_i^a \partial_a$, $i = 1, \dots, n$. Splitting

$$D(P) = VD \oplus H_\nabla D \quad (4)$$

determines a splitting of the de Rham differential $d : \Lambda(P) \longrightarrow \Lambda(P)$ into a horizontal part $d_\nabla : \Lambda(P) \longrightarrow \Lambda(P)$, and a vertical part $d_\nabla^V : \Lambda(P) \longrightarrow \Lambda(P)$, $d = d_\nabla + d_\nabla^V$,

¹This last notation is motivated by the fact that A -modules $E_0^{p, \bullet}$ are columns of the first term of the (cohomological) Leray-Serre spectral sequence of the fiber bundle α (see [mvv??]).

where d_∇ (resp. d_∇^V) is the Lie derivative along the horizontal-form valued vector field (resp. the form valued vertical vector field) $H_\nabla : A \longrightarrow \bar{\Lambda}^1(P)$ (resp. $V_\nabla : A \longrightarrow \Lambda^1(P)$) determined by ∇ . H_∇ (resp. V_∇) is locally given by $H_\nabla = dx^i \nabla_i$ (resp. $V_\nabla = (dy^a - \nabla_i^a dx^i) \partial_a$). Notice that $(\Lambda(P), d_\nabla^V, d_\nabla)$ is not a bi-complex unless ∇ is flat. Splitting (4) also determines an isomorphism $\phi_\nabla : V\Lambda \otimes_A \bar{\Lambda} \longrightarrow \Lambda(P)$ locally given by

$$\phi_\nabla(d^V y^{a_1} \dots d^V y^{a_q} \otimes dx^{i_1} \dots dx^{i_q}) = d_\nabla^V y^{a_1} \dots d_\nabla^V y^{a_q} dx^{i_1} \dots dx^{i_q}.$$

In particular, for any q, p , there is an obvious projection $\mathfrak{p}_\nabla^{q,p} : \Lambda(P) \longrightarrow V\Lambda^q \otimes_A \bar{\Lambda}^p$.

For any $k \geq 0$ put $'\Omega^{k+1} := \text{Aff}_A(C, V\Lambda^k \otimes_A \bar{\Lambda}^n)$. An element $'\vartheta \in '\Omega^{k+1}$ is locally given by

$$' \vartheta(\nabla) = (' \vartheta_{a, a_1 \dots a_k}^i \nabla_i^a + ' \vartheta_{a_1 \dots a_k}) d^V y^{a_1} \dots d^V y^{a_k} \otimes d^n x, \quad \nabla \in C,$$

$\dots, ' \vartheta_{a, a_1 \dots a_k}^i, \dots, ' \vartheta_{a_1 \dots a_k}, \dots$ local functions on P . The linear part $' \underline{\vartheta}$ of an element $' \vartheta \in '\Omega^{k+1}$ is an element in the A -module

$$\begin{aligned} \text{Hom}_A(\bar{\Lambda}^1 \otimes_A V\mathbb{D}, V\Lambda^k \otimes_A \bar{\Lambda}^n) &\simeq \text{Hom}_A(V\mathbb{D}, V\Lambda^k \otimes_A \bar{\Lambda}^{n-1}) \\ &\supset V\Lambda^{k+1} \otimes_A \bar{\Lambda}^{n-1} \\ &=: \underline{\Omega}^{k+1}(P, \alpha) \\ &\equiv \underline{\Omega}^{k+1}, \end{aligned}$$

where we identified $V\Lambda^k \otimes \bar{\Lambda}^{n-1}$ and $\text{Hom}_A(\bar{\Lambda}^1, V\Lambda^k \otimes \bar{\Lambda}^n)$ via the isomorphism

$$V\Lambda^k \otimes \bar{\Lambda}^{n-1} \ni \sigma \otimes \rho \longmapsto \varphi_{\sigma \otimes \rho} \in \text{Hom}_A(\bar{\Lambda}^1, V\Lambda^k \otimes \bar{\Lambda}^n),$$

$\sigma \in V\Lambda^k, \rho \in \bar{\Lambda}^{n-1}$, defined by putting

$$\varphi_{\sigma \otimes \rho}(\eta) := (-)^k \sigma \otimes (\eta \cdot \rho) \in V\Lambda^k \otimes \bar{\Lambda}^n, \quad \eta \in \bar{\Lambda}^1,$$

and embedded $V\Lambda^{k+1} \otimes_A \bar{\Lambda}^{n-1}$ into $\text{Hom}_A(V\mathbb{D}, V\Lambda^k \otimes_A \bar{\Lambda}^{n-1})$ via the correspondence

$$V\Lambda^{k+1} \otimes_A \bar{\Lambda}^{n-1} \ni \sigma' \otimes \rho \longmapsto \varphi'_{\sigma' \otimes \rho} \in \text{Hom}_A(V\mathbb{D}, V\Lambda^k \otimes_A \bar{\Lambda}^{n-1}),$$

$\sigma' \in V\Lambda^k, \rho \in \bar{\Lambda}^{n-1}$, defined by putting

$$\varphi'_{\sigma' \otimes \rho}(Y) := i_Y \sigma' \otimes \rho \in V\Lambda^k \otimes_A \bar{\Lambda}^{n-1}, \quad Y \in V\mathbb{D}.$$

Put also $\Omega^0 \equiv \Omega^0(P, \alpha) := \bar{\Lambda}^{n-1}$, $\underline{\Omega}^0 \equiv \underline{\Omega}^0(P, \alpha) := \Omega^0$, and for $k \geq 0$,

$$\Omega^{k+1} \equiv \Omega^{k+1}(P, \alpha) := \{\vartheta \in '\Omega^{k+1} \mid \underline{\vartheta} \in \underline{\Omega}^{k+1}\},$$

$\Omega \equiv \Omega(P, \alpha) := \bigoplus_{q \geq 0} \Omega^q$ and $\underline{\Omega} \equiv \underline{\Omega}(P, \alpha) := \bigoplus_{q \geq 0} \underline{\Omega}^q$. Elements in Ω^k will be called *affine k -forms* over α . It is easy to show that an element $\vartheta \in '\Omega^{k+1}$ is an affine $(k+1)$ -form iff it is locally given by

$$\vartheta(\nabla) = (\vartheta_{aa_1 \dots a_k}^i \nabla_i^a + \vartheta_{a_1 \dots a_k}) d^V y^{a_1} \dots d^V y^{a_k} \otimes d^n x, \quad \nabla \in C.$$

$\dots, \vartheta_{aa_1 \dots a_k}^i, \dots, \vartheta_{a_1 \dots a_k}, \dots$ being local functions on P such that $\vartheta_{aa_1 \dots a_k}^i = \vartheta_{[aa_1 \dots a_k]}^i$, $i = 1, \dots, n, a, a_1, \dots, a_k = 1, \dots, m$.

According to the above said, the linear part $\underline{\vartheta} \in \underline{\Omega}^{k+1}$ of ϑ is implicitly defined by the formula

$$\vartheta(\nabla + \eta \otimes Y)(Y_1, \dots, Y_k) - \vartheta(\nabla)(Y_1, \dots, Y_k) = (-)^k \eta \cdot \underline{\vartheta}(Y, Y_1, \dots, Y_k) \in \overline{\Lambda}^n,$$

$\nabla \in C$, $\eta \in \overline{\Lambda}^{n-1}$, $Y, Y_1, \dots, Y_k \in VD$, and it is locally given by

$$\underline{\vartheta} = \frac{(-)^k}{k+1} \vartheta_{a_1 \dots a_{k+1}}^i d^V y^{a_1} \dots d^V y^{a_{k+1}} \otimes d^{n-1} x_i. \quad (5)$$

2.3. Affine Forms and Differential Forms. Let $\Omega_0(P, \alpha) = \bigoplus_{q \geq 0} \Omega_0^q(P, \alpha)$ (or, simply, $\Omega_0 \equiv \bigoplus_{q \geq 0} \Omega_0^q$) be the kernel of the projection $\Omega \ni \vartheta \mapsto \underline{\vartheta} \in \underline{\Omega}$. Clearly, Ω_0^q is canonically isomorphic to $V^{q-1} \Lambda \otimes_A \overline{\Lambda}^n$ for $q > 0$ (in the following we will understand such isomorphism), while $\Omega_0^0 = 0$. Moreover, Ω^q (resp. Ω_0^q) is the module of sections of an $[n \binom{q}{m} + \binom{q-1}{m}]$ (resp. $\binom{q-1}{m}$)-dimensional vector bundle over P .

Theorem 1. *There are canonical isomorphisms of A -modules*

$$\begin{aligned} \iota_{0,q} : \Lambda_n^{q+n-1} &\longrightarrow \Omega_0^q, \\ \iota_q : \Lambda_{n-1}^{q+n-1} &\longrightarrow \Omega^q, \\ \underline{\iota}_q : E_0^{n-1,q} &\longrightarrow \underline{\Omega}^q, \end{aligned}$$

$q \geq 0$, such that diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda_n & \longrightarrow & \Lambda_{n-1} & \longrightarrow & E_0^{n-1} \longrightarrow 0 \\ & & \downarrow \iota_0 & & \downarrow \iota & & \downarrow \underline{\iota} \\ 0 & \longrightarrow & \Omega_0 & \longrightarrow & \Omega & \longrightarrow & \underline{\Omega} \longrightarrow 0 \end{array} \quad (6)$$

commutes, where $\iota_0 := \bigoplus_q \iota_{0,q}$, $\iota := \bigoplus_q \iota_q$ and $\underline{\iota} := \bigoplus_q \underline{\iota}_q$.

Proof. Let $q > 0$. First of all, denote by $\underline{\iota}_q : E_0^{n-1,q} \longrightarrow \underline{\Omega}^q$ the already mentioned natural isomorphism (3) and notice that for any $\omega \in \Lambda_n^{q+n-1}$ and $Y_1, \dots, Y_{q-1} \in VD$, $(i_{Y_1} \circ \dots \circ i_{Y_{q-1}})(\omega) \in \overline{\Lambda}^n$. Therefore, it is well defined an element $\iota_{0,q}(\omega) \in \Omega_0^q$ by putting $\iota_{0,q}(\omega)(Y_1, \dots, Y_{q-1}) := (i_{Y_1} \circ \dots \circ i_{Y_{q-1}})(\omega) \in \overline{\Lambda}^n$, $Y_1, \dots, Y_{q-1} \in VD$. Moreover, the correspondence $\Lambda_n^{q+n-1} \ni \omega \mapsto \iota_{0,q}(\omega) \in \Omega_0^q$ is an isomorphism of A -modules. Indeed, let $\omega \in \Lambda_n^{q+n-1}$ and $(i_{Y_1} \circ \dots \circ i_{Y_{q-1}})(\omega) = 0$ for all $Y_1, \dots, Y_{q-1} \in VD$, then $\omega \in \Lambda_{n+1}^{q+n-1} = \mathbf{0}$, so that $\iota_{0,q}$ is injective. Moreover, Λ_n^{q+n-1} and Ω_0^q are locally free A -modules of the same local dimension. We conclude that

$$\iota_0 := \bigoplus_q \iota_{0,q} : \Lambda_n \longrightarrow \Omega_0$$

is a canonical isomorphism of A -modules as well. Moreover, it sends Λ_n^{q+n-1} into Ω_0^q , $q \geq 0$. Finally, if $\omega \in \Lambda_n^{q+n-1}$ is locally given by

$$\omega = \omega_{a_1 \dots a_{q-1}} dy^{a_1} \dots dy^{a_{q-1}} d^n x,$$

then $\iota_0(\omega) \in \Omega_0$ is locally given by $\iota_0(\omega) = \omega_{a_1 \dots a_{q-1}} d^V y^{a_1} \dots d^V y^{a_{q-1}} \otimes d^n x$.

Now, for $\omega \in \Lambda_{n-1}^{q+n-1}$ and $\nabla \in C$ put

$$\iota_q(\omega)(\nabla) := \mathfrak{p}_{\nabla}^{q-1,n}(\omega).$$

If ω is locally given by

$$\omega = \omega_{a_1 \dots a_q}^i dy^{a_1} \dots dy^{a_q} d^{n-1}x_i + \omega_{a_1 \dots a_{q-1}} dy^{a_1} \dots dy^{a_{q-1}} d^n x,$$

then

$$\begin{aligned} \omega &= \omega_{a_1 \dots a_q}^i (d_{\nabla}^V + d_{\nabla})(y^{a_1}) \dots (d_{\nabla}^V + d_{\nabla})(y^{a_q}) d^{n-1}x_i \\ &\quad + \omega_{a_1 \dots a_{q-1}} d_{\nabla}^V y^{a_1} \dots d_{\nabla}^V y^{a_{q-1}} d^n x \\ &= \omega_{a_1 \dots a_q}^i d_{\nabla}^V y^{a_1} \dots d_{\nabla}^V y^{a_q} d^{n-1}x_i \\ &\quad + \sum_s (-)^{p-s} \omega_{a_1 \dots a_q}^i \nabla_j^{a_s} d_{\nabla}^V y^{a_1} \dots \widehat{d_{\nabla}^V y^{a_s}} \dots d_{\nabla}^V y^{a_q} dx^j d^{n-1}x_i \\ &\quad + \omega_{a_1 \dots a_{q-1}} d_{\nabla}^V y^{a_1} \dots d_{\nabla}^V y^{a_{q-1}} d^n x \\ &= \omega^{q,n-1} + [q(-)^{q-1} \omega_{aa_1 \dots a_{q-1}}^i \nabla_i^a + \omega_{a_1 \dots a_{q-1}}] d_{\nabla}^V y^{a_1} \dots d_{\nabla}^V y^{a_{q-1}} d^n x \end{aligned}$$

where a cap “ $\widehat{}$ ” denotes omission of the factor below it, and $\omega^{q,n-1} \in \Lambda(P)$ is a suitable form such that $\mathfrak{p}_{\nabla}^{q-1,n}(\omega^{q,n-1}) = 0$. Therefore, locally

$$\begin{aligned} \iota_q(\omega)(\nabla) &= \mathfrak{p}_{\nabla}^{q-1,n}(\omega) \\ &= [q(-)^{q-1} \omega_{aa_1 \dots a_{q-1}}^i \nabla_i^a + \omega_{a_1 \dots a_{q-1}}] d_{\nabla}^V y^{a_1} \dots d_{\nabla}^V y^{a_{q-1}} \otimes d^n x. \end{aligned} \quad (7)$$

This shows simultaneously that $\iota_q(\omega)$ is affine, that it is in Ω^q and that ι_q is injective. Since Λ_{n-1}^{p+n} and Ω^q are locally free A -modules of the same local dimension, then the correspondence $\iota_q : \Lambda_{n-1}^{q+n-1} \ni \omega \mapsto \iota_q(\omega) \in \Omega^q$ is an isomorphism. Commutativity of diagram (6) immediately follows from local formulas (5) and (7). \square

Notice that isomorphism ι generalizes considerably the well known isomorphism $\Lambda_{n-1}^n \simeq \text{Aff}_A(C, \overline{\Lambda}^n)$ [5].

Finally, let $\pi : E \rightarrow M$ be a fiber bundle and \dots, q^A, \dots fiber coordinates on E . Notice that $\Omega^1(E, \pi)$ (resp. $\underline{\Omega}^1(E, \pi)$) is the $C^\infty(E)$ -module of sections of a vector bundle $\mu_0 \pi : \mathcal{M} \pi \rightarrow E$ (resp. $\tau_0^\dagger \pi : J^\dagger \pi \rightarrow E$). Recall that there is a distinguished element Θ in $\Omega^1(\mathcal{M} \pi, \mu \pi)$ (resp. $\underline{\Theta} \in \underline{\Omega}^1(J^\dagger \pi, \tau^\dagger \pi)$), the tautological one [5], which in standard coordinates

$$\dots, x^i, \dots, q^A, \dots, p_A^i, \dots, p$$

on $\mathcal{M} \pi$ (resp. $\dots, x^i, \dots, q^A, \dots, p_A^i, \dots$ on $J^\dagger \pi$) is given by

$$\Theta = p_A^i dq^A d^{n-1}x_i - p d^n x \quad (\text{resp. } \underline{\Theta} = p_A^i d^V q^A \otimes d^{n-1}x_i).$$

3. AFFINE FORM CALCULUS

3.1. Natural Operations with Affine Forms. In this section we derive the main formulas of calculus on affine forms. Such formulas will turn useful in generalizing proofs from the context of hamiltonian systems to the context of PD-hamiltonian systems (see Section 5).

Let $\alpha : P \longrightarrow M$ be as in the previous section. Isomorphism ι (resp. $\iota_0, \underline{\iota}$) can be used to “transfer structures” from Λ_{n-1} (resp. Λ_n, E_0^{n-1}) to Ω (resp. $\Omega_0, \underline{\Omega}$) and back. As an instance, notice that Ω has got a natural structure of $\Lambda(P)$ -module given by

$$\lambda\vartheta := \iota(\lambda\omega),$$

$\lambda \in \Lambda(P)$, $\vartheta = \iota(\omega) \in \Omega$, $\omega \in \Lambda_{n-1}$. Moreover, Ω is generated by Ω^0 as a $\Lambda(P)$ -module. Similarly, Λ_n (resp. E_0^{n-1}) has a structure of $V\Lambda$ -module given by

$$\lambda^V \omega_0 := \iota_0^{-1}(\lambda^V \rho^V \otimes \nu) \text{ (resp. } \lambda^V \underline{\omega} := \iota_0^{-1}(\lambda^V \rho^V \otimes \sigma)),$$

$\lambda^V \in V\Lambda$, $\omega_0 = \iota_0^{-1}(\rho^V \otimes \nu)$ (resp. $\underline{\omega} = \underline{\iota}^{-1}(\rho^V \otimes \sigma)$), $\rho^V \in V\Lambda$, $\nu \in \overline{\Lambda}^n$ (resp. $\sigma \in \overline{\Lambda}^{n-1}$), so that $\rho^V \otimes \nu \in V\Lambda \otimes_A \overline{\Lambda}^n = \Omega_0$ (resp. $\rho^V \otimes \sigma \in V\Lambda \otimes_A \overline{\Lambda}^{n-1} = \underline{\Omega}$). Clearly, Λ_n (resp. E_0^{n-1}) is generated by $\overline{\Lambda}^n$ (resp. $\overline{\Lambda}^{n-1}$) as a $V\Lambda$ -module. Finally, the presented structures are compatible in the sense that for $\omega_0 \in \Lambda_n$, $\omega \in \Lambda_{n-1}$ and $\lambda \in \Lambda(P)$, we have

$$\lambda^V \omega_0 = \lambda \omega_0 \text{ and } \underline{\lambda \omega} = \lambda^V \underline{\omega}.$$

As a last instance of how to use isomorphisms in (6) to transfer a structure from one space to the other we define the insertion of a connection $\nabla \in C$ into a differential form $\omega \in \Lambda_n$ as follows

$$i_{\nabla} \omega := \iota_0^{-1}(\vartheta(\nabla)) = (\iota_0^{-1} \circ \mathfrak{p}_{\nabla}^{q-1, n})(\omega) \in \Omega_0,$$

$\vartheta = \iota(\omega)$. Notice that the just defined insertion of a connection in an element $\omega \in \Lambda_n$ has been already discussed in [30]. In the following we will always understand isomorphisms $\iota, \iota_0, \underline{\iota}$.

Notice that $\underline{\Omega}$ inherits many operations from Ω . Indeed, let $\nabla \in C$, $Z \in \overline{\Lambda}^1 \otimes_A VD \subset \Lambda(P) \otimes_A D(P)$, $Y \in VD$, $X \in D_V$, $q \geq 0$. Then

- $i_Z(\Omega) \subset \Omega_0$ and $i_Z(\Omega_0) = 0$ so that it is well defined an operator, which, abusing the notation, we again denote by $i_Z : \underline{\Omega} \longrightarrow \Omega_0$, via the formula

$$i_Z \underline{\omega} := i_Z \omega \in \Omega_0,$$

$\omega \in \Omega$. Moreover, it is easy to show that

$$i_Z \underline{\omega} = i_{\nabla+Z} \omega - i_{\nabla} \omega.$$

Finally, for $Z = \eta \otimes Y_1$, and $\underline{\omega} = \rho^V \otimes \sigma$, $\eta \in \overline{\Lambda}^1$, $Y_1 \in VD$, $\rho^V \in V\Lambda^q$ and $\sigma \in \overline{\Lambda}^{n-1}$, we have

$$i_Z \underline{\omega} = (-)^{q-1} i_{Y_1} \rho^V \otimes \eta \sigma.$$

- $i_Y(\Omega) \subset \Omega$ (resp. $L_X(\Omega) \subset \Omega$) and $i_Y(\Omega_0) \subset \Omega_0$ (resp. $L_X(\Omega_0) \subset \Omega_0$) so that it is well defined the quotient map, which, abusing the notation, we again denote by $i_Y : \underline{\Omega} \longrightarrow \underline{\Omega}$ (resp. $L_X : \underline{\Omega} \longrightarrow \underline{\Omega}$), via the formula

$$i_Y \underline{\omega} := \underline{i_Y \omega} \in \underline{\Omega} \text{ (resp. } L_X \underline{\omega} = \underline{L_X \omega} \in \underline{\Omega}\text{)}.$$

Finally, for $\underline{\omega} = \rho^V \otimes \sigma$, $\rho^V \in V\Lambda^q$ and $\sigma \in \bar{\Lambda}^{n-1}$, we have

$$i_Y \underline{\omega} = i_Y \rho^V \otimes \sigma.$$

- $d_{\nabla}(\Omega) \subset \Omega_0$ and $d_{\nabla}(\Omega_0) = 0$ so that it is well defined an operator, which, abusing the notation, we again denote by $d_{\nabla} : \underline{\Omega} \longrightarrow \underline{\Omega}$, via the formula

$$d_{\nabla} \underline{\omega} := d_{\nabla} \omega \in \Omega_0,$$

$\omega \in \Omega$.

Remark 2. Notice that the insertion $i_{\nabla} \omega$, being affine in ∇ , is actually point wise, i.e., if $\nabla' \in C$ is such that $\nabla'_y = \nabla_y \in C|_y = \alpha_{1,0}^{-1}(y)$ for some $y \in P$, then $(i_{\nabla'} \omega)_y = (i_{\nabla} \omega)_y$. Therefore, it is well defined the insertion $i_c \omega_y$ of an element $c \in \alpha_{1,0}^{-1}(y)$, $y \in P$, into ω_y . Similar considerations apply to both the above defined insertions i_Z and i_Y . Finally, for all $y \in P$, the projection $\Omega \longrightarrow \underline{\Omega}$ as well determines a well defined linear map $\Omega|_y \ni \omega_y \longmapsto \underline{\omega}_y \in \underline{\Omega}|_y$ whose kernel is $\Omega_0|_y$.

In the following we will denote by $\delta : \Omega \longrightarrow \Omega$ (resp. $\delta_0 : \Omega_0 \longrightarrow \Omega_0$) the restricted de Rham differential, i.e., for $\omega \in \Omega$ (resp. $\omega_0 \in \Omega_0$), $\delta \omega := d\omega \in \Omega$ (resp. $\delta_0 \omega_0 := d\omega_0 \in \Omega_0$) and with $\underline{\delta} : \underline{\Omega} \longrightarrow \underline{\Omega}$ the quotient differential. Then, for $\omega_0 = \rho^V \otimes \alpha^*(\nu_0)$ (resp. $\omega = \rho^V \otimes \alpha^*(\sigma_0)$), $\rho^V \in V\Lambda$, $\nu_0 \in \Lambda^n(M)$ (resp. $\sigma_0 \in \Lambda^{n-1}(M)$), we have

$$\delta_0 \omega_0 = d^V \rho^V \otimes \alpha^*(\nu_0) \text{ (resp. } \underline{\delta} \omega = d^V \rho^V \otimes \alpha^*(\sigma_0)\text{)}.$$

In other words δ_0 (resp. $\underline{\delta}$) is isomorphic to the differential $d^V \otimes \text{id} : V\Lambda \otimes_{A_0} \Lambda^n(M) \longrightarrow V\Lambda \otimes_{A_0} \Lambda^n(M)$ (resp. $d^V \otimes \text{id} : V\Lambda \otimes_{A_0} \Lambda^{n-1}(M) \longrightarrow V\Lambda \otimes_{A_0} \Lambda^{n-1}(M)$).

All the above mentioned formulas can be proved by straightforward computations.

Now, let ∇ , Y and X be as above. denote by $[[\cdot, \cdot]]$ the Frölicher-Nijenhuis bracket in $\Lambda(P) \otimes_A D(P)$. It is easy to see that $[[H_{\nabla}, X]] \in \bar{\Lambda}^1 \otimes_A VD \subset \Lambda(P) \otimes_A D(P)$. It holds the following

Theorem 3. Let $\omega \in \Omega$, then

$$\begin{aligned} [i_{\nabla}, \delta] \omega &:= (i_{\nabla} \circ \delta - \delta_0 \circ i_{\nabla}) \omega = d_{\nabla} \omega \in \Omega_0, \\ [i_{\nabla}, i_Y] \omega &:= (i_{\nabla} \circ i_Y - i_Y \circ i_{\nabla}) \omega = 0 \in \Omega_0, \\ [i_{\nabla}, L_X] \omega &:= (i_{\nabla} \circ L_X - L_X \circ i_{\nabla}) \omega = i_{[[H_{\nabla}, X]]} \omega \in \Omega_0. \end{aligned} \tag{8}$$

Proof. First prove that $i_{\nabla} : \Omega \longrightarrow \Omega_0$ satisfies the ‘‘Leibnitz rule’’

$$i_{\nabla}(\lambda \omega) = \lambda \cdot i_{\nabla} \omega + i_{H_{\nabla}} \lambda \cdot \omega, \tag{9}$$

$\lambda \in \Lambda(P)$, $\omega \in \Omega$. For $\rho \in \Lambda(P)$, denote $\rho_{\nabla}^{\bullet,p} := \sum_q \mathfrak{p}_{\nabla}^{q,p}(\rho)$, so that $\rho = \sum_p \rho_{\nabla}^{\bullet,p}$. Notice that for $\omega \in \Omega$ and $\lambda \in \Lambda(P)$, we have $\omega = \omega_{\nabla}^{\bullet,n} + \omega_{\nabla}^{\bullet,n-1}$ so that

$$i_{\nabla}(\lambda\omega) = \mathfrak{p}_{\nabla}^{\bullet,n}(\lambda\omega) = \lambda_{\nabla}^{l,0} \omega_{\nabla}^{\bullet,n} + \lambda_{\nabla}^{\bullet,1} \omega_{\nabla}^{\bullet,n-1} = \lambda \cdot i_{\nabla}\omega + \lambda_{\nabla}^{\bullet,1} \cdot \omega_{\nabla}^{\bullet,n-1}.$$

Moreover, $i_{H_{\nabla}}\lambda = \sum_p i_{H_{\nabla}}\lambda_{\nabla}^{\bullet,p} = \sum_p p\lambda_{\nabla}^{\bullet,p}$, which in turn implies $\lambda_{\nabla}^{\bullet,p} = i_{H_{\nabla}}\lambda - \sum_{p>1} p\lambda_{\nabla}^{\bullet,p}$. Therefore

$$\begin{aligned} i_{\nabla}(\lambda\omega) &= \lambda \cdot i_{\nabla}\omega + \lambda_{\nabla}^{\bullet,1} \cdot \omega_{\nabla}^{\bullet,n-1} \\ &= \lambda \cdot i_{\nabla}\omega + i_{H_{\nabla}}\lambda \cdot \omega_{\nabla}^{\bullet,n-1} - \sum_{p>1} p\lambda_{\nabla}^{\bullet,p} \omega_{\nabla}^{\bullet,n-1} \\ &= \lambda \cdot i_{\nabla}\omega + i_{H_{\nabla}}\lambda \cdot \omega. \end{aligned}$$

In view of (9), the above defined operators $[i_{\nabla}, \delta]$, $[i_{\nabla}, i_Y]$, $[i_{\nabla}, L_X] : \Omega \longrightarrow \Omega_0$, satisfy analogous ‘‘Leibnitz rules’’

$$\begin{aligned} [i_{\nabla}, \delta](\lambda\omega) &= d_{\nabla}\lambda \cdot \omega + (-)^l \lambda \cdot [i_{\nabla}, \delta](\omega), \\ [i_{\nabla}, i_Y](\lambda\omega) &= \lambda \cdot [i_{\nabla}, i_Y](\omega), \\ [i_{\nabla}, L_X](\lambda\omega) &= i_{\llbracket H_{\nabla}, X \rrbracket} \lambda \cdot \omega + \lambda \cdot [i_{\nabla}, L_X](\omega). \end{aligned} \tag{10}$$

Since Ω is generated by $\overline{\Lambda}^{n-1}$ as a $\Lambda(P)$ -module, in view of (10), it is enough to prove (8) for $\omega \in \overline{\Lambda}^{n-1}$. In this case

$$\begin{aligned} [i_{\nabla}, \delta]\omega &= i_{\nabla}d\omega = (d\omega)_{\nabla}^{\bullet,n} = d_{\nabla}\omega, \\ [i_{\nabla}, i_Y]\omega &= 0, \\ [i_{\nabla}, L_X]\omega &= i_{\nabla}L_X\omega = (L_X\omega)_{\nabla}^{\bullet,n} = 0 = i_{\llbracket H_{\nabla}, X \rrbracket}\omega. \end{aligned}$$

□

We now discuss the interaction between affine forms and bundle morphisms. Let $\alpha' : P' \longrightarrow M$ be another fiber bundle and $G : P \longrightarrow P'$ a bundle morphism. Clearly, G preserves the ideals Λ_p , $p \geq 0$, i.e., $G^*(\Lambda_p(P', \alpha')) \subset \Lambda_p(P, \alpha)$. In particular,

$$G^*(\Omega(P', \alpha')) \subset \Omega(P, \alpha) \text{ and } G^*(\Omega_0(P', \alpha')) \subset \Omega_0(P, \alpha).$$

We conclude that it is well defined the quotient map which, abusing the notation, we again denote by $G^* : \underline{\Omega}(P', \alpha') \longrightarrow \underline{\Omega}(P, \alpha)$. Now, consider G -compatible connections $\nabla \in C(P, \alpha)$ and $\nabla' \in C(P', \alpha')$. It is easy to show that

$$G^* \circ i_{\nabla'} = i_{\nabla} \circ G^* : \Omega(P', \alpha') \longrightarrow \Omega_0(P, \alpha). \tag{11}$$

3.2. Cohomology.

Remark 4. (see [41]) In the following we denote by \mathcal{F} the abstract fiber of α . Notice that, for any $q \geq 0$, $VH^q \equiv VH^q(P, \alpha) := H^q(V\Lambda, d^V)$ is the A_0 -module of sections of a (pro-finite) vector bundle $\alpha^q : P^q \longrightarrow M$ over M whose abstract fiber is $H^q(\mathcal{F})$. Moreover, α^q is endowed with a canonical flat connection ∇^q (∇^q is a smooth analogue

of Gauss-Manin connection in algebraic geometry). Correspondingly, there is a de Rham like complex

$$\cdots \longrightarrow \Lambda^{p-1} \otimes_{A_0} VH^q \xrightarrow{d_1^{p-1,q}} \Lambda^p \otimes_{A_0} VH^q \xrightarrow{d_1^{p,q}} \cdots,$$

whose cohomology we denote by $E_2^{\bullet,q} := \bigoplus_p E_2^{p,q}$, $E_2^{p,q} := H^p(\Lambda(M) \otimes_{A_0} V\Lambda^q, d_1^{\bullet,q})$ ², $q \geq 0$. It can be proved that, if α is trivial or M is simply connected, then there is a (generically non-canonical), isomorphism

$$E_2^{p,q} \approx H^p(M) \otimes H^q(\mathcal{F}), \quad p, q \geq 0.$$

Finally, notice also that, for any $q \geq 0$,

$$\begin{aligned} H^q(\Omega_0, \delta_0) &\simeq \Lambda^n(M) \otimes_{A_0} VH^q, \\ H^q(\underline{\Omega}, \underline{\delta}) &\simeq \Lambda^{n-1}(M) \otimes_{A_0} VH^q. \end{aligned}$$

Proposition 5. *Let $\alpha : P \rightarrow M$ be a fiber bundle. Then, for any $q \geq 0$, there exists a short exact sequence of vector spaces*

$$0 \longrightarrow \text{coker } d_1^{n,q-1} \longrightarrow H^q(\Omega, \delta) \longrightarrow \ker d_1^{n-1,q} \longrightarrow 0.$$

In particular, $H^q(\Omega, \delta) \approx \text{coker } d_1^{n,q-1} \oplus \ker d_1^{n-1,q} = E_2^{n,q-1} \oplus \ker d_1^{n-1,q}$.

Proof. Consider the short exact sequence of complexes

$$0 \longrightarrow \Omega_0 \longrightarrow \Omega \longrightarrow \underline{\Omega} \longrightarrow 0,$$

and the associated long sequence in cohomology

$$\cdots \longrightarrow H^{q-1}(\underline{\Omega}, \underline{\delta}) \xrightarrow{\partial} H^q(\Omega_0, \delta_0) \longrightarrow H^q(\Omega, \delta) \longrightarrow H^q(\underline{\Omega}, \underline{\delta}) \xrightarrow{\partial} \cdots. \quad (12)$$

We already commented, in the above remark, that, for any q , $H^q(\Omega_0, \delta_0)$ identifies with $\Lambda^n(M) \otimes_{A_0} VH^q$ and $H^q(\underline{\Omega}, \underline{\delta})$ identifies with $\Lambda^{n-1}(M) \otimes_{A_0} VH^q$. Similarly, it is easy to show that the connecting operator

$$\partial : H^{q-1}(\underline{\Omega}, \underline{\delta}) \longrightarrow H^q(\Omega_0, \delta_0)$$

identifies with the de Rham-like differential

$$d_1^{n-1,q} : \Lambda^{n-1}(M) \otimes_{A_0} VH^q \longrightarrow \Lambda^n(M) \otimes_{A_0} VH^q.$$

The thesis then follows from exactness of (12). \square

Corollary 6. *If \mathcal{F} is connected, then $H^0(\Omega, \delta) \simeq \ker d_M^{n-1}$,*

$$d_M^{n-1} : \Lambda^{n-1}(M) \longrightarrow \Lambda^n(M)$$

being the last de Rham differential of M .

²Similarly as above, this last notations are motivated by the fact that the differentials $d_1^{\bullet,q}$ (resp. the vector spaces $E_2^{\bullet,q}$) are the ones in the first term (resp. are rows of the second term) of the (cohomological) Leray-Serre spectral sequence of the fiber bundle α [41].

Proof. If \mathcal{F} is connected $VH^0 \simeq A_0$ and $d_1^{n-1,0}$ identifies with d_M^{n-1} . \square

Corollary 7. *Let $q \geq 0$ and $\omega \in \Omega^q$ be δ -closed, i.e., $\delta\omega = 0$. Then, 1) if $q = 0$, ω is locally of the form $\alpha^*(\eta)$ for some $\eta \in \Lambda^{n-1}(M)$, 2) if $q > 0$, then ω is locally δ -exact, i.e., ω is locally of the form $\delta\theta$, θ being a local element in Ω^{q-1} .*

Proof. If \mathcal{F} is contractible, then $VH^q = 0$, and therefore $H^q(\Omega, \delta) = 0$, for all $q > 0$. \square

Let $\omega \in \Omega$ and $\theta \in \Omega$ be such that $\omega = \delta\theta$. Then θ will be said a *potential* of ω .

4. PD-HAMILTONIAN SYSTEMS

4.1. PD-Hamiltonian Systems and PD-Hamilton Equations. In this section we introduce what we think should be understood as the partial differential, i.e., field theoretic, (1st order) analogue of a hamiltonian (mechanical) system on an abstract symplectic manifold.

Let $\alpha : P \rightarrow M$ be as in the previous section and $\omega \in \Omega^2(P, \alpha)$ be such that $\delta\omega = 0$. Put

$$\begin{aligned} \ker \omega &:= \{Y \in VD \mid i_Y \omega = 0\}, & \ker \underline{\omega} &:= \{Y \in VD \mid i_Y \underline{\omega} = 0\}, \\ \text{Ker } \omega &:= \{\nabla \in C \mid i_{\nabla} \omega = 0\}, & \text{Ker } \underline{\omega} &:= \{Z \in VD \otimes_A \bar{\Lambda}^1 \mid i_Z \underline{\omega} = 0\}. \end{aligned}$$

Since ω is closed, both $\ker \omega$ and $\ker \underline{\omega}$ are modules of smooth sections of involutive α -vertical distributions D^ω and \underline{D}^ω on P , where, for $y \in P$,

$$D_y^\omega := \{\xi \in V_y P \mid i_\xi \omega_y = 0\}, \quad \underline{D}_y^\omega := \{\xi \in V_y P \mid i_\xi \underline{\omega}_y = 0\}.$$

Similarly, $\text{Ker } \underline{\omega}$ is a sub-module in $VD \otimes_A \bar{\Lambda}^1$. As a minimal regularity requirement assume that \underline{D}^ω has got constant rank \underline{r} . Then, it is easy to check that, as a consequence, $\text{Ker } \underline{\omega}$ is the module of sections of a smooth vector bundle $\varpi : W \rightarrow P$. For $y \in P$, denote $r(y) = \dim D_y^\omega$. In general, $r(y)$ will change from point to point $y \in P$. However, we are proving in brief that $r(y)$ cannot change that much. First of all, since, obviously, $D^\omega \subset \underline{D}^\omega$, then $r(y) \leq \underline{r}$ for all $y \in P$. Now, for $y \in P$, denote

$$\text{Ker } \omega_y := \{c \in \alpha_{1,0}^{-1}(y) \mid i_c \omega_y = 0\}.$$

Then, $\text{Ker } \omega_y$ is either empty or an affine space modelled over $\varpi^{-1}(y)$. It holds the

Proposition 8. *For any $y \in P$, $\underline{r} - r(y) \leq 1$ (see also Theorem 4 of [23]).*

Proof. Let $y \in P$ and suppose $r(y) < \underline{r}$. If $\xi \in \underline{D}_y^\omega$ then (see Remark 2) $i_\xi \underline{\omega}_y = i_\xi \omega_y = 0$ so that $i_\xi \omega_y \in \Omega_0^1|_y = \bar{\Lambda}^n|_y$. Then consider the map $\gamma_y : \underline{D}_y^\omega \ni \xi \mapsto \gamma_y(\xi) := i_\xi \omega_y \in \bar{\Lambda}^n|_y$. Since $r(y) < \underline{r}$, γ_y is surjective and the sequence of vector spaces $0 \rightarrow D_y^\omega \rightarrow \underline{D}_y^\omega \xrightarrow{\gamma_y} \bar{\Lambda}^n|_y \rightarrow 0$ is exact. Since $\bar{\Lambda}^n|_y$ is 1-dimensional, it follows that $\underline{r} - r(y) = 1$. \square

The following proposition characterizes the case $r(y) = \underline{r}$.

Proposition 9. *Let ω be as above. Then $r(y) = \underline{r}$ iff $\text{Ker } \omega_y \neq \emptyset$.*

Proof. The result is nothing more than an application of the Rouché-Capelli theorem. We here propose a dual proof. Let $\xi \in V_y P$ be given by $\xi = \xi^a \partial_a |_y$. Then $\xi \in \underline{D}_y^\omega$ iff

$$\omega_{ab}^i(y) \xi^a = 0, \quad a = 1, \dots, m, \quad i = 1, \dots, n. \quad (13)$$

Similarly, $\xi \in D_y^\omega$ iff they are satisfied both (13) and

$$\omega_a(y) \xi^a = 0. \quad (14)$$

Therefore, $D_y^\omega = \underline{D}_y^\omega$ iff Equation (14) linearly depends on Equations (13), i.e., iff there are real numbers h_i^b , $b = 1, \dots, m, i = 1, \dots, n$, such that

$$\omega_a(y) = \omega_{ab}^i(y) h_i^b,$$

$a = 1, \dots, m$. Now, let $c \in \alpha_{1,0}^{-1}(y)$ be given by $y_i^a(c) = -\frac{1}{2} h_i^a$. Then $i_c \omega_y$ is given by

$$\begin{aligned} i_c \omega_y &= (-2\omega_{ba}^i(y) y_i^a(c) + \omega_a(y)) dy^a d^n x |_y \\ &= -(\omega_{ab}^i(y) h_i^b - \omega_a(y)) dy^a d^n x |_y \\ &= 0. \end{aligned}$$

□

Definition 10. A PD-prehamiltonian system on the fiber bundle $\alpha : P \rightarrow M$ is a δ -closed element $\omega \in \Omega^2(P, \alpha)$. A PD-hamiltonian system on α is a PD-prehamiltonian system ω such that $\ker \underline{\omega} = 0$ (and, therefore, $\ker \omega = 0$ as well).

Let $\theta \in \Omega^1$ be locally given by $\theta = \theta_a^i dy^a d^{n-1} x_i - H d^n x, \dots, \theta_a^i, \dots, H$ being local functions on P . Then $\delta\theta$ is locally given by

$$\delta\theta = \partial_{[a} \theta_{b]}^i dy^a dy^b d^{n-1} x_i - (\partial_i \theta_a^i + \partial_a H) dy^a d^n x.$$

Similarly, let $\omega \in \Omega^2$ and $Y \in VD$ be locally given by $\omega = \omega_{ab}^i dy^a dy^b d^{n-1} x_i + \omega_a dy^a d^n x$ and $Y = Y^a \partial_a$, respectively. Then $\delta\omega$, $i_Y \omega$ and $i_Y \underline{\omega}$ are locally given by

$$\begin{aligned} \delta\omega &= \partial_{[a} \omega_{bc]}^i dy^a dy^b dy^c d^{n-1} x_i + (\partial_i \omega_{ab}^i + \partial_{[a} \omega_{b]}^i) dy^a dy^b d^n x, \\ i_Y \omega &= 2\omega_{ab}^i Y^a dy^b d^{n-1} x_i + \omega_a Y^a d^n x, \\ i_Y \underline{\omega} &= 2\omega_{ab}^i Y^a d^V y^b \otimes d^{n-1} x_i, \end{aligned}$$

so that ω is a PD-prehamiltonian system iff

$$\partial_{[a} \omega_{bc]}^i = 0, \quad \partial_i \omega_{ab}^i + \partial_{[a} \omega_{b]}^i = 0, \quad (15)$$

or, which is the same (see Corollary 7),

$$\omega_{ab}^i = \partial_{[a} \theta_{b]}^i, \quad \omega_a = -\partial_a H - \partial_i \theta_a^i, \quad (16)$$

for some $\dots, \theta_a^i, \dots, H$ local functions on P . Moreover, ω is a PD-hamiltonian system iff

$$\omega_{ab}^i Y^a = 0 \implies Y^a = 0. \quad (17)$$

In its turn (17) implies $\omega_a = \omega_{ab}^i f_i^b$ for some \dots, f_i^b, \dots local functions on P (see the proof of Proposition 9).

Let ω be a PD-prehamiltonian system on α , $\sigma : U \longrightarrow P$ a local section of α , $U \subset M$ an open subset. The first jet prolongation $\dot{\sigma} : U \longrightarrow J^1\alpha$ of σ may be interpreted as a “connection in α along σ ”, i.e., a section of the restricted bundle $\alpha_{1,0}|_\sigma : J^1\alpha|_\sigma \longrightarrow P$. Moreover, elements in $\Omega|_\sigma$ may be interpreted as affine maps from $C|_\sigma$ to $\Omega_0|_\sigma \simeq V\Lambda|_\sigma \otimes_{A_0} \Lambda^n(M)$ whose linear part is in $\underline{\Omega}|_\sigma \simeq V\Lambda|_\sigma \otimes_{A_0} \Lambda^{n-1}(M)$. Namely, an element $\diamond \in C|_\sigma$ can “be inserted” into an element $\rho|_\sigma \in \Omega|_\sigma$, $\rho \in \Omega$, giving an element $i_{\diamond}\rho|_\sigma \in \Omega_0|_\sigma$. Thus, we can search for local sections σ of α such that

$$i_{\dot{\sigma}}\omega|_\sigma = 0 \quad (18)$$

Definition 11. *Equations (18) are called the PD-Hamilton equations (of the PD-prehamiltonian system ω).*

If ω is locally given by $\omega = \omega_{ab}^i dy^a dy^b d^{n-1}x_i + \omega_a dy^a d^n x$, then the associated PD-Hamilton equations are locally given by

$$2\omega_{ab}^i \partial_i y^a - \omega_b = 0. \quad (19)$$

Conversely, a system of PDEs in the form (19) is a PD-Hamilton equation for some PD-prehamiltonian (resp. PD-hamiltonian) system iff coefficients $\dots, \omega_{ab}^i, \dots, \omega_b, \dots$ satisfy (15) (or, which is the same, (16)) (resp. (15) and (17)). Notice that, in view of (19) a general PD-prehamiltonian system ω encode “kinematical information”, which can be identified with $\underline{\omega}$, and “dynamical information”, which can be identified with the specific choice of ω in the class of those PD-hamiltonian systems with linear part $\underline{\omega}$ (see the comment at the end of Section 2.3, Remark 17 and Example 18).

Searching for solutions of PD-Hamilton equations of a PD-prehamiltonian system ω , we could proceed in two steps:

- (1) search for a connection $\nabla \in \text{Ker } \omega$,
- (2) search for n -dimensional integral submanifolds of the horizontal distribution $H_{\nabla}P$.

However, a solution to the first step of the above mentioned procedure exists iff $\ker \omega = \ker \underline{\omega}$ which is not always the case. Therefore, in general, we are led to weaken 1 and search for connections ∇' in a subbundle $P' \subset P$ such that $i_{\nabla'}\omega|_{P'} = 0$. As showed in the next proposition, there is always an algorithmic way to find a maximal subbundle $\check{\alpha} : \check{P} \longrightarrow M$ of α such that the affine equation $i_{\check{\nabla}}\omega|_{\check{P}} = 0$, $\check{\nabla} \in C(\check{P}, \check{\alpha})$ admits at least one solution. We will refer to the above mentioned algorithm as the PD-constraint algorithm (see also [19, 31, 32, 33, 34]).

Proposition 12. *Let ω be as above and $\text{Ker } \omega = \emptyset$ (i.e., $D_y^\omega \neq \underline{D}_y^\omega$ for some $y \in P$). Under suitable regularity conditions on ω (to be specified in the proof), there exists a (maximal) subbundle $\check{P} \subset P$ such that $i_{\check{\nabla}}\omega|_{\check{P}} = 0$ for some $\check{\nabla} \in C(\check{P}, \check{\alpha})$.*

Proof. For $s = 1, 2, \dots$ define recursively

$$P_{(s)} := \{y \in P_{(s-1)} \mid \text{Ker } \omega_y \cap (\alpha_{(s-1)})_1^{-1}(y) \neq \emptyset\} \subset P,$$

$$\alpha_{(s)} := \alpha|_{P_{(s)}} : P_s \longrightarrow M,$$

where $P_{(0)} := P$, $\alpha_{(0)} := \alpha$ (in particular $P_1 = \{y \in P \mid r(y) = \underline{r}\}$). We assume that $\alpha_{(s)} : P_{(s)} \longrightarrow M$ is a smooth (closed) subbundle for all s (regularity conditions). Then, for dimensional reasons, there exists \bar{s} such that $P_{(s)} = P_{(\bar{s})}$ for all $s \geq \bar{s}$. Put $\check{P} := P_{(\bar{s})}$. \square

$\check{\alpha} := \alpha|_{\check{P}} : \check{P} \longrightarrow M$ will be called the *constraint subbundle*. Notice that \check{P} can be empty (for instance when $r(y) = \underline{r} - 1$ for all $y \in P$) and, in this case, PD-Hamilton equations do not possess solutions.

Corollary 13. *Let ω be a PD-prehamiltonian system on α and σ a solution of PD-Hamilton equations. Then $\text{im } \sigma \subset \check{P}$.*

Proof. By induction on s , $\text{im } \sigma \subset P_{(s)}$ for all $s = 1, 2, \dots$ \square

The converse of the above corollary is, a priori, only true for $n = 1$. Namely, we may wonder if for any $y \in \check{P}$ there is a solution σ of PD-Hamilton equations such that $y \in \text{im } \sigma$. We know that there is a connection $\check{\nabla}$ in \check{P} which is “a solution of PD-Hamilton equations up to first order”, i.e., $i_{\check{\nabla}}\check{\omega}|_{\check{P}} = 0$. n -dimensional integral manifolds of the horizontal distribution $H_{\check{\nabla}}\check{P}$ determined on \check{P} by $\check{\nabla}$ are clearly images of solutions of PD-Hamilton equations. If $n = 1$, $\check{\nabla}$ is trivially flat and Fröbenius theorem guarantees that for any $y \in \check{P}$ there is a solution “through y ”. The same is a priori untrue for $n > 2$. Integrability conditions on $H_{\check{\nabla}}\check{P}$ will be discussed in a separate paper.

Clearly, standard examples of PD-prehamiltonian systems come from Lagrangian field theory. Let us briefly recall how. Let $\pi : E \longrightarrow M$ be a fiber bundle and consider a lagrangian density $\mathcal{L} \in \overline{\Lambda}^n(J^1\pi, \pi_1)$. \mathcal{L} determines two Legendre transforms (see, for instance, [4]) which we denote by $F\mathcal{L} : J^1\pi \longrightarrow \mathcal{M}\pi$ and $\underline{F}\mathcal{L} : J^1\pi \longrightarrow J^1\pi$. Consider the submanifold $C_0 := \text{im } \underline{F}\mathcal{L} \subset J^1\pi$ of primary hamiltonian constraints of the lagrangian theory. $F\mathcal{L}$ factorizes as $F\mathcal{L} = \mathcal{H} \circ \underline{F}\mathcal{L}$, $\mathcal{H} : C_0 \longrightarrow \mathcal{M}\pi$ being the hamiltonian section. Then $\omega_{\mathcal{L}} := F\mathcal{L}^*(\delta\Theta)$ (resp. $\mathcal{H}^*(\delta\Theta)$) is a PD-prehamiltonian system in π_1 (resp. in $C_0 \longrightarrow M$). Moreover, as it is well known [3], if $s : M \longrightarrow E$ is a solution of the Euler-Lagrange equations, then $\dot{s} : M \longrightarrow J^1\pi$ (resp. $\underline{F}\mathcal{L} \circ \dot{s} : M \longrightarrow C_0$) is a solution of PD-Hamilton equations of $\omega_{\mathcal{L}}$ (resp. $\mathcal{H}^*(\delta\Theta)$).

4.2. PD-Hamiltonian Systems and Multisymplectic Geometry *à la* Forger.

Forger and Gomes have recently proposed a definition of multipresymplectic structure on a fiber bundle [23]. Their work aims to define such a structure so that 1) the differential $d\Theta$ of the tautological n -form Θ on the affine dual bundle of the first jet bundle is multisymplectic 2) every multipresymplectic structure is locally isomorphic to the pull-back of Θ along a fibration (Darboux lemma). Since, in our opinion, 1) [23] is the best motivated and established work about fundamentals of multisymplectic geometry, 2) abstract fiber bundles play in [23] a similar role as in this paper, we

analyze in this subsection the relationship between PD-prehamiltonian systems and multipresymplectic structures *à la* Forger, referring to [23] for the main definition. Here we just mention two of the main results of [23] (which can eventually be understood as definitions of polypresymplectic structure and multipresymplectic structure on a fiber bundle, respectively)

Theorem 14 (Forger and Gomes I). *Let $\alpha : P \longrightarrow M$ be a fiber bundle, \dots, x^i, \dots local coordinates on M , $i = 1, \dots, n = \dim M$ and $\underline{\omega} \in \underline{\Omega}^2$. $\underline{\omega}$ is a polypresymplectic structure on α iff, around every point of P , there are local fiber coordinates $\dots, q^A, \dots, p_A^i, \dots, z^1, \dots, z^s$, $A = 1, \dots, m$, $i = 1, \dots, n$ (so that $\dim P = n + m + mn + s$) such that $\underline{\omega}$ is locally given by*

$$\omega = d^V p_A^i d^V q^A \otimes d^{n-1} x_i.$$

Theorem 15 (Forger and Gomes II). *Let $\alpha : P \longrightarrow M$ be a fiber bundle, \dots, x^i, \dots local coordinates on M , $i = 1, \dots, n = \dim M$, and $\omega \in \Omega^2$. ω is a multipresymplectic structure on α iff, around every point of P , there are local fiber coordinates $\dots, q^A, \dots, p_A^i, \dots, p, z^1, \dots, z^r$, $A = 1, \dots, m$, $i = 1, \dots, n$ (so that $\dim P = (n + 1)(m + 1) + r$) such that ω is locally given by*

$$\omega = dp_A^i dq^A d^{n-1} x_i - dp d^n x.$$

Proposition 16. *Let ω be a PD-prehamiltonian system on α . The following two conditions are equivalent.*

- (1) $\underline{\omega}$ is a polypresymplectic structure and $r(y) = \underline{r} - 1$ for all $y \in P$.
- (2) ω is a multipresymplectic structure.

Proof. Recall that, in view of Proposition 5, ω is locally δ -exact. Suppose $\underline{\omega}$ is polypresymplectic and $r(y) = \underline{r} - 1$ for all $y \in P$. Then $\underline{r} > 0$ and, in view of Theorem 14, (around every point in P) there are α -adapted local coordinates

$$\dots, x^i, \dots, q^A, \dots, p_A^i, z^0, z^1, \dots, z^{\underline{r}-1},$$

such that, locally, $\underline{\omega} = d^V p_A^i d^V q^A \otimes d^{n-1} x_i$ (in particular, $\ker \underline{\omega}$ is locally spanned by $\dots, \partial/\partial z^\alpha, \dots$). Therefore, $\underline{\omega} = \delta \underline{\theta}_0$, where $\underline{\theta}_0 := p_A^i d^V q^A \otimes d^{n-1} x_i$ is a local element of $\underline{\Omega}^1$. A general (local) potential of ω is then $\theta' \in \Omega^1$ such that $\underline{\theta}' = \underline{\theta}_0 + d^V \nu$, ν being a local element in $\underline{\Omega}^0 = \overline{\Lambda}^{n-1}$. The (local) potential $\theta := \theta' - \delta \nu$ is locally in the form $\theta = p_A^i dq^A d^{n-1} x_i - p d^n x$, where p is a local function on P . Therefore ω is locally given by

$$\omega = dp_A^i dq^A d^{n-1} x_i - dp d^n x.$$

$\ker \omega$ is locally spanned by those local elements $Y^\alpha \frac{\partial}{\partial z^\alpha}$ in $\ker \underline{\omega}$ such that $Y^\alpha \frac{\partial h}{\partial z^\alpha} = 0$. Since $\ker \omega \neq \ker \underline{\omega}$, then $\frac{\partial p}{\partial z^\alpha} dz^\alpha \neq 0$. Let, for instance, be $\frac{\partial p}{\partial z^0} \neq 0$. Then $\dots, x^i, \dots, q^A, \dots, p_A^i, p, z^1, \dots, z^{\underline{r}-1}$ is a new local coordinate system on P . In view of Theorem 15, ω is then multipresymplectic.

On the other hand, let ω be multipresymplectic. Then $\underline{\omega}$ is polypresymplectic. Moreover (around every point in P) there are α -adapted local coordinates

$$\dots, x^i, \dots, q^A, \dots, p_A^i, \dots, p, z^1, \dots, z^r \quad (20)$$

such that, locally, $\omega = dp_A^i dq^A d^{n-1}x_i - dp d^n x$ and $\underline{\omega} = d^V p_A^i d^V q^A \otimes d^{n-1}x_i$. This shows that for all $y \in P$,

$$D_y^\omega = \langle \dots, \frac{\partial}{\partial z^\alpha} \Big|_y, \dots \rangle \neq \underline{D}_y^\omega = \langle \dots, \frac{\partial}{\partial z^\alpha} \Big|_y, \dots, \frac{\partial}{\partial p} \Big|_y \rangle.$$

□

Remark 17. Let ω be a PD-prehamiltonian system ω . First of all, notice that, if ω is a multipresymplectic structure then, in view of Proposition 16, PD-Hamilton equations of ω do not possess solutions. In this sense, multipresymplectic structures do not contain any dynamical information.

Now, the proof of Proposition 16 also shows that if $\underline{\omega}$ is a polypresymplectic structure and $\ker \omega = \ker \underline{\omega}$ then ω is locally in the form

$$\omega = dp_A^i dq^A d^{n-1}x_i - dH d^n x$$

where $\frac{\partial H}{\partial z^\alpha} = 0$, $\alpha = 1, 2, \dots$, i.e., H is constant along the leaves of the distribution $D^\omega = \underline{D}^\omega$.

Example 18. Let $\omega \in \Omega^2$ be a multisymplectic structure on α . In this case $\ker \omega = 0$, while \underline{D}^ω is a 1-dimensional (involutive) distribution. Leaves of \underline{D}^ω are submanifolds in the fibers of α . denote by \underline{P} the set of leaves of \underline{D}^ω . There is an obvious projection $\underline{\alpha} : \underline{P} \rightarrow M$. Suppose that $\underline{\alpha} : \underline{P} \rightarrow M$ is a smooth fiber bundle and $\mathfrak{p} : P \rightarrow \underline{P}$ a smooth submersion (which is always true locally). There is a distinguished class of (local) PD-hamiltonian systems on $\underline{\alpha}$. Indeed, let $U \subset \underline{P}$ be an open subbundle and $\mathcal{H} : U \rightarrow P$ a local section of \mathfrak{p} . Then $\omega' := \mathcal{H}^*(\omega) \in \Omega^2(U, \underline{\alpha})$ is a PD-hamiltonian system. In particular, if we choose coordinates on P as in (20) (here $r = 0$), then $\dots, x^i, \dots, q^A, \dots, p_A^i, \dots$ are coordinates on \underline{P} , \mathcal{H} is given by

$$\mathcal{H}^*(p) = H,$$

H being a local function on \underline{P} , and ω' is locally given by

$$\omega' = dp_A^i dq^A d^{n-1}x_i - dH d^n x,$$

in particular $\theta' := p_A^i dq^A d^{n-1}x_i - H d^n x$ is a local potential of ω' . Finally, PD-Hamilton equations of ω' read

$$\begin{aligned} q_i^A &= \frac{\partial H}{\partial p_A^i}, \\ p_{A,i}^i &= -\frac{\partial H}{\partial q^A}, \end{aligned}$$

which are de Donder-Weyl equations (see, for instance, [3]).

4.3. PD-Hamiltonian Systems and Variational Calculus. We show in this subsection that PD-Hamilton equations are locally variational. First of all, an element $\theta \in \Omega^1$ may be understood as a (fiber-wise affine) horizontal n -form over $J^1\alpha$, i.e., as an element $\mathcal{L}^\theta \in \overline{\Lambda}^n(J^1\alpha, \alpha_1)$ via

$$\mathcal{L}_c^\theta := i_c \theta_y, \quad c \in J^1\alpha, \quad y = \alpha_{1,0}(c) \in P.$$

In its turn \mathcal{L}^θ is a 1st order lagrangian density in the fiber bundle α determining an action functional which we denote by $S^\theta = \int \mathcal{L}^\theta$. If θ is locally given by $\theta = \theta_a^i dy^a d^{n-1}x_i - Hd^n x$, then \mathcal{L}^θ is locally given by $\mathcal{L}^\theta = L^\theta d^n x$, where L^θ is the local function on $J^1\alpha$ given by

$$L^\theta = (\theta_b^i y_i^b - H).$$

In particular, if $\theta = \delta\nu$ for some $\nu \in \Omega^0 = \overline{\Lambda}^{n-1}$ locally given by $\nu = \nu^i d^{n-1}x_i$, then

$$L^\theta = (\partial_i + y_i^a \partial_a) \nu^i, \quad (21)$$

i.e., L^θ is a total divergence.

Proposition 19. *Let $\omega \in \Omega^2$ be a δ -exact PD-prehamiltonian system. Then PD-Hamilton equations of ω coincide with Euler-Lagrange equations associated with the action $S^\theta := \int \mathcal{L}^\theta$, where $\theta \in \Omega^1$ is the opposite of any potential of ω , i.e., $-\delta\theta = \omega$. Moreover, if $H^1(\Omega, \delta) = 0$ then S^θ is independent of the choice of θ and does only depend on ω .*

Proof. The first part of the proposition can be proved in local coordinates. Indeed, compute variational derivatives of L^θ ,

$$\begin{aligned} \frac{\delta}{\delta y^b} L^\theta &:= \partial_b L^\theta - (\partial_i + y_i^a \partial_a) \frac{\partial}{\partial y_i^b} L^\theta \\ &= y_i^a (\partial_b \theta_a^i - \partial_a \theta_b^i) - \partial_a H - \partial_i \theta_a^i \\ &= -2\omega_{ab}^i y_i^a + \omega_b. \end{aligned}$$

where we used (16). To prove the second part of the proposition, use (21) to conclude that, for $\nu \in \Omega^0$, $\delta L^{\delta\nu} / \delta y^a = 0$. \square

Remark 20. *Condition $H^1(\Omega, \delta) = 0$ depends on the topology of the fiber bundle α . It is satisfied, for instance, if $H^n(M) = 0$ and $H^1(\mathcal{F}) = 0$, \mathcal{F} being, as above, the abstract fiber of α . Indeed, if $H^1(\mathcal{F}) = 0$ then $H^1(\underline{\Omega}, \underline{\delta}) = 0$ so that, the first part of the exact sequence (12) reads*

$$0 \longrightarrow H^0(\Omega, \delta) \longrightarrow \Lambda^{n-1}(M) \xrightarrow{d_M^{n-1}} \Lambda^n(M) \longrightarrow H^1(\Omega, \delta) \longrightarrow 0$$

and $H^1(\Omega, \delta) \simeq H^n(M) = 0$.

5. PD-NOETHER SYMMETRIES AND CURRENTS

5.1. PD-Noether Theorem and PD-Poisson Bracket. The multisymplectic analogues of 1) hamiltonian vector fields and 2) Poisson bracket in symplectic geometry have been longly investigated [6, 7, 8, 9, 10, 36]. We here propose the natural definitions for general PD-hamiltonian systems. Notice that, even if they look formally identical to (or possibly less general than) the ones proposed in [6, 8, 9, 10], our definitions have actually got a dynamical content, not only a kinematical one (see Remark 17), so that, for instance, we can prove a PD-version of (hamiltonian) Noether theorem. That's why, e.g., we will better speak about PD-Noether symmetries rather than hamiltonian (multi)vector fields [42].

Let ω be a PD-prehamiltonian system on the bundle $\alpha : P \longrightarrow M$. In the following we assume α to have connected fiber.

Definition 21. *Let $Y \in VD$ and $f \in \Omega^0$. If $i_Y\omega = \delta f$, then Y and f are said to be a PD-Noether symmetry and a PD-Noether current of ω (relative to each other), respectively.*

Denote by $\mathcal{S}(\omega)$ and $\mathcal{C}(\omega)$ the sets of PD-Noether symmetries and PD-Noether currents of ω , respectively. A PD-Noether symmetry Y (relative to a PD-Noether current f) is a symmetry of ω in the sense that

$$L_Y\omega = i_Y\delta\omega + \delta i_Y\omega = \delta\delta f = 0.$$

The next proposition clarifies in what sense a PD-Noether current is a conserved current for ω .

Proposition 22 (PD-Noether theorem). *Let $Y \in \mathcal{S}(\omega)$ and $f \in \mathcal{C}(\omega)$ be a PD-Noether symmetry and a PD-Noether current of ω relative to each other. Then $\sigma^*(f) \in \Lambda^{n-1}(M)$ is a closed form for every solution σ of PD-Hamilton equations.*

Proof. First of all, let $\varrho \in \Omega^1$ and τ be a (local) section of α . It is easy to show (for instance, using local coordinates) that $\tau^*(\varrho) = i_\tau\varrho|_\tau \in \Lambda^n(M)$. Then

$$\begin{aligned} d\sigma^*(f) &= \sigma^*(df) \\ &= \sigma^*(\delta f) \\ &= i_\delta\delta f|_\sigma \\ &= i_\delta i_Y\omega|_\sigma \\ &= i_Y|_\sigma i_\delta\omega|_\sigma \\ &= 0. \end{aligned}$$

□

We are now in the position to introduce a Lie bracket among PD-Noether currents.

Proposition 23. *Let $Y_1, Y_2 \in \mathcal{S}(\omega)$ be PD-Noether symmetries relative to the PD-Noether currents $f_1, f_2 \in \mathcal{C}(\omega)$, respectively. Then $[Y_1, Y_2] \in \mathcal{S}(\omega)$ and $f := L_{Y_1}f_2 \in$*

$\mathcal{C}(\omega)$ and they are relative to each other. Moreover, f is independent of the choice of Y_1 among the PD-Noether symmetries relative to the PD-Noether current f_1 .

Proof. Compute

$$\begin{aligned}\delta L_{Y_1} f_2 &= L_{Y_1} \delta f_2 \\ &= L_{Y_1} i_{Y_2} \omega \\ &= i_{[Y_1, Y_2]} \omega + i_{Y_2} L_{Y_1} \omega \\ &= i_{[Y_1, Y_2]} \omega.\end{aligned}$$

Now, let $V \in \ker \omega$. Then $L_V f_2 = i_V \delta f_2 = i_V i_{Y_2} \omega = 0$. This proves the second part of the proposition. \square

Let Y_1, Y_2, f_1, f_2 be as in the above proposition.

Proposition 24. *The \mathbb{R} -bilinear map*

$$\mathcal{C}(\omega) \times \mathcal{C}(\omega) \ni (f_1, f_2) \longmapsto \{f_1, f_2\} := L_{Y_1} f_2 \in H(\omega),$$

Y_1 being a PD-Noether symmetry relative to f_1 , is a Lie bracket.

Proof. Let $Y_2 \in \mathcal{S}(\omega)$ be a PD-Noether symmetry relative to $f_2 \in \mathcal{C}(\omega)$. Skew-symmetry of $\{\cdot, \cdot\}$ immediately follows from the remark:

$$\begin{aligned}\{f_1, f_2\} &= L_{Y_1} f_2 \\ &= i_{Y_1} \delta f_2 + \delta i_{Y_1} f_2 \\ &= i_{Y_1} i_{Y_2} \omega.\end{aligned}$$

Now, check Leibnitz rule. Let $Y_3 \in \mathcal{S}(\omega)$ and $f_3 \in \mathcal{C}(\omega)$ be another pair of PD-Noether symmetry, PD-Noether current relative to each other. Then

$$\begin{aligned}\{f_1, \{f_2, f_3\}\} &= L_{Y_1} \{f_2, f_3\} \\ &= L_{Y_1} L_{Y_2} f_3 \\ &= L_{[Y_1, Y_2]} f_3 + L_{Y_2} L_{Y_1} f_3 \\ &= \{\{f_1, f_2\}, f_3\} + \{f_2, \{f_1, f_3\}\}.\end{aligned}$$

\square

PD-Noether symmetries and PD-Noether currents of a PD-hamiltonian system constitute very small Lie subalgebras of the Lie algebras of higher symmetries and conservation laws of PD-Hamilton equations, for which they have been given fully satisfactory definitions and have been developed many infinite jet based computational techniques [2]. Nevertheless, it is worth a definition and a careful analysis, independent on infinite jets, in view of the possibility of developing a “(multi)symplectic theory” of higher symmetries and conservation laws (see, for instance, [37]). In Section 6 we propose some specific examples.

Finally, notice that, in general, nor a PD-Noether current is uniquely determined by the relative PD-Noether symmetry nor vice versa (unless $\ker \omega = 0$). However, “non-trivial PD-Noether symmetries” are in one to one correspondence with “non-trivial PD-Noether currents” in the following sense. Clearly, $\ker \omega \subset \mathcal{S}(\omega)$ and $H^0(\Omega, \delta) \subset \mathcal{C}(\omega)$. We will call elements in $\ker \omega$ *gauge PD-Noether symmetries* (see below) and elements in $H^0(\Omega, \delta)$ (i.e., closed $(n-1)$ -forms on M , see Corollary 6) *trivial PD-Noether currents*.

Remark 25. *It is easy to see that $\ker \omega$ and $H^0(\Omega, \delta)$ are ideals in the Lie algebras $\mathcal{S}(\omega)$ and $\mathcal{C}(\omega)$, respectively. Let $\overline{\mathcal{S}}(\omega) := \mathcal{S}(\omega)/\ker \omega$ and $\overline{\mathcal{C}}(\omega) := \mathcal{C}(\omega)/H^0(\Omega, \delta)$ be the quotient Lie algebras. Then the map*

$$\overline{\mathcal{S}}(\omega) \ni Y + \ker \omega \longmapsto f + H^0(\Omega, \delta) \in \overline{\mathcal{C}}(\omega),$$

where $Y \in \mathcal{S}(\omega)$ and $f \in \mathcal{C}(\omega)$ are relative to each other, is a well defined isomorphism of Lie algebras. It is natural to call elements in $\overline{\mathcal{S}}(\omega)$ and $\overline{\mathcal{C}}(\omega)$ non-trivial PD-Noether symmetries and non-trivial PD-Noether currents, respectively. Indeed, elements in $\ker \omega$ are trivial symmetries in that they are infinitesimal gauge transformations (see next subsection), and elements in $H^0(\Omega, \delta)$ are trivial conserved currents in that they are conserved currents for every PD-prehamiltonian system ω , independently of ω .

5.2. Gauge Reduction of PD-Hamiltonian Systems. From a physical point of view, elements in $\ker \omega$ are infinitesimal gauge transformations and therefore should be quotiented out via a reduction of the system. In this section we assume $\ker \omega = \ker \underline{\omega}$ or, which is the same, $\text{Ker } \omega \neq \emptyset$. As a further regularity condition we assume that the leaves of $D^\omega = \underline{D}^\omega$ form a smooth fiber bundle \tilde{P} over M , whose projection we denote by $\tilde{\alpha} : \tilde{P} \rightarrow M$, in such a way that the canonical projection $\mathfrak{p} : P \rightarrow \tilde{P}$ is a smooth bundle. The last condition is always fulfilled at least locally. Notice, also, that, by construction, \mathfrak{p} has connected fiber.

Theorem 26. *There exists a unique PD-hamiltonian system $\tilde{\omega}$ in $\tilde{\alpha}$ such that 1) $\omega = \mathfrak{p}^*(\tilde{\omega})$, 2) $\ker \tilde{\omega} = \ker \underline{\tilde{\omega}} = 0$ and 3) a local section σ of $\tilde{\alpha}$ is a solution of the PD-Hamilton equation of ω iff $\mathfrak{p} \circ \sigma$ (which is a local section of $\tilde{\alpha}$) is a solution of PD-Hamilton equations of $\tilde{\omega}$.*

Proof. Let $\tilde{\nabla} \in C(\tilde{P}, \tilde{\alpha})$. There exists a (non-unique) connection $\nabla \in C(P, \alpha)$ such that ∇ and $\tilde{\nabla}$ are \mathfrak{p} -compatible. To prove this, choose a connection \square in \mathfrak{p} and lift the planes of $\tilde{\nabla}$ to P by means of \square . It is easy to show that the so obtained distribution on P defines a connection ∇ in α with the required property. Similarly, every vector field $\tilde{X} \in \text{VD}(\tilde{P}, \tilde{\alpha})$ can be lifted to a (non-unique) \mathfrak{p} -projectable vector field $X \in \text{VD}(P, \alpha)$ such that \tilde{X} is its projection. Then $X \in \text{D}_V(P, \mathfrak{p})$. Consider $\eta := \omega(\nabla)(X) \in \Omega^0(P, \alpha)$ and prove that $L_Y \eta = 0$ for any $Y \in \text{VD}(P, \mathfrak{p})$. Indeed, let $Y \in \text{VD}(P, \mathfrak{p})$. Then $[Y, X] \in \text{VD}(P, \mathfrak{p})$. Similarly $\llbracket Y, H_\nabla \rrbracket \in \overline{\Lambda}^1(P, \alpha) \otimes \text{VD}(P, \mathfrak{p}) \subset \overline{\Lambda}^1(P, \alpha) \otimes \text{VD}(P, \alpha)$.

Now, $VD(P, \mathfrak{p}) = \ker \omega$ by construction, and therefore

$$\begin{aligned}
L_Y \eta &= L_Y i_X i_{\nabla} \omega \\
&= [L_Y, i_X] i_{\nabla} \omega + i_X L_Y i_{\nabla} \omega \\
&= i_{[Y, X]} i_{\nabla} \omega + i_X i_{\nabla} L_Y \omega + i_X [L_Y, i_{\nabla}] \omega \\
&= i_{\nabla} i_{[Y, X]} \omega + i_X i_{[[Y, H_{\nabla}]]} \omega \\
&= 0.
\end{aligned}$$

Since fibers of \mathfrak{p} are connected we conclude that $\eta = \mathfrak{p}^*(\tilde{\eta})$ for a unique $\tilde{\eta} \in \Omega^0(\tilde{P}, \tilde{\alpha})$. Put

$$\tilde{\omega}(\tilde{\nabla})(\tilde{X}) := \tilde{\eta}.$$

$\tilde{\omega}$ is a well defined element in $\Omega^2(\tilde{P}, \tilde{\alpha})$. Indeed, let $\nabla' \in C(P, \alpha)$ be also \mathfrak{p} -compatible with $\tilde{\nabla}$ and $X' \in VD(P, \alpha)$ be another \mathfrak{p} -projectable vector field projecting onto \tilde{X} . Then $\nabla' - \nabla \in \bar{\Lambda}^1(P, \alpha) \otimes VD(P, \mathfrak{p})$ and $X' - X \in VD(P, \mathfrak{p})$. Therefore,

$$\begin{aligned}
\omega(\nabla')(X') &= i_{X'} i_{\nabla'} \omega \\
&= i_{X'} i_{\nabla} \omega + i_{X'} i_{\nabla' - \nabla} \omega \\
&= i_X i_{\nabla} \omega + i_{X' - X} i_{\nabla} \omega \\
&= i_X i_{\nabla} \omega + i_{\nabla} i_{X' - X} \omega \\
&= \omega(\nabla)(X).
\end{aligned}$$

Moreover, $\omega = \mathfrak{p}^*(\tilde{\omega})$ by construction.

Let us compute $\ker \tilde{\omega}$. Thus, let $\tilde{X} \in VD(\tilde{P}, \tilde{\alpha})$ be such that $i_{\tilde{X}} \tilde{\omega} = 0$ and $X \in VD(P, \alpha)$ be as above. Then $i_X \omega = \mathfrak{p}^*(i_{\tilde{X}} \tilde{\omega}) = 0$. This shows that $X \in VD(P, \mathfrak{p})$ and then $\tilde{X} = 0$.

Finally, let σ be a local section of α , $\tilde{\sigma} := \mathfrak{p} \circ \sigma$, $\tilde{X} \in VD(\tilde{P}, \tilde{\alpha})$ and X as above. Compute

$$\begin{aligned}
(i_{\tilde{\sigma}} \tilde{\omega}|_{\tilde{\sigma}})(\tilde{X}|_{\tilde{\sigma}}) &= i_{\tilde{\sigma}}(i_{\tilde{X}} \tilde{\omega})|_{\tilde{\sigma}} \\
&= \tilde{\sigma}^*(i_{\tilde{X}} \tilde{\omega}) \\
&= (\sigma^* \circ \mathfrak{p}^*)(i_{\tilde{X}} \tilde{\omega}) \\
&= \sigma^*(i_X \omega) \\
&= i_{\tilde{\sigma}}(i_X \omega)|_{\sigma} \\
&= (i_{\tilde{\sigma}} \omega|_{\sigma})(X|_{\sigma}).
\end{aligned}$$

This shows that $i_{\tilde{\sigma}} \omega|_{\sigma} = 0$ iff $i_{\tilde{\sigma}} \tilde{\omega}|_{\tilde{\sigma}} = 0$. □

Proposition 27. *There are natural isomorphisms of Lie algebras*

$$\begin{aligned}
\overline{\mathcal{F}}(\omega) &\simeq \mathcal{S}(\tilde{\omega}), \\
\mathcal{L}(\omega) &\simeq \mathcal{L}(\tilde{\omega}).
\end{aligned}$$

Proof. First of all let $f \in \mathcal{C}(\omega)$ and $X \in \mathcal{S}(\omega)$ be relative to each other. Then $f = \mathfrak{p}^*(\tilde{f})$ for some $\tilde{f} \in \Omega^0(\tilde{P}, \tilde{\alpha})$ and X is \mathfrak{p} -projectable. Indeed, for all $Y \in \ker \omega$,

$$L_Y f = i_Y \delta f + \delta i_Y f = i_Y i_X \omega = i_{[Y, X]} \omega = 0.$$

Moreover,

$$\mathfrak{p}^*(\delta \tilde{f}) = \delta \mathfrak{p}^*(\tilde{f}) = \delta f = i_X \omega = \mathfrak{p}^*(i_{\tilde{X}} \tilde{\omega}),$$

where \tilde{X} denotes the \mathfrak{p} -projection of X , and, therefore, $\delta \tilde{f} = i_{\tilde{X}} \tilde{\omega}$, i.e., $\tilde{f} \in \mathcal{C}(\tilde{\omega})$ and $\tilde{X} \in \mathcal{S}(\tilde{\omega})$ is a PD-Noether symmetry relative to it. Thus, they are well defined maps

$$\overline{\mathcal{S}}(\omega) \ni X + \ker \omega \longmapsto \tilde{X} \in \mathcal{S}(\tilde{\omega}), \quad (22)$$

$$\mathcal{C}(\omega) \ni f \longmapsto \tilde{f} \in \mathcal{C}(\tilde{\omega}). \quad (23)$$

Conversely, let $\tilde{X}_1 \in \mathcal{S}(\tilde{\omega})$, $\tilde{f}_1 \in \mathcal{C}(\tilde{\omega})$ be relative to each other, $X_1 \in \text{VD}(P, \alpha)$ be any \mathfrak{p} -projectable vector field, $\tilde{X}_1 \in \text{VD}(\tilde{P}, \tilde{\alpha})$ be its projection, and $f_1 := \mathfrak{p}^*(\tilde{f}_1) \in \Omega^0(P, \alpha)$. Then $X_1 \in \mathcal{S}(\omega)$ and $f_1 \in \mathcal{C}(\omega)$ is a PD-Noether current relative to it. Indeed,

$$i_{X_1} \omega = \mathfrak{p}^*(i_{\tilde{X}_1} \tilde{\omega}) = \mathfrak{p}^*(\delta \tilde{f}_1) = \delta \mathfrak{p}^*(\tilde{f}_1) = \delta f_1.$$

We conclude that (22) and (23) are inverted by

$$\mathcal{S}(\tilde{\omega}) \ni \tilde{X}_1 \longmapsto X_1 + \ker \omega \in \overline{\mathcal{S}}(\omega),$$

$$\mathcal{C}(\tilde{\omega}) \ni \tilde{f}_1 \longmapsto f_1 \in \mathcal{C}(\omega),$$

respectively. □

6. EXAMPLES

6.1. Non-Degenerate Examples. Let $\alpha : \mathbb{R}^{2n+1} \ni (x^1, \dots, x^n, u, u_1, \dots, u_n) \longmapsto (x^1, \dots, x^n) \in \mathbb{R}^n$, $n > 1$. Consider $T, V \in C^\infty(\mathbb{R}^{2n+1})$ of the form $T = T(u_1, \dots, u_n)$ and $V = V(u)$, respectively. The form

$$\omega := \frac{\partial^2 T}{\partial u_i \partial u_j} du_i (du d^{n-1} x_j - u_j d^n x) - V' du d^n x,$$

is a PD-prehamiltonian system in α (here and in what follows a prime “ ’ ” denotes differentiation with respect to u). The associated PD-Hamilton equations read

$$\begin{aligned} \frac{\partial^2 T}{\partial u_i \partial u_j} \partial_j u_i + V' &= 0, \\ \partial_i u &= u_i, \end{aligned}$$

which are in turn equivalent to

$$\begin{aligned} \frac{\partial^2 T}{\partial u_i \partial u_j} \partial_{ij}^2 u + V' &= 0, \\ \partial_i u &= u_i, \end{aligned} \quad (24)$$

$\partial_{ij}^2 := \partial_i \partial_j$, $i, j = 1, \dots, n$. Moreover, for

$$\det \left(\frac{\partial^2 T}{\partial u_i \partial u_j} \right) \neq 0,$$

ω is a PD-hamiltonian system. We will only consider this case in the following. Thus, put $T^{ij} := \frac{\partial^2 T}{\partial u_i \partial u_j}$, $i, j = 1, \dots, n$, and let (T_{ij}) be the inverse matrix of (T^{ij}) . As examples notice that

(1) For $T = \frac{1}{2}g^{ij}u_i u_j$,

$$(g^{ij}) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

(resp., $g^{ij} = \delta^{ij}$, $i, j = 1, \dots, n$), (24) reduces to the wave equation (resp., the Poisson equation) with a u -dependent potential V (including the f -Gordon equation as a particular example, if $n = 2$ and $f = -V'$).

(2) For $n = 2$, $T = \sqrt{1 + g^{ij}u_i u_j}$, $g^{ij} = \delta^{ij}$, $i, j = 1, 2$, and $V = 0$, (24) reduces to the equation for minimal surfaces in \mathbb{R}^3 transversal to the projection $\mathbb{R}^3 \ni (x_1, x_2, u) \mapsto (x_1, x_2) \in \mathbb{R}^2$.

Let us search for PD-Noether symmetries and currents of ω . Let $Y = U \frac{\partial}{\partial u} + U_i \frac{\partial}{\partial u^i} \in VD$ and $f = f^i d^{n-1}x_i \in \Omega^0$. Then

$$\begin{aligned} i_Y \omega &= T^{ij}(U_i du - U du_i) d^{n-1}x_j - (T^{ij}u_i U_j + V'U) d^n x, \\ \delta f &= \partial_i f^i d^n x + \frac{\partial}{\partial u} f^i du d^{n-1}x_i + \frac{\partial}{\partial u_k} f^i du_k d^{n-1}x_i. \end{aligned}$$

Recall that Y and f are a PD-Noether symmetry and a PD-Noether current relative to each other, respectively, iff $i_Y \omega = \delta f$, i.e.,

$$\partial_i f^i + T^{ij}u_i U_j + V'U = 0, \quad (25)$$

$$\frac{\partial}{\partial u} f^i - T^{ij}U_j = 0, \quad (26)$$

$$\frac{\partial}{\partial u_j} f^i + T^{ij}U = 0. \quad (27)$$

It follows from (27) that $\frac{\partial}{\partial u_j} f^i = \frac{\partial}{\partial u_i} f^j$, $i, j = 1, \dots, n$, and then

$$\frac{\partial^2}{\partial u_k \partial u_j} f^i = \frac{\partial^2}{\partial u_k \partial u_i} f^j, \quad i, j, k = 1, \dots, n.$$

Now,

$$\begin{aligned} \frac{\partial^2}{\partial u_k \partial u_j} f^i &= \frac{\partial}{\partial u_k} \frac{\partial}{\partial u_j} f^i \\ &= -\frac{\partial}{\partial u_k} (T^{ij}U) \\ &= -\frac{\partial^3 T}{\partial u_k \partial u_i \partial u_j} U - T^{ij} \frac{\partial}{\partial u_k} U \end{aligned}$$

Similarly,

$$\frac{\partial^2}{\partial u_k \partial u_i} f^j = \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_k} f^j = -\frac{\partial^3 T}{\partial u_i \partial u_j \partial u_k} U - T^{jk} \frac{\partial}{\partial u_i} U.$$

Therefore,

$$T^{ij} \frac{\partial}{\partial u_k} U - T^{jk} \frac{\partial}{\partial u_i} U = 0$$

Contracting with T_{ij} we find $(n-1)\frac{\partial}{\partial u_i}U = 0$ and, therefore,

$$U = U(x^1, \dots, x^n, u),$$

so that (27) can be rewritten as

$$\frac{\partial}{\partial u_j} \left(f^i + \frac{\partial T}{\partial u_i} U \right) = 0.$$

We conclude that

$$f^i = -\frac{\partial T}{\partial u_i} U + A^i \quad (28)$$

for some $A^i = A^i(x^1, \dots, x^n, u)$, $i = 1, \dots, n$. Notice that (26) can be used to determine the U_j 's from the f^i 's via

$$U_j = T_{ji} \frac{\partial}{\partial u} f^i.$$

It remains to solve (25) which, in view of (28), reduces to

$$\left(\partial_i + u_i \frac{\partial}{\partial u} \right) A^i - \frac{\partial T}{\partial u_i} \left(\partial_i + u_i \frac{\partial}{\partial u} \right) U + V'U = 0. \quad (29)$$

We cannot go further on in solving (29) without better specifying T . In the following we will only consider two special cases.

- (1) $T = \frac{1}{2} g^{ij} u_i u_j$, (g^{ij}) being a constant, non degenerate, symmetric matrix with inverse (g_{ij}) . In this case (29) reads

$$\partial_i A^i + V'U + \left(\frac{\partial}{\partial u} A^i - g^{ij} \partial_j U \right) u_i - \left(g^{ij} \frac{\partial}{\partial u} U \right) u_i u_j = 0. \quad (30)$$

The left hand side of (30) is polynomial in u_1, \dots, u_n . Thus, all the corresponding coefficients must vanish, i.e.,

$$\frac{\partial}{\partial u} U = 0, \quad (31)$$

$$\frac{\partial}{\partial u} A^i - g^{ij} \partial_j U = 0, \quad (32)$$

$$\partial_i A^i + V'U = 0. \quad (33)$$

From (31), $U = U(x^1, \dots, x^n)$ and, then, from (32), $\frac{\partial^2}{\partial u^2} A^i = 0$, $i = 1, \dots, n$, which in turn implies, using (32) again,

$$A^i = (g^{ij} \partial_j U) u + B^i$$

for some $B^i = B^i(x^1, \dots, x^n)$. Finally, (33) implies

$$(g^{ij} \partial_{ij}^2 U) u + \partial_i B^i + V'U = 0$$

and differentiating once more with respect to u

$$g^{ij} \partial_{ij}^2 U + V''U = 0.$$

Since U doesn't depend on u , if

(a) $V''' \neq 0$. Then $U = 0$ so that

$$f^i = \frac{1}{2} \partial_j B^{ji}, \quad U_j = 0$$

for some $B^{ij} = -B^{ji} = B^{ij}(x^1, \dots, x^n)$, i.e.,

$$Y = 0 \quad \text{and} \quad f = d\beta,$$

$\beta = B^{ji} d^{n-2} x_{ji}$, where $d^{n-2} x_{ji} := i_{\partial_j} d^{n-1} x_i$, $i, j = 1, \dots, n$. Therefore, ω doesn't possess PD-Noether symmetries nor non-trivial PD-Noether currents.

(b) $V''' = 0$. Then $V = \frac{1}{2} \mu u^2$ for some constant μ and

$$g^{ij} \partial_{ij}^2 U + \mu U = 0, \quad f^i = g^{ij} (u \partial_j U - u_j U) + \frac{1}{2} \partial_j B^{ji}, \quad U_j = \partial_j U,$$

for some $B^{ij} = -B^{ji} = B^{ij}(x^1, \dots, x^n)$. Thus,

$$Y = U \frac{\partial}{\partial u} + \partial_j U \frac{\partial}{\partial u_j} \quad \text{and} \quad f = g^{ij} (u \partial_j U - u_j U) d^{n-1} x_i + d\beta,$$

$\beta = B^{ji} d^{n-2} x_{ji}$, where U is any solution of the PD-Hamilton equation

$$g^{ij} \partial_{ij}^2 u + \mu u = 0. \tag{34}$$

Let us compute the PD-Poisson bracket. Consider two solutions of (34), say U_1, U_2 , the corresponding PD-Noether symmetries Y_1, Y_2 and associated PD-Noether currents f_1, f_2 . Then

$$\{f_1, f_2\} = L_{Y_1} f_2 = g^{ij} (U_1 \partial_j U_2 - U_2 \partial_j U_1) d^{n-1} x_i,$$

which, as can be easily checked, is a trivial conservation law.

(2) $n = 2$, $T = \sqrt{1 + \delta^{ij} u_i u_j}$ and $V = 0$. In this case (29) reads

$$\tau^{1/2} \left(\partial_i + u_i \frac{\partial}{\partial u} \right) A^i = \delta^{ij} u_j \left(\partial_i + u_i \frac{\partial}{\partial u} \right) U, \tag{35}$$

where $\tau = 1 + \delta^{ij} u_i u_j$. Squaring both sides of (35) we get

$$\tau \left[\left(\partial_i + u_i \frac{\partial}{\partial u} \right) A^i \right]^2 - \left[\delta^{ij} u_j \left(\partial_i + u_i \frac{\partial}{\partial u} \right) U \right]^2 = 0,$$

whose left hand side is polynomial in u_1, u_2 . Collecting homogeneous terms we get

$$\begin{aligned} & \left[\left(\frac{\partial}{\partial u} U \right)^2 \delta^{ij} - \left(\frac{\partial}{\partial u} A^i \right) \left(\frac{\partial}{\partial u} A^j \right) \right] \delta^{kl} u_i u_j u_k u_l \\ & + 2\delta^{ij} \left[\delta^{kl} \left(\frac{\partial}{\partial u} U \right) (\partial_l U) - \left(\frac{\partial}{\partial u} A^k \right) (\partial_l A^l) \right] u_i u_j u_k \\ & - \left[\delta^{ij} (\partial_k A^k)^2 + \left(\frac{\partial}{\partial u} A^i \right) \left(\frac{\partial}{\partial u} A^j \right) - \delta^{ik} \delta^{jl} (\partial_k U) (\partial_l U) \right] u_i u_j \\ & + 2 \left(\partial_j A^j \right)^2 \left(\frac{\partial}{\partial u} A^i \right) u_i + (\partial_i A^i)^2 = 0. \end{aligned} \tag{36}$$

All coefficient of the left hand side of (36) must vanish. It follows that

$$\frac{\partial}{\partial u} U = \partial_1 U = \partial_2 U = 0, \quad \frac{\partial}{\partial u} A^1 = \frac{\partial}{\partial u} A^2 = 0, \quad \partial_1 A^1 + \partial_2 A^2 = 0,$$

i.e., U is a constant while $A^1 = \partial_2 B$, $A^2 = -\partial_1 B$ for some $B = B(x^1, x^2)$. Thus,

$$Y = U \frac{\partial}{\partial u}, \quad f = U \tau^{-1/2} (u_2 dx^1 - u_1 dx^2) + dB.$$

It is obvious that the PD-Poisson bracket is also trivial in this case.

6.2. A Degenerate, Constrained Example. The example in this subsection is taken from [43]. Let $\alpha : \mathbb{R}^{3m+2} \times \mathbb{R}_+ \ni (q^1, \dots, q^m, s_1, \dots, s_m, t_1, \dots, t_m, s, t; e) \mapsto (s, t) \in \mathbb{R}^2$. The form

$$\omega := -dt_\alpha dq^\alpha ds + ds_\alpha dq^\alpha dt - \delta^{\alpha\beta} (et_\alpha dt_\beta - s_\alpha ds_\beta) ds dt - \varepsilon deds dt,$$

where $\varepsilon := \frac{1}{2}(\delta^{\alpha\beta} t_\alpha t_\beta - 1)$, is a PD-prehamiltonian system in α . The associated PD-Hamilton equation reads

$$\begin{aligned} \frac{\partial}{\partial t} t_\alpha + \frac{\partial}{\partial s} s_\alpha &= 0, \\ \frac{\partial}{\partial t} q^\alpha &= e \delta^{\alpha\beta} t_\beta, \\ \frac{\partial}{\partial s} q^\alpha &= -\delta^{\alpha\beta} s_\beta, \\ \varepsilon &= 0, \end{aligned}$$

$\alpha = 1, \dots, m$, which is in turn equivalent to

$$\begin{aligned} e^{-1} \frac{\partial^2}{\partial t^2} q^\alpha - \frac{\partial^2}{\partial s^2} q^\alpha &= e^{-2} \left(\frac{\partial}{\partial t} q^\alpha \right) \left(\frac{\partial}{\partial t} e \right), \\ e^2 &= \delta_{\alpha\beta} \left(\frac{\partial}{\partial t} q^\alpha \right) \left(\frac{\partial}{\partial t} q^\beta \right), \\ t_\alpha &= e^{-1} \delta_{\alpha\beta} \frac{\partial}{\partial t} q^\beta, \\ s_\alpha &= \delta_{\alpha\beta} \frac{\partial}{\partial s} q^\beta, \end{aligned} \quad ,$$

Notice that \underline{D}^ω is generated by $\frac{\partial}{\partial e}$, while

$$D_y^\omega = \begin{cases} \mathbf{0} & \text{for } \varepsilon(y) \neq 0 \\ \left\langle \frac{\partial}{\partial e} \Big|_y \right\rangle & \text{for } \varepsilon(y) = 0 \end{cases}, \quad y \in P$$

we conclude that $P_{(1)}$ is the hypersurface defined by $\delta^{\alpha\beta} t_\alpha t_\beta = 1$. It is easy to see that, actually, $\check{P} = P_{(1)}$.

Let us search for PD-Noether symmetries and currents of ω . Let $Y = Q^\alpha \frac{\partial}{\partial q^\alpha} + S_\alpha \frac{\partial}{\partial s_\alpha} + T_\alpha \frac{\partial}{\partial t_\alpha} + E \frac{\partial}{\partial e} \in \text{VD}$ and $f = \alpha ds + \beta dt \in \Omega^0$. Then $i_Y \omega = \delta f$ iff

$$\begin{aligned} \frac{\partial}{\partial s} \beta - \frac{\partial}{\partial t} \alpha &= \delta^{\beta\gamma} (s_\beta S_\gamma - et_\beta T_\gamma) - \varepsilon E, \\ \frac{\partial}{\partial q^\alpha} \alpha &= -T_\alpha, \quad \frac{\partial}{\partial q^\alpha} \beta = S_\alpha, \quad \frac{\partial}{\partial t_\alpha} \alpha = \frac{\partial}{\partial s_\alpha} \beta = Q^\alpha \\ \frac{\partial}{\partial s_\alpha} \alpha &= \frac{\partial}{\partial t_\alpha} \beta = 0, \quad \frac{\partial}{\partial e} \alpha = \frac{\partial}{\partial e} \beta = 0, \end{aligned} \quad (37)$$

$\alpha = 1, \dots, m$. Equations (37) can be easily solved and give quite large $\mathcal{S}(\omega)$ and $\mathcal{C}(\omega)$. Namely,

$$\alpha = C^\alpha t_\alpha + A, \quad \beta = C^\alpha s_\alpha + B,$$

and

$$\begin{aligned} Q^\alpha &= C^\alpha, \quad T_\alpha = -\frac{\partial C^\beta}{\partial q^\alpha} t_\beta - \frac{\partial A}{\partial q^\alpha}, \quad S_\alpha = \frac{\partial C^\beta}{\partial q^\alpha} s_\beta + \frac{\partial B}{\partial q^\alpha}, \\ \varepsilon E &= \frac{\partial C^\alpha}{\partial s} s_\alpha - \frac{\partial C^\alpha}{\partial t} t_\alpha + \frac{\partial B}{\partial s} - \frac{\partial A}{\partial t} - \delta^{\alpha\beta} \left[s_\alpha \left(\frac{\partial C^\gamma}{\partial q^\beta} s_\gamma + \frac{\partial B}{\partial q^\beta} \right) + et_\alpha \left(\frac{\partial C^\gamma}{\partial q^\beta} t_\gamma + \frac{\partial A}{\partial q^\beta} \right) \right] \end{aligned}$$

where $A, B, \dots, C^\alpha, \dots, D^{\alpha\beta}, \dots, E^\alpha, \dots$ are arbitrary functions of the only $s, t, \dots, q^\beta, \dots$

Compute the PD-Poisson bracket. Let f_1, f_2 be PD-Noether currents determined by functions $A_1, B_1, \dots, C_1^\alpha, \dots$ and $A_2, B_2, \dots, C_2^\alpha, \dots$ respectively. A straightforward computation shows that

$$\{f_1, f_2\} = (C^\alpha t_\alpha + A)ds + (C^\alpha s_\alpha + B)dt$$

with

$$\begin{aligned} A &= C_1^{\beta} \frac{\partial}{\partial q^\beta} A_2 - C_2^{\beta} \frac{\partial}{\partial q^\beta} A_1, \\ B &= C_1^{\beta} \frac{\partial}{\partial q^\beta} B_2 - C_2^{\beta} \frac{\partial}{\partial q^\beta} B_1, \\ C^\alpha &= C_1^{\beta} \frac{\partial}{\partial q^\beta} C_2^\alpha - C_2^{\beta} \frac{\partial}{\partial q^\beta} C_1^\alpha, \end{aligned}$$

$\alpha = 1, \dots, m.$

6.3. A Degenerate, Unconstrained Example. Finally, we propose an example of reduction. Consider the cotangent bundle $\pi : T^*\mathbb{M} \ni A_i dx^i|_{(x^1, \dots, x^n)} \mapsto (x^1, \dots, x^n) \in \mathbb{M}$ and let $\alpha := \pi_1 : (x^1, \dots, x^n, \dots, A_i, \dots, A_{i,j}, \dots) \ni J^1\pi \mapsto (x^1, \dots, x^n) \in \mathbb{M}$, \mathbb{M} being the n -dimensional Minkowski space. As such \mathbb{M} is endowed with the metric $g := g_{ij} dx^i \cdot dx^j$ where

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In the following we will raise and lower indexes using g . Let

$$\omega := 2dA^{[j,i]} \left(\frac{1}{2} A_{i,j} d^n x - dA_i d^{n-1} x_j \right)$$

Then ω is a PD-prehamiltonian system on π whose PD-Hamilton equation reads

$$\begin{aligned} \partial_k A^{[i,k]} &= 0, \\ \partial_{[j} A_{i]} &= A_{[i,j]}, \end{aligned}$$

$i, j = 1, \dots, n$, which are equivalent to Maxwell equations for the vector potential

$$\begin{aligned} (\partial_k \partial^k) A_i - \partial_i \partial_k A^k &= 0, \\ A_{[i,j]} &= \partial_{[j} A_{i]}. \end{aligned}$$

Notice that

$$\ker \omega = \ker \underline{\omega} = \left\langle \dots, \frac{\partial}{\partial A_{i,j}} + \frac{\partial}{\partial A_{j,i}}, \dots \right\rangle.$$

Therefore ω is “degenerate and unconstrained”. Moreover, leaves of $D^\omega = \underline{D}^\omega$ are given by $A_{[i,j]} = \text{const}$. We conclude that $J^1\pi$ “reduces” via

$$\begin{aligned} \mathfrak{p} : J^1\pi &\longrightarrow T^*\mathbb{M} \times_M \wedge^2 T^*\mathbb{M} \simeq \mathbb{R}^{n(n+3)/2} \\ (x^1, \dots, x^n, \dots, A_i, \dots, A_{i,j}, \dots) &\longmapsto (x^1, \dots, x^n, \dots, A_i, \dots, F_{ij}, \dots) \end{aligned}$$

where $F_{ij} = F_{[ij]}$, $\mathbf{p}^*(F_{ij}) := 2A_{[j,i]}$ and $\omega = \mathbf{p}^*(\tilde{\omega})$, with

$$\tilde{\omega} := dF^{ij} \left(\frac{1}{4} F_{ji} d^n x - dA_i d^{n-1} x_j \right)$$

is a PD-hamiltonian system on

$$\tilde{\alpha} : \mathbb{R}^{n(n+3)/2} \ni (x^1, \dots, x^n, \dots, A_i, \dots, F_{ij}, \dots) \longmapsto (x^1, \dots, x^n) \in \mathbb{R}^n,$$

whose PD-Hamilton equations read

$$\begin{aligned} \partial_k F^{ik} &= 0, \\ \partial_{[j} A_{i]} &= 2F_{ji}, \end{aligned}$$

which are Maxwell equations for the field strength.

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