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DIFFERENTIAL INVARIANTS OF GENERIC PARABOLIC MONGE-AMPERE EQUATIONS

D. CATALANO FERRAIOLI AND A. M. VINOGRADOV

ABSTRACT. Some new results on geometry of classical parabolic Monge-Ampere equations (PMA) are presented. PMAs are either *integrable*, or *nonintegrable* according to integrability of its characteristic distribution. All integrable PMAs are locally equivalent to the equation $u_{xx} = 0$. We study nonintegrable PMAs by associating with each of them a 1-dimensional distribution on the corresponding first order jet manifold, called the *directing distribution*. According to some property of these distributions, nonintegrable PMAs are subdivided into three classes, one *generic* and two *special* ones. Generic PMAs are uniquely characterized by their directing distributions. To study directing distributions we introduce their canonical models, *projective curve bundles* (PCB). A PCB is a 1-dimensional subbundle of the projectivized cotangent bundle to a 4-dimensional manifold. Differential invariants of projective curves composing such a bundle are used to construct a series of contact differential invariants for corresponding PMAs. These give a solution of the equivalence problem for PMAs with respect to contact transformations.

1. INTRODUCTION

Since Monge's "Application de l'Analyse à la Géométrie" Monge-Ampere equations periodically attract attention of geometers. This is not only due to the numerous applications to geometry, mechanics and physics. Geometry of these equations being tightly related with various parts of the modern differential geometry has all merits to be studied as itself. Last 2-3 decades manifested a return of interest to geometry of Monge-Ampere equations, mostly to elliptic and hyperbolic ones. The reader will find an account of recent results together with an extensive bibliography in [5].

In this article we study geometry of classical parabolic Monge-Ampere equations (PMAs) on the basis of a new approach sketched in [8]. According to it, a PMA $\mathcal{E} \subset J^2(\pi)$, π being a 1-dimensional fiber bundle over a bidimensional manifold, is completely characterized by its *characteristic distribution* $\mathcal{D}_{\mathcal{E}}$ which is a 2-dimensional Lagrangian distribution on $J^1(\pi)$, and vice versa. Such distributions and, accordingly, the corresponding to them PMAs, are naturally subdivided into four classes, *integrable, generic* and two types of *special* ones (see [8] and sec.4). All integrable Lagrangian foliations are locally contact equivalent. A consequence of it is that a PMA $\mathcal{E}$ is locally

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contact equivalent to the equation $u_{xx} = 0$ iff the distribution $\mathcal{D}_{\mathcal{E}}$ is integrable. This exhausts the integrable case. On the contrary, nonintegrable Lagrangian distributions are very diversified and our main goal here is to describe their multiplicity, i.e., more precisely, equivalence classes of PMAs with respect to contact transformations.

With this purpose we associate with a Lagrangian distribution a *projective curve bundle* (shortly, PCB) over a 4-dimensional manifold N. A PCB over N is a 1-dimensional smooth subbundle of the "projectivized" cotangent bundle $PT^*(N)$ of N. Under some regularity conditions such a bundle possesses a canonical contact structure and, as a consequence, a canonically inscribed in it Lagrangian distribution. There exists a one-to-one correspondence between generic Lagrangian distributions and *regular* PCBs. This way the equivalence problem for generic PMAs is reformulated as the equivalence problem for PCBs (see [8]) and this is the key point of our approach. The fiber of such a bundle over a point $x \in N$ is a curve γ_x in the projective space $PT^*_x(N)$. The curve γ_x can be characterized by its scalar differential invariants with respect to the group of projective transformations. By putting such invariants for single curves γ_x together for all $x \in N$ one obtains scalar differential invariants for the considered PCB and, consequently, for the corresponding PMA.

This kind of invariants resolve the equivalence problem for generic PMAs on the basis of the "principle of $n-$invariants" (see [1, 10]). We have chosen them among others for their transparent geometrical meaning. It should be stressed, however, that there are other choices, maybe, less intuitive but more efficient in practice. We shall discuss this point separately.

Special Lagrangian distributions admit a similar interpretation in terms of 2-dimensional distributions on 4-dimensional manifolds supplied with additional structures, called *fringes* (see [8]). Differential invariants of fringes, also coming from projective differential geometry, allow to construct basic scalar differential invariants for special PMAs. They will be discussed in a separate paper. It is worth mentioning that all linear PMAs are special.

Our approach is based on the theory of solution singularities for nonlinear PDEs (see [9]). Indeed, a Lagrangian distribution or, equivalently, the associated PCB, represents equations that describe fold type singularities of multivalued solutions of the corresponding PMA. An advantage of this point of view is that it allows a similar analysis of higher order PDEs and, in particular, to understand what are higher order analogues of Monge-Ampere equations. These topics will be discussed in a forthcoming joint paper by M. Bachtold and the second author.

The paper is organized as follows. The notations and generalities concerning jet spaces and Monge-Ampere equations, we need throughout the paper, are collected in sections 2 and 3, respectively. In particular, the interpretation of PMAs as Lagrangian distributions is presented here. Section 4 contains some basic facts on geometry of Lagrangian distributions. The central of them is the notion of the *directing distribution* of a Lagrangian distribution. The above mentioned subdivision of nonitegrable PMAs into generic and special types reflects some contact properties of this distribution. Projective curve bundles that are canonical models of directing distributions are introduced and studied subsequently in section 6. For completeness in section 5 we give a short proof of the known fact that integrable PMAs are locally equivalent one to another. Finally, basic scalar differential invariants of generic PCBs and hence of generic PMAs are constructed and discussed in section 7.

Throughout the paper we use the following notations and conventions:

- all objects in this paper, e.g., manifolds, mappings, functions, vector fields, etc, are supposed to be smooth;
- $C^\infty(M)$ stands for the algebra of smooth functions on the manifold M and $C^\infty(M)$-modules of all vector fields and differential k-forms are denoted by $D(M)$ and $\Lambda^k(M)$, respectively;
- the evaluation of $X \in D(M)$ (resp., of $\alpha \in \Lambda^k(M)$) at $p \in M$ is denoted by $X|_p$ (resp., $\alpha|_p$);
- $d_pf : T_pM \longrightarrow T_{f(p)}N$ stands for the differential of the map $f : M \to N$,M and N being two manifolds;
- For $X, Y \in D(M)$ and $\alpha \in \Lambda^k(M)$ we shall use the shorten notation $X^r(Y)$ for $L^r_X(Y)$ and $X^r(\alpha)$ for $L^r_X(\alpha)$ (with $X^0(Y) = Y$ and $X^0(\alpha) = \alpha$) for the r-th power of the Lie derivative L_X.
- depending on the context by a *distribution* on a manifold M we understand either a subbundle $\mathcal{D}$ of the tangent bundle TM, whose fiber over $p \in M$ is denoted by $\mathcal{D}(p)$, or the $C^\infty(M)$-module of its sections. In particular, $X \in \mathcal{D}$ means that $X_p \in \mathcal{D}(p)$, $\forall p \in M$;
- we write $\mathcal{D} = \langle X_1, ..., X_2\rangle$ if the distribution $\mathcal{D}$ is generated by vector fields $X_1, ..., X_r \in D(M)$; similarly, $\mathcal{D} = Ann(\alpha_1, ..., \alpha_s)$ means that $\mathcal{D}$ is constituted by vector fields annihilated by forms $\alpha_1, ..., \alpha_s \in \Lambda^1(M)$;
- if M is equipped with a contact distribution $\mathcal{C}$ and $\mathcal{S} \subset \mathcal{C}$ is a subdistribution of $\mathcal{C}$, then $\mathcal{S}^\perp$ denotes the $\mathcal{C}$-orthogonal complement to $\mathcal{S}$.

2. Preliminaries

In this section the notations and basic facts we need throughout the paper are collected. The reader is referred to [1, 3, 4] for further details.

2.1. **Jet bundles.** Let E be an $(n + m)$-dimensional manifold. The manifold of k-th order jets, $k \geq 0$, of n-dimensional submanifolds of E is denoted by $J^k(E, n)$ and $\pi_{k,l} : J^k(E, n) \longrightarrow J^l(E, n)$, $k \geq l$, stands for the canonical projection. If E is fibered by a map $\pi : E \to M$ over an n-dimensional manifold M, then $J^k\pi$ denotes the k-th order jet manifold of local sections of π. $J^k\pi$ is an open domain in $J^k(E, n)$. The $k-$th order jet of an $n-$dimensional submanifold $L \subset E$ at a point $z \in L$ is denoted by $[L]^k_z$. Similarly, if σ is a (local) section of π and $x \in M$, then $[\sigma]^k_x = [\sigma(U)]^k_{\sigma(x)}$, U being the domain of σ, stands for the $k-$th order jet of σ at x. The correspondence $z \mapsto [L]^k_z$

defines the $k-$th *lift* of L

$$j_k L : \quad L \quad \longrightarrow \quad J^k(E,n).$$

Similarly, the $k-$th *lift* of a local section σ of π

$$j_k\sigma : U \to J^k\pi$$

sends $x \in U$ to $[\sigma]_x^k$, i.e., $j_k\sigma = j_k(\sigma(U)) \circ \sigma$. Put

$$L^{(k)} = \mathrm{Im}(j_k L), \qquad\qquad M_\sigma^k = \mathrm{Im}(j_k\sigma).$$

Let $\theta_{k+1} = [L]_z^{k+1}$ be a point of $J^{k+1}(E,n)$. Then the *R-plane* associated with θ_{k+1} is the subspace

$$R_{\theta_{k+1}} = T_{\theta_k}(L^{(k)})$$

of $T_{\theta_k}(J^k(E,n))$ with $\theta_k = [L]_z^k$. The correspondence $\theta_{k+1} \mapsto R_{\theta_{k+1}}$ is biunique. Put

$$V_{\theta_{k+1}} = T_{\theta_k}(J^k(E,n))/R_{\theta_{k+1}}.$$

and denote by

$$pr_{\theta_{k+1}} : T_{\theta_k}(J^k(E,n)) \longrightarrow V_{\theta_{k+1}}$$

the canonical projection. The vector bundle

$$\nu_{k+1} : V_{(k+1)} \longrightarrow J^{k+1}(E,n), \qquad k \geq 0,$$

whose fiber over θ_{k+1}, is $V_{\theta_{k+1}}$ is naturally defined. By $\nu_{k,r}$ denote the pullback of ν_k via $\pi_{r,k}, r \geq k$.

Let $\mathcal{C}(\theta_k) \subset T_{\theta_k}(J^k\pi)$ be the span of all $R-$planes at θ_k. Then $\theta_k \mapsto \mathcal{C}(\theta_k)$ is the *Cartan distribution* on $J^k(E,n)$ denoted by $\mathcal{C}_k$. This distribution can be alternatively defined as the kernel of the ν_k-valued *Cartan form* U_k on $J^k(E,n)$:

$$U_k(\xi) = pr_{\theta_k}(d_{\theta_k}\pi_{k,k-1}(\xi)) \in V_{\theta_k}, \quad \xi \in T_{\theta_k}(J^k(E,n)).$$

A diffeomorphism $\varphi : J^k(E,n) \to J^k(E,n)$ is called *contact* if it preserves the Cartan distribution. Similarly, a vector field Y on $J^k(E,n)$ is called *contact* if $[Y, \mathcal{C}_k] \subset \mathcal{C}_k$. A contact diffeomorphism φ (respectively, a contact field Y) canonically lifts to a contact diffeomorphism $\varphi^{(l)}$ (respectively, a contact field $Y^{(l)}$) on $J^{k+l}(E,n)$).

Below the above constructions will be mainly used for $n = 2, m = 1, k = 1, 2$. In this case $\mathcal{C}_1$ is the canonical contact structure on $J^1(E,2), \dim E = 3$, and the bundle ν_1 is 1-dimensional. ν_1 is canonically isomorphic to the bundle whose fiber over $\theta \in J^1(E,2)$ is
$T_\theta(J^1(E,2))/\mathcal{C}(\theta)$.

A vector field X on E defines a section $s_X \in \Gamma(\nu_1), s_X(\theta) = pr_\theta(X)$. Since ν_1 is 1-dimensional, the ν_1-valued form U_1 can be presented as

$$U_1 = U_X \cdot s_X, \quad U_X \in \Lambda^1(J^1(E,2)), \tag{1}$$

in the domain where $s_X \neq 0$.

Let $\mathcal{M}$ be a manifold supplied with a contact distribution $\mathcal{C}$. An almost everywhere nonvanishing differential form $U \in \Lambda^1(\mathcal{M})$ is called *contact* if it vanishes on $\mathcal{C}$. A vector field $Y \in D(\mathcal{M})$ is contact iff

$$L_X(U) = \lambda U, \quad \lambda \in C^\infty(\mathcal{M}), \tag{2}$$

for a contact form U. For instance, U_X (see (1)) is a contact form.

If a vector field $X \in D(J^1(E,2)$ is contact, then $f = X \lrcorner U_1 \in \Gamma(\nu_1)$ is called the *generating function* of X. X is completely determined by f and is denoted by X_f in order to underline this fact. If U is a contact form on a contact manifold $(\mathcal{M}, \mathcal{C})$ and $X \in D(\mathcal{M})$ is contact, then $f = X \lrcorner U \in C^\infty(\mathcal{M})$ is the *generating function* of X with respect to U.

Vector fields $X, Y \in D(\mathcal{M})$ belonging to $\mathcal{C}$ are called $\mathcal{C}$*-orthogonal* if $[X, Y]$ also belongs to $\mathcal{C}$. Obviously, this is equivalent to $dU(X, Y) = 0$ for a contact form U. Observe that $\mathcal{C}$-orthogonality is a $C^\infty(\mathcal{M})$-linear property. A subdistribution $\mathcal{D}$ of $\mathcal{C}$ is called *Lagrangian* if any two fields $X, Y \in \mathcal{D}$ are $\mathcal{C}$-orthogonal and $\mathcal{D}$ is not contained in another distribution of bigger dimension possessing this property. If dim $\mathcal{M} = 2n + 1$, then dim $\mathcal{D} = n$.

A local chart (x, y, u) in E, where (x, y) are interpreted as independent variables and u as the dependent one, extends canonically to a local chart

$$(x, y, u, u_x = p, u_y = q, u_{xx} = r, u_{xy} = s, u_{yy} = t) \tag{3}$$

on $J^2(E, 2)$. Functions (x, y, u, p, q) form a (standard) chart in $J^1(E, 2)$. The local contact form $U = U_{\partial_u}$ (see (1)) in this chart reads

$$U = du - pdx - qdy.$$

Accordingly, in this chart the contact vector field corresponding to the generating with respect to U function f reads

$$X_f = -f_p\partial_x - f_q\partial_y + (f - pf_p - qf_q)\partial_u + (f_x + pf_x)\partial_p + (f_y + qf_x)\partial_q. \tag{4}$$

In the sequel we shall use $\mathcal{C}$ and U for the contact distribution and a contact form in the current context, respectively.

3. Parabolic MA equations

3.1. **MA equations.** Let E be a 3-dimensional manifold. A k-th order differential equation imposed on bidimensional submanifolds of E is a hypersurface $\mathcal{E} \subset J^k(E, 2)$. In a standard jet chart it is seen as a k-th order equation for one unknown function in two variables. In the sequel we shall deal only with second order equations of this kind. In a jet chart (3) on $J^2(E, 2)$ such an equation reads

$$F(x, y, u, p, q, r, s, t) = 0. \tag{5}$$

The standard subdivision of equations (5) into hyperbolic, parabolic and elliptic ones is intrinsically characterized by the nature of singularities of their multi-valued solutions (see [9]). Some elementary facts from solution singularity theory we need in this paper are brought below.

Let $\theta_2 \in J^2(E,2)$, $\theta_1 = \pi_{2,1}(\theta_2)$ and $F_{\theta_1} = \pi_{2,1}^{-1}(\theta_1)$. Recall that an R-plane at θ_1 is a *Lagrangian* plane in $\mathcal{C}(\theta_1)$, i.e., a bidimensional subspace $R \subset \mathcal{C}(\theta_1)$ such that $d\omega|_R = 0$ for a contact 1-form ω on $J^1(E,2)$. If $P \subset \mathcal{C}(\theta_1)$ is a 1-dimensional subspace, then

$$l(P) = \{\theta \in F_{\theta_1} | R_\theta \supset P\} \subset F_{\theta_1} \tag{6}$$

is the 1-*ray* corresponding to P.

A local chart on F_{θ_1} is formed by restrictions of r, s, t to F_{θ_1}. By abusing the notation we shall use r, s, t for these restrictions as well. Denote by $(\tilde{r}, \tilde{s}, \tilde{t})$ coordinates in $T_{\theta_2}(F_{\theta_1})$ with respects to the basis $\partial_r|_{\theta_2}, \partial_s|_{\theta_2}, \partial_t|_{\theta_2}$. Cones

$$\mathcal{V}_{\theta_2} = \left\{\tilde{r}\tilde{t} - \tilde{s}^2 = 0\right\} \subset T_{\theta_2}(F_{\theta_1}),\ \theta_2 \in F_{\theta_1},$$

define a distribution of cones $\mathcal{V} : \theta_2 \mapsto \mathcal{V}_{\theta_2}$ on F_{θ_1} called the *ray distribution*. This distribution is invariant with respect to contact transformations.

Lemma 1. *If P is spanned by the vector*

$$w = \zeta_1\partial_x + \zeta_2\partial_y + \mu\partial_u + \eta_1\partial_p + \eta_2\partial_q \in T_{\theta_1}(J^1(E,2))$$

then the 1-ray $l(P)$ is described by equations

$$\begin{cases} \zeta_1 r + \zeta_2 s = \eta_1, \\ \zeta_1 s + \zeta_2 t = \eta_2. \end{cases} \tag{7}$$

In particular, $l(P)$ is tangent to $\mathcal{V}$.

Proof. Put $p_0 = p(\theta_1), q_0 = q(\theta_1)$. Since $P \subset \mathcal{C}(\theta_1)$ we have

$$w \lrcorner U = 0 \Leftrightarrow \mu = \zeta_1 p_0 + \zeta_2 q_0, \tag{8}$$

and hence

$$w = \zeta_1(\partial_x + p_0\partial_u) + \zeta_2(\partial_y + q_0\partial_u) + \eta_1\partial_p + \eta_2\partial_q.$$

Moreover, $R_{\theta_2} \supset P$ iff

$$w \lrcorner (dp - rdx - sdy)_{\theta_2} = w \lrcorner (dq - sdx - tdy)_{\theta_2} = 0$$

and these relations are identical to (7).

Obviously, the components of the tangent to $l(P)$ vector at θ_1 are

$$(\tilde{r}, \tilde{s}, \tilde{t}) = \left(\zeta_2^2, -\zeta_1\zeta_2, \zeta_1^2\right) \tag{9}$$

and manifestly satisfy the equation $\tilde{r}\tilde{t} - \tilde{s}^2 = 0$. □

Put

$$\mathcal{E}_{\theta_1} = \mathcal{E} \cap F_{\theta_1}$$

An equation $\mathcal{E}$ is of *principal type* if it intersects transversally fibers of the projection $\pi_{2,1}$. In such a case $\mathcal{E}_{\theta_1}$ is a bidimensional submanifold of $F_{\theta_1}, \forall\, \theta_1 \in J^1(E,2)$. Further on we assume $\mathcal{E}$ to be of principal type.

The *symbol of* $\mathcal{E}$ at $\theta_2 \in \mathcal{E}$ is the bidimensional subspace

$$Smbl_{\theta_2}(\mathcal{E}) := T_{\theta_2}\mathcal{E}_{\theta_1}$$

of $T_{\theta_2}(F_{\theta_1})$.

A point $\theta_2 \in \mathcal{E}$ is *elliptic* (resp., *parabolic*, or *hyperbolic*) if $\mathcal{V}_{\theta_2}$ intersects $Smbl_{\theta_2}(\mathcal{E})$ in its vertex only (resp., along a line, or along two lines). So, if θ_2 is parabolic, then $Smbl_{\theta_2}(\mathcal{E})$ is a tangent to the cone $\mathcal{V}_{\theta_2}$ plane. In other words, in this case $\mathcal{E}_{\theta_1}$ is tangent to the ray distribution on F_{θ_1}.

Definition 1. *An equation $\mathcal{E}$ is called* elliptic *(resp.,* parabolic*, or* hyperbolic*) if all its points are* elliptic *(resp.,* parabolic*, or* hyperbolic*).*

Lemma 2. *If a 1-ray $l(P)$ is tangent to a parabolic equation $\mathcal{E}$ at a point θ_2, then $l(P) \subset \mathcal{E}_{\theta_1}$.*

Proof. The 1-rays distribution on F_{θ_1} may be viewed as the distribution of Monge's cones of a first order PDE for one unknown function in two variables. As it is easy to see, this equation in terms of coordinates $x_1 = r, x_2 = t, y = s$ on F_{θ_1} is $\frac{\partial y}{\partial x_1} \cdot \frac{\partial y}{\partial x_2} = \frac{1}{4}$. A banal computation then shows that characteristics of this equation are exactly 1-rays and hence its solutions are ruled surfaces composed of 1-rays. □

Corollary 1. *If $\mathcal{E}$ is a parabolic equation, then $\mathcal{E}_{\theta_1}$ is a ruled surface in F_{θ_1} composed of 1-rays.*

For a parabolic equation $\mathcal{E}$ and a point $\theta_1 \in J^1(E, 2)$ consider all 1-dimensional subspaces $P \subset \mathcal{C}(\theta_1)$ such that $l(P) \subset \mathcal{E}_{\theta_1}$. This is a 1-parametric family of lines and, so, their union is a bidimensional conic surface $\mathcal{W}_{\theta_1}$ in $\mathcal{C}(\theta_1)$. Then

$$\theta_1 \mapsto \mathcal{W}_{\theta_1}, \quad \theta_1 \in J^1(E, 2),$$

is the *Monge distribution* of $\mathcal{E}$. Integral curves of this distribution are curves along which multivalued solutions of $\mathcal{E}$ fold up. It is worth mentioning that tangent planes to a surface $\mathcal{W}_{\theta_1}$ are all Lagrangian. We omit the proof but note that Lagrangian planes are simplest surfaces possessing this property.

Given type singularities of multivalued solutions of a PDE are described by corresponding subsidiary equations. If $\mathcal{E}$ is a parabolic equation, then integral curves of the corresponding to $\mathcal{E}$ Monge distribution describe loci of fold type singularities of its solutions.

Intrinsically, the class of Monge-Ampere (MA) equations is characterized by the property that these subsidiary equations are as simple as possible. More precisely, this means that conic surfaces $\mathcal{W}_{\theta_1}$s' must be geometrically simplest. As we have already noticed, for parabolic equations the simplest are Lagrangian planes. Thus parabolic MA equations (PMAs) are conceptually defined as parabolic equations whose Monge distributions are distributions of Lagrangian planes. It will be shown below that this definition coincides with the traditional one.

Recall that, according to the traditional descriptive point of view, MA equations are defined as equations of the form

$$N(rt - s^2) + Ar + Bs + Ct + D = 0 \tag{10}$$

with N, A, B, C and D being some functions of variables x, y, u, p, q. MA equations with $N = 0$ are called *quasilinear*. $\Delta = B^2 - 4AC + 4ND$ is the *discriminant* of (10).

Proposition 1. *Equation (10) is elliptic (resp., parabolic, or hyperbolic) if* $\Delta < 0$ $(resp., \Delta = 0,$ *or* $\Delta > 0)$.

Proof. As it is easy to see, the symbol of equation (10) at a point θ_2 of coordinates (r, s, t) is described by the equation

$$N(t\tilde{r} + r\tilde{t} - 2s\tilde{s}) + A\tilde{r} + B\tilde{s} + C\tilde{t} = 0. \tag{11}$$

with $(\tilde{r}, \tilde{s}, \tilde{t})$ subject to the relation $\tilde{r}\tilde{t} - \tilde{s}^2 = 0$. Now one directly extracts the result from this relation and (11). □

Finally, we observe that all above definitions and constructions are contact invariant.

3.2. **Parabolic MA equations as Lagrangian distributions.** In this section it will be shown that the conceptual definition of PMA equations coincides with the traditional one.

First of all, we have

Proposition 2. *Equation (10) is parabolic in the sense of definition 1 if and only if* $\Delta = 0$. *The Monge distribution of parabolic equation (10) is a distribution of Lagrangian planes, i.e., a Lagrangian distribution.*

Proof. First, let $\mathcal{E}$ be equation (10). A simple direct computation shows that $\mathcal{E}$ is tangent to the 1-ray distribution iff $\Delta = 0$.

Second, coefficients of equation (10) may be thought as functions on $J^1(E, 2)$. Let $A_0, \ldots, N_0$ be their values at a point $\theta_1 \in J^1(E, 2)$. Then

$$N_0(t\tilde{r} + r\tilde{t} - 2s\tilde{s}) + A_0\tilde{r} + B_0\tilde{s} + C_0\tilde{t} = 0$$

is the equation of $\mathcal{E}_{\theta_1}$ in F_{θ_1}. If $N_0 \neq 0$ this equation describes a standard cone with the vertex at the point θ_2 of coordinates $r = -C_0/N_0, s = B_0/2N_0, t = -A_0/N_0$. So, $\mathcal{E}_{\theta_1}$ is the union of 1-rays $l(P)$ passing through θ_2. By definition this implies that $P \subset R_{\theta_2}$ and hence $\mathcal{W}_{\theta_1}$ is the union of lines P that belong to R_{θ_2}. This shows that $\mathcal{W}_{\theta_1} = R_{\theta_2}$.

Finally, note that the case $N_0 = 0$ can be brought to the previous one by a suitable choice of jet coordinates. □

From now on we shall denote by $\mathcal{D}_{\mathcal{E}}$ the Monge distribution of a PMA equation $\mathcal{E}$. From the proof of the above Proposition we immediately extract geometrical meaning of the correspondence $\mathcal{E} \mapsto \mathcal{D}_{\mathcal{E}}$.

Corollary 2. *The distribution* $\mathcal{D}_{\mathcal{E}}$ *associates with a point* $\theta_1 \in J^1(E, 2)$ *the* R*-plane* R_{θ_2} *with* θ_2 *being the vertex of the cone* $\mathcal{E}_{\theta_1}$.

A coordinate description of $\mathcal{D}_\mathcal{E}$ is as follows.

Proposition 3.

$$\mathcal{D}_\mathcal{E} = \left\langle \partial_x + p\partial_u - \frac{C}{N}\partial_p + \frac{B}{2N}\partial_q,\ \partial_y + q\partial_u + \frac{B}{2N}\partial_p - \frac{A}{N}\partial_q \right\rangle, \quad \textit{if } N \neq 0 \tag{12}$$

or

$$\mathcal{D}_\mathcal{E} = \left\langle A\partial_x + \frac{B}{2}\partial_y + (Ap + Bq)\partial_u - D\partial_p,\ \frac{B}{2}\partial_p - A\partial_q \right\rangle, \quad \textit{if } N = 0. \tag{13}$$

Proof. If $(x_0, \ldots, t_0)$ are coordinates of $\theta_2 \in J^2(E, 2)$, then the R-plane R_{θ_2} is generated by vectors

$$\partial_x + p_0\partial_u + r_0\partial_p + s_0\partial_q \qquad \text{and} \qquad \partial_y + q_0\partial_u + s_0\partial_p + t_0\partial_q$$

at the point θ_1. By Corollary 2 one gets the needed result for $N \neq 0$ just by specializing coordinates of θ_2 in these expressions to that of the vertex of the cone $\mathcal{E}_{\theta_1}$ (see the proof of Proposition 2). The case $N = 0$ is reduced to the previous one by a suitable transformation of jet coordinates. □

In its turn, the distribution $\mathcal{D}_\mathcal{E}$ completely determines the equation $\mathcal{E}$. More exactly, we have

Proposition 4. *Let $\mathcal{D}$ be a Lagrangian subdistribution of $\mathcal{C}$. Then the submanifold*

$$\mathcal{E}_\mathcal{D} = \{\theta_2 \in J^2(E, 2) : \dim(R_{\theta_2} \cap \mathcal{D}(\theta_1)) > 0\}$$

of $J^2(E, 2)$ is a parabolic Monge-Ampere equation and $\mathcal{D} = \mathcal{D}_{\mathcal{E}_\mathcal{D}}$.

Proof. There is the only one point $\theta_2 \in (\mathcal{E}_\mathcal{D})_{\theta_1}$ such that $R_{\theta_2} = \mathcal{D}(\theta_1)$. With this exception $\dim(R_{\theta_2} \cap \mathcal{D}(\theta_1)) = 1$. Hence $(\mathcal{E}_\mathcal{D})_{\theta_1}$ is the union of all 1-rays $l(P)$ such that $P \subset \mathcal{D}(\theta_1)$. They all pass through the exceptional point θ_2 and hence constitute a cone in F_{θ_1}. But cones composed of 1-rays are tangent to the 1-ray distribution on F_{θ_1} and, so, all their points are parabolic. Finally, the last assertion directly follows from Corollary 2. □

Results of this section are summed up in the following Theorem which is the starting point of our subsequent discussion of parabolic Monge-Ampere equations.

Theorem 1. *The correspondence $D \longmapsto \mathcal{E}_D$ between Lagrangian distributions on $J^1(E, 2)$ and parabolic Monge-Ampere equations is one-to-one.*

The meaning of this Theorem is that it decodes the geometrical problem hidden under analytical condition (10). Namely, this problem is to find Legendrian submanifolds S of a 5-dimensional contact manifold $(\mathcal{M}, \mathcal{C})$ that intersect a given Lagrangian distribution $\mathcal{D} \subset \mathcal{C}$ in a nontrivial manner, i.e.,

$$\dim\{T_\theta(S) \cap \mathcal{D}(\theta)\} > 0,\ \forall\, \theta \in S.$$

The triple $\mathcal{E} = (\mathcal{M}, \mathcal{C}, \mathcal{D})$ encodes this problem. By this reason, in the rest of this paper the term "parabolic Monge-Ampere equation" will refer to such a triple. In particular, equivalence and classification problems for PMAs are interpreted as such problems for Lagrangian distributions on 5-dimensional contact manifolds.

4. Geometry of Lagrangian distributions

In this section we deduce some basic facts about Lagrangian distributions on 5-dimensional contact manifolds which allow to reveal four natural classes of them. We fix the notation $(\mathcal{M}, \mathcal{C})$ for the considered contact manifold and $\mathcal{D}$ for a Lagrangian distribution on it.

First of all, Lagrangian distributions are subdivided onto *integrable* and *nonintegrable* ones. Accordingly, the corresponding parabolic Monge-Ampere equations are called *integrable*, or *non-integrable*. In the subsequent section it will be shown that all integrable PMAs are locally contact equivalent to the equation $u_{xx} = 0$ and we shall concentrate on nonintegrable PMAs.

If $\mathcal{D}$ is nonintegrable, then its *first prolongation* $\mathcal{D}_{(1)}$, i.e., the span of all vector fields belonging to $\mathcal{D}$ and their commutators, is 3-dimensional. Moreover, we have

Lemma 3.
(1) $\mathcal{D}_{(1)} \subset \mathcal{C}$;
(2) *the* $\mathcal{C}$*-orthogonal complement* $\mathcal{R}$ *of* $\mathcal{D}_{(1)}$ *is a* 1*-dimensional subdistribution of* $\mathcal{D}$.

Proof. (1) If (locally) $\mathcal{D} = \langle X, Y \rangle$, then (locally) $\mathcal{D}_{(1)} = \langle X, Y, [X, Y] \rangle$. But, by definition of $\mathcal{C}$-orthogonality, $[X, Y] \in \mathcal{C}$.

(2) The form dU restricted to $\mathcal{C}$ is nondegenerate. So, the assertion follows from the fact that in a symplectic linear space a Lagrangian subspace in a hyperplane contains the skew-orthogonal complement of the hyperplane. □

The 1-dimensional distribution $\mathcal{R}$ will be called *the directing distribution of* $\mathcal{D}$ (alternatively, of $\mathcal{E}$).

This way one gets the following flag of distributions

$$\mathcal{R} \subset \mathcal{D} \subset \mathcal{D}_{(1)} \subset \mathcal{C}.$$

The directing distribution $\mathcal{R}$ uniquely defines $\mathcal{D}_{(1)}$, which is its $\mathcal{C}$-orthogonal complement, and the distribution

$$\mathcal{D}' = \left\{ X \in \mathcal{D}_{(1)} | \, [X, \mathcal{R}] \subset D_{(1)} \right\}.$$

Since $[X, \mathcal{R}] \subset D_{(1)}$ for any $X \in \mathcal{D}$, we see that $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}_{(1)}$. So, by obvious dimension arguments, only one of the following two possibilities may occur locally: either $\mathcal{D}' = \mathcal{D}$, or $\mathcal{D}' = \mathcal{D}_{(1)}$. A non-integrable Lagrangian distribution $\mathcal{D}$ is called *generic* in the first case and *special* in the second. Accordingly, the corresponding PMAs are called generic, or special.

The following assertion directly follows from the above definitions.

Proposition 5. *A generic Lagrangian distribution $\mathcal{D}$ is completely determined by its directing distribution $\mathcal{R}$. It is no longer so for a special distribution and in this case $\mathcal{R}$ is the characteristic distribution of $\mathcal{D}_{(1)}$.*

In this paper we shall concentrate on generic PMA equations with a special attention to the equivalence problem. In view of Proposition 5 this problem takes part of the equivalence problem for 1-dimensional subdistributions of $\mathcal{C}$. To localize this part a criterion allowing to distinguish directing distributions of generic PMA equations from other 1-dimensional subdistributions of $\mathcal{C}$. The introduced below notion of the *type* of a 1-dimensional subdistribution of $\mathcal{C}$ gives such a criterion.

Fix a 1-dimensional distribution $\mathcal{S} \subset \mathcal{C}$ and put

$$\mathcal{S}_r = \{Y \in \mathcal{C} \mid X^s(Y) \in \mathcal{C}, \forall\, X \in \mathcal{S} \text{ and } 0 \leq \mathrm{s} \leq \mathrm{r}\}\,.$$

In the following lemma we list without a proof some obvious properties of $\mathcal{S}_r$.

Lemma 4.

- $\mathcal{S}_{r+1} \subset \mathcal{S}_r$ *and* $\mathcal{S} \subset \mathcal{S}_r$*;*
- *If (locally)* $\mathcal{S} = \langle X \rangle$*, then (locally)*
$$\mathcal{S}_r = \{Y \in \mathcal{C} \mid X^s(Y) \in \mathcal{C},\ 0 \leq s \leq r\}$$
- $\mathcal{S}_r$ *is a finitely generated* $C^\infty(\mathcal{M})$*-module;*
- *If (locally)* $\mathcal{C} = Ann(U),\ U \in \Lambda^1(\mathcal{M})$*, and (locally)* $\mathcal{S} = \langle X \rangle$*, then*
$$\mathcal{S}_r = Ann(U, X(U), ..., X^r(U)).$$

Let $\mathcal{T}$ be a $C^\infty(\mathcal{M})$-module. An open domain $B \subset \mathcal{M}$ is called *regular* for $\mathcal{T}$ if the localization of $\mathcal{T}$ to B is a projective $C^\infty(B)$-module. If $\mathcal{T}$ is finitely generated (see [7]), then this localization is isomorphic to the $C^\infty(B)$-module of smooth sections of a finite dimensional vector bundle over B. In such a case the dimension of this bundle is called the *rank* of $\mathcal{T}$ on B and denoted by $\mathrm{rank_B}\mathcal{T}$. Moreover, the manifold $\mathcal{M}$ is subdivided into a number of open domains that are regular for $\mathcal{T}$ and the set of its *singular points* which is closed and thin. In particular, vector bundles representing localizations of $\mathcal{S}_r$ to its regular domains are distributions contained in $\mathcal{C}$ and containing $\mathcal{S}$ (Lemma 4).

Lemma 5. *Let $B \subset \mathcal{M}$ be a common regular domain for modules $\mathcal{S}_r$ and $\mathcal{S}_{r+1}$. Then either* $\mathrm{rank_B}\mathcal{S}_\mathrm{r} = \mathrm{rank_B}\mathcal{S}_{\mathrm{r+1}} + 1$*, or* $\mathrm{rank_B}\mathcal{S}_\mathrm{r} = \mathrm{rank_B}\mathcal{S}_{\mathrm{r+1}}$*. In the latter case B is regular for $\mathcal{S}_p, p \geq r$, and* $\mathrm{rank_B}\mathcal{S}_\mathrm{p} = \mathrm{rank_B}\mathcal{S}_\mathrm{r}$*.*

Proof. The first alternative takes place iff $X^{r+1}(U)$ is $C^\infty(\mathcal{M})$-independent of $U, X(U), \ldots, X^r(U)$ as it is easily seen from the last assertion of Lemma 4. In the second case the equality of ranks implies that localizations of $\mathcal{S}_r$ and $\mathcal{S}_{r+1}$ to B coincide. This shows that the localization of $\mathcal{S}_p$ to B stabilizes by starting from $p = r$. □

Corollary 3. *With the exception of a thin closed set the manifold $\mathcal{M}$ is subdivided into open domains each of them is regular for all $\mathcal{S}_p, p \geq 0$. Moreover, in such a domain B,* $\mathrm{rank_B}\mathcal{S}_p = 4 - p$*, if $p \leq r$, and* $\mathrm{rank_B}\mathcal{S}_p = 4 - r$*, if $p \geq r$, for an integer $r = r(B)$, $1 \leq r \leq 3$.*

Proof. It immediately follows from Lemma 5 that the function $p \mapsto \mathrm{rank_B}\mathcal{S}_\mathrm{p}$ steadily decreases up to the instance, say $p = r$, when
$\mathrm{rank_B}\mathcal{S}_\mathrm{r} = \mathrm{rank_B}\mathcal{S}_\mathrm{r+1}$ occurs for the first time, and stabilizes after. Since $\mathcal{S} \subset \mathcal{S}_p$, the last assertion of Lemma 4 shows that this instance happens at most for $p = 3$, i.e., that $r \leq 3$. On the other hand, forms U and $X(U)$ are independent. Indeed, the equality $X(U) = fU$ for a function$f$ means that X is a contact field with generating function $i_X(U) \equiv 0$. But only the zero field is such one. Hence $r \geq 1$.

Finally, observe that regular domains for $\mathcal{S}_{p+1}$ are obtained from those for $\mathcal{S}_p$ by removing from latters some thin subsets. Since, as we have seen before, the situation stabilizes after at most four steps by starting from $p = 0$, the existence of common regular domains for all $\mathcal{S}_p$'s whose union is everywhere dense in $\mathcal{M}$ is guaranteed. □

Definition 2. *If $\mathcal{M}$ is the only regular domain for all $\mathcal{S}_p$'s, then $\mathcal{S}$ is called* regular *on $\mathcal{M}$ and the integer r by starting from which the* $\mathrm{rank_B}\mathcal{S}_\mathrm{r}$ *stabilizes is called the* type *of $\mathcal{S}$ and also the* type *of X, if $\mathcal{S} = \langle X \rangle$.*

Hence Lemma 5 tells that the type of $\mathcal{S}$ can be only one of the numbers $1, 2$, or 3, and that the union of domains in which $\mathcal{S}$ is regular is everywhere dense in $\mathcal{M}$.

It is not difficult to exhibit vector fields of each these three types. For instance, vector fields of the form $X_f - fX_1$ on $\mathcal{M} = J^1(E, 2)$ with everywhere non-vanishing function f are of type 1. Fields $\partial_x + p\partial_z + q\partial_p$ and $\partial_x + p\partial_z + (xy + q)\partial_p$ are of type 2 and 3, respectively.

Now we can characterize directing distributions of generic PMA equations

Proposition 6. *A 1-dimensional regular distribution $\mathcal{S} \subset \mathcal{C}$ is the directing distribution of a generic PMA equation iff it is of type 3.*

Proof. Let $\mathcal{S}$ be a regular distribution of type 3 and (locally) $\mathcal{S} =< X >$. Then by definition $\mathcal{S}_1 = \mathcal{S}^\perp$ and $\mathcal{D} = \mathcal{S}_2$ is a bidimensional subdistribution of $\mathcal{C}$. $\mathcal{D}$ is Lagrangian, since $\mathcal{D} \subset \mathcal{S}_1 = \mathcal{S}^\perp$ and $\mathcal{S} \subset \mathcal{D}$. If (locally) $\mathcal{D} = \langle X, Y \rangle$, then $[X, Y] \notin \mathcal{D}$. Indeed, assuming that $[X, Y] \in \mathcal{D}$ one sees that $X^p(Y) \in \mathcal{D} \subset \mathcal{C}, \forall p$, and hence $Y \in \mathcal{S}_p, \forall p$. In particular, this implies that $\mathcal{D} = \langle X, Y \rangle \subset \mathcal{S}_3$ in contradiction with the fact that $\dim \mathcal{S}_3 = 1$ if $\mathcal{S}$ is of type 3. So, $\dim \langle \mathrm{X}, \mathrm{Y}, [\mathrm{X}, \mathrm{Y}] \rangle = 3$, and $\langle X, Y, [X, Y] \rangle \subset \mathcal{S}^\perp$, since $X(Y), X^2(Y) \in \mathcal{C}$. Now, by dimension arguments, we conclude that $\langle X, Y, [X, Y] \rangle = \mathcal{S}^\perp$ and, therefore, $\mathcal{S}$ is the directing distribution of $\mathcal{D} = \mathcal{S}_2$.

Conversely, assume that (locally) $\mathcal{R} = \langle X \rangle$ is the directing distribution of a generic PMA equation corresponding to the Lagrangian distribution $\mathcal{D} = \langle X, Y \rangle$ (locally). Then, by definition, $\mathcal{R}_1 = \mathcal{R}^\perp$. Moreover, $\mathcal{D} = \mathcal{D}'$ implies that $X^2(Y) \notin \mathcal{R}^\perp$ and hence fields $X, Y, X(Y), X^2(Y)$ form a local basis of $\mathcal{C}$. This shows that $X^3(Y) \notin \mathcal{C}$. Indeed, the assumption $X^3(Y) \in \mathcal{C}$ implies $[X, \mathcal{C}] \subset \mathcal{C}$, i.e., that X is a nonzero contact field with zero generating function. Hence $Y \notin \mathcal{R}_3 \Leftrightarrow [X, Y] \notin \mathcal{R}_2$. From one side, this shows that $\mathcal{D} = \mathcal{R}_2$ and, from other side, that $\mathcal{R}_2 \neq \mathcal{R}_3$. So, $\mathcal{R}_3 = \langle \mathcal{R} \rangle \Rightarrow \mathrm{rank}\, \mathcal{R} = 3$. □

From now on we denote by Z the directing distribution of the considered PMA $\mathcal{E}$. A coordinate description of $\mathcal{D} = \mathcal{D}_{\mathcal{E}}, \mathcal{D}_1$ and Z is easily obtained by a direct computation (see Proposition 3):

Proposition 7. *Let $\mathcal{E}$ be a quasilinear nonintegrable PMA of the form (10) with $A \neq 0$. By normalizing its coefficients to $A = 1$ one has: $\mathcal{D} = \langle X_1, X_2 \rangle$ and $\mathcal{D}_{(1)} = \langle X_1, X_2, X_3 \rangle$ with*

$$X_1 := \partial_x + p\partial_u + \frac{B}{2}(\partial_y + q\partial_u) - D\partial_p,$$

$$X_2 := \frac{B}{2}\partial_p - \partial_q,$$

$$X_3 := [X_1, X_2] = M_1 (\partial_y + q\partial_u) + M_2\partial_p,$$

where

$$M_1 = -\frac{1}{2}X_2(B), \qquad M_2 = \frac{1}{2}X_1(B) + X_2(D),$$

and $\mathcal{R} = \langle Z \rangle$ with

$$Z := M_1X_1 - M_2X_2.$$

Proposition 8. *Let $\mathcal{E}$ be a nonintegrable PMA of the form (10) with $N \neq 0$. By normalizing its coefficients to $N = 1$ one has: $\mathcal{D} = \langle X_1, X_2 \rangle$ and $\mathcal{D}_{(1)} = \langle X_1, X_2, X_3 \rangle$ with*

$$X_1 := \partial_x + p\partial_u - C\partial_p + \frac{B}{2}\partial_q,$$

$$X_2 := \partial_y + q\partial_u + \frac{B}{2}\partial_p - A\partial_q,$$

$$X_3 := [X_1, X_2] = M_1\partial_q + M_2\partial_p,$$

where

$$M_1 = -X_1(A) - \frac{1}{2}X_2(B), \qquad M_2 = \frac{1}{2}X_1(B) + X_2(C),$$

and $\mathcal{R} = \langle Z \rangle$ with

$$Z := M_1X_1 - M_2X_2.$$

5. Classification of integrable PMAs

For completeness we shall prove here the following, essentially known, result in a manner that illustrate the idea of our further approach.

Theorem 2. *With the exception of singular points all integrable PMA equations are locally contact equivalent each other and, in particular, to the equation $u_{xx} = 0$.*

Proof. Let $\mathcal{E} = (\mathcal{M}, \mathcal{C}, \mathcal{D})$ be an integrable PMA equation, i.e., the Lagrangian distribution $\mathcal{D}$ is integrable and, as such, define a 2-dimensional Legendrian foliation of $\mathcal{M}$. Locally this foliation can be viewed as a fibre bundle $\Pi : \mathcal{M} \to W$ over a 3-dimensional manifold W. The differential $d_\theta(\Pi) : T_\theta\mathcal{M} \to T_yW, y = \Pi(\theta)$, sends $\mathcal{C}(\theta)$ to a bidimensional subspace $P_\theta \subset T_yW$, since $\ker \mathrm{d}_\theta(\Pi) = \mathcal{D}(\theta)$. This way one gets the map

$\Pi_y : \Pi^{-1}(y) \to G_{3,2}(y)$, $\theta \mapsto P_\theta$, where $G_{3,2}(y)$ is the Grassmanian of 2-dimensional subspaces in T_yW. Note that $\dim \Pi^{-1}(\mathrm{y}) = \dim \mathrm{G}_{3,2}(\mathrm{y}) = 2$ and, so, the local rank of Π_y may vary from 0 to 2. We shall show that, with the exception of a thin set of singular points, Π_y's are of rank 2, i.e., Π_y's are local diffeomorphisms.

First, assume that this rank is zero for all $y \in W$, i.e., Π_y's are locally constant maps. In this case $\mathcal{P}_\theta$ does not depend on $\theta \in \Pi^{-1}(y)$ and we can put $\mathcal{P}(y) = P_\theta$, for a $\theta \in \Pi^{-1}(y)$. Hence $y \mapsto \mathcal{P}(y)$ is a distribution on W and $\mathcal{C}$ is its pullback via Π. This shows that the distribution tangent to fibers of Π, i.e., $\mathcal{D}$, is characteristic for $\mathcal{C}$. But a contact distribution does not admit nonzero characteristics.

Second, if the rank of Π_y's equals to one for all $y \in W$, then $\mathcal{M}$ is foliated by curves

$$\gamma_P = \{\theta \in \Pi^{-1}(y) | d_\theta \Pi(\mathcal{C}(\theta)) = P\}$$

with P being a bidimensional subspace of T_yW. Locally, this foliation may be seen as a fibre bundle $\Pi_0 : \mathcal{M} \to N$ over a 4-dimensional manifold N, and Π factorizes into the composition

$$\mathcal{C} \xrightarrow{\Pi_0} N \xrightarrow{\Pi_1} W$$

with Π_1 uniquely defined by Π and Π_0. By construction the 3-dimensional subspace $d_\theta \Pi_0(\mathcal{C}(\theta)) \subset T_zN$, $z = \Pi_0(\theta)$, does not depend on $\theta \in \Pi_0^{-1}(z) = \gamma_P$ and one can put $\mathcal{Q}(z) = d_\theta \Pi_0(\mathcal{C}(\theta))$ for a $\theta \in \Pi_0^{-1}(z)$. As before we see that $\mathcal{C}$ is the pullback via Π_0 of the 3-dimensional distribution $z \mapsto \mathcal{Q}(z)$ in contradiction with the fact that $\mathcal{C}$ does not admit nonzero characteristics.

Thus, except singular points, Π is of rank 2 and hence a local diffeomorphism. So, locally, Π_y identifies $\Pi^{-1}(y)$ and an open domain in $G_{3,2}(y)$. By observing that $G_{3,2}(y) = \pi_{1,0}^{-1}(y)$, with $\pi_{1,0} : J^1(W, 2) \to W$ being a natural projection, one gets a local identification of $\mathcal{M}$ with an open domain in $J^1(W, 2)$. It is easy to see that this identification is a contact diffeomorphism. In other words, we have proven that any integrable Lagrangian distribution on a 5-dimensional contact manifold is locally equivalent to the distribution of tangent planes to fibers of the projection $J^1(\mathbb{R}^3, 2) \to \mathbb{R}^3$.

Finally, we observe that $\mathcal{D}_\mathcal{E} = \langle \partial_x, \partial_y \rangle$ for the equation $\mathcal{E} = \{u_{xx} = 0\}$ and hence this equation is integrable. □

6. Projective curve bundles and non-integrable generic PMAs

In this section non-integrable generic Lagrangian distributions and, therefore, the corresponding PMAs are represented as 4-parameter families of curves in the projective 3-space or, more exactly, as *projective curve bundles*. Differential invariants of single curves composing such a bundle (say, projective curvature, torsion,etc) put together give differential invariants of the whole bundle and consequently of the corresponding PMA. This basic geometric idea is developed in details in the subsequent section.

Let N be a 4-dimensional manifold. Denote by PT_a^*N the 3-dimensional projective space of all 1-dimensional subspaces of the cotangent to N space T_a^*N at the point

$a \in N$. The *projectivization* $p\tau^* : PT^*N \to N$ of the cotangent to N bundle $\tau^* : T^*N \to N$ is the bundle whose total space is $PT^*N = \bigcup_{a \in N} PT_a^*N$ and the fiber over $a \in N$ is PT_a^*N, i.e., $(p\tau^*)^{-1}(a) = PT_a^*N$. A *projective curve bundle* (PCB) over N is a 1-dimensional subbundle $\pi : K \to N$ of $p\tau^*$:

$$\begin{array}{ccc} K & \hookrightarrow & PT^*N \\ \pi \downarrow & & \downarrow p\tau^* \\ N & \xrightarrow{id} & N \end{array}$$

The fiber $\pi^{-1}(y)$, $y \in N$, is a smooth curve in the projective space PT_y^*N. A diffeomorphism $\Phi : N \to N'$ lifts canonically to a diffeomorphism $PT^*N \to PT^*N'$. This lift sends a PCB π over N to a PCB over N', denoted by $\Phi\pi$. A PCB π over N and a PCB π' over N' are *equivalent* if there exist a diffeomorphism $\Phi : N \to N'$ such that $\pi' = \Phi\pi$.

Let $\pi : K \to N$ be a PCB and $\theta =< \rho >\in K$ with $\rho \in T_{\pi(\theta)}^*N$. Denote by W_θ the 3-dimensional subspace of $T_{\pi(\theta)}N$ annihilated by θ, i.e.,

$$W_\theta = \left\{ \xi \in T_{\pi(\theta)}N \,|\, \rho(\xi) = 0 \right\}.$$

Two distributions are canonically defined on K. First of them is the 1-dimensional distribution $\mathcal{R}_\pi$ formed by all vertical with respect to π vectors. The second one, denoted by $\mathcal{C}_\pi$, is defined by

$$(\mathcal{C}_\pi)_\theta = \{ \eta \in T_\theta K \,|\, d_\theta\pi(\eta) \in W_\theta \}.$$

Obviously, $\dim \mathcal{C}_\pi = 4$ and $\mathcal{R}_\pi \subset \mathcal{C}_\pi$. If, locally, $\mathcal{R}_\pi = \langle Z \rangle$, $Z \in D(K)$, and $\mathcal{C}_\pi = Ann(U_\pi)$, $U_\pi \in \Lambda^1(K)$, then the *osculating distributions* of π are defined as

$$\mathcal{Z}_s^\pi = Ann(U_\pi, Z(U_\pi), ..., Z^s(U_\pi)), \qquad s = 0, 1, 2, 3.$$

It is easy to see that this definition does not depend on the choice of Z and U_π. Note that $\mathcal{C}_\pi = \mathcal{Z}_0^\pi$. Also, $\mathcal{R}_\pi \subset \mathcal{Z}_s^\pi$, $\forall\, s \geq 0$, as it easily follows from $[i_Z, L_Z] = 0$ and $U_\pi(Z) = 0$. Moreover, generically forms $U_\pi, Z(U_\pi), ..., Z^3(U_\pi)$ are independent and so, by dimension arguments, $\mathcal{R}_\pi = \mathcal{Z}_3^\pi$.

We say that π is a *regular PCB* iff the following two conditions are satisfied: (i) $\mathcal{R}_\pi = \mathcal{Z}_3^\pi$ and (ii) $\mathcal{C}_\pi$ is a contact structure on K. We emphasize that regularity is a generic condition. Moreover, conditions (i)-(ii) are equivalent to the fact that $\mathcal{R}_\pi$ is of type 3 with respect to the contact distribution $\mathcal{C}_\pi$. So, by Proposition 6, the distribution

$$\mathcal{D}_\pi = \{ X \in \mathcal{R}_\pi^\perp | L_X(\mathcal{R}_\pi) \subset \mathcal{R}_\pi^\perp \}.$$

with $\mathcal{R}_\pi^\perp$ being the $\mathcal{C}_\pi$-orthogonal complement of $\mathcal{R}_\pi$ is bidimensional and Lagrangian for a regular PCB π. Thus we have

Theorem 3. *If π is a regular PCB, then $\mathcal{D}_\pi$ is a Lagrangian subdistribution of $\mathcal{C}_\pi$ and $(K, \mathcal{C}_\pi, \mathcal{D}_\pi)$ is a generic PMA whose directing distribution is $\mathcal{R}_\pi$. Conversely, a generic PMA locally determines a regular PCB.*

Proof. The first assertion of the Theorem is already proved. It remains to represent a generic PMA $(\mathcal{M}, \mathcal{C}, \mathcal{D})$ as a regular PCB. Integral curves of its directing distribution $\mathcal{R}$ foliate $\mathcal{M}$. Locally, this foliation may be considered as a fiber bundle $\pi : \mathcal{M} \to N$ over a 4-dimensional manifold N. Since $\mathcal{R}(\theta) \subset \mathcal{C}(\theta)$, the subspace $V_\theta = d_\theta\pi(\mathcal{C}(\theta)) \subset T_yN$, $y = \pi(\theta)$, is 3-dimensional. Put $\Gamma_y = \pi^{-1}(y)$. The map $\pi_y : \Gamma_y \to G_{4,3}(y)$, $\theta \mapsto V_\theta$, with $G_{4,3}(y)$ being the Grassmanian of 3-dimensional subspaces in T_yN, is almost everywhere of rank 1. Indeed, the assumption that locally this rank is zero leads, as in the proof of Theorem 2, to conclude that locally the contact distribution $\mathcal{C}$ is the pullback via π of a 3-dimensional distribution on N.

The correspondence $\iota_y : G_{4,3}(y) \to PT_y^*N$ that sends a 3-dimensional subspace $V \subset T_yN$ to $Ann(V)$ is, obviously, a diffeomorphism. Hence the composition $\iota_y \circ \pi_y$ is a local embedding with exception of a thin subset of singular points. Now, it is easy to see that images of Γ_y's via $\iota_y \circ \pi_y$'s give the required PCB. □

The above construction associating a PCB with a given PMA is manifestly functorial, i.e., an equivalence $F : (\mathcal{M}, \mathcal{C}, \mathcal{D}) \longrightarrow (\mathcal{M}', \mathcal{C}', \mathcal{D}')$ of PMAs induces an equivalence $\Phi : (N, K, \pi) \longrightarrow (N', K', \pi')$ of associated PCBs. Indeed, F sends $\mathcal{R}$ to $\mathcal{R}'$ and hence integral curves of $\mathcal{R}$ (locally, fibers of π) to integral curves of $\mathcal{R}'$ (locally, fibers of π'). This defines a map Φ of the variety N of fibers of π to the variety N' of fibers of π', etc. Thus we have

Corollary 4. *The problem of local contact classification of generic PMAs is equivalent to the problem of local classification of regular PCBs with respect to diffeomorphisms of base manifolds.*

Now we observe that there is another, in a sense, dual PCB associated with a given PMA equation. Namely, associate with a point $\theta \in K$ the line $L_\theta = d_\theta\pi(D_\pi(\theta)) \subset T_yN$, $y = \pi(\theta)$. The correspondence $\theta \mapsto L_\theta \in PT_yN$, where PT_yN denotes the projective space of lines in T_yN, defines a map of $\pi^{-1}(y)$ to PT_yN, i.e., a (singular) curve in PT_yN. As before this defines locally a 1-dimensional subbundle in the projectivization PTN of TN. It will be called the *second* PCB associated with the considered PMA.

PCBs may be considered as canonical models of PMAs. Besides other they suggest a geometrically transparent construction of scalar differential invariants of PMAs.

Let $\mathcal{I}$ be a scalar projective differential invariant of curves in $\mathbb{R}P^3$, say, the *projective curvature* (see [11, 2]), $\theta \in K$ and $y = \pi(\theta)$. The value of this invariant for the curve $\Gamma_y = \pi^{-1}(y)$ in PT_y^* is a function on Γ_y. Denote it by $\mathcal{I}_{\pi,y}$ and put $\mathcal{I}_\pi(\theta) = \mathcal{I}_{\pi,y}(\theta)$ if $\theta \in \Gamma_y \subset K$. Then, obviously, $\mathcal{I}_\pi \in C^\infty(K)$ is a differential invariant of the PCB π and hence of the PMA represented by π.

Theorem 4. *The differential invariants of the form $\mathcal{I}_\Psi$ are sufficient for a complete classification of generic PMA equations on the basis of the "principle of n-invariants".*

Proof. According to the "principle of n-invariants", it is sufficient to construct $n = \dim \mathcal{M} = 5$ independent differential invariants of PMAs in order to solve the classification problem. Such invariants of the required form will be constructed in the next section. □

For the "principle of n-invariants" the reader is referred to [1, 10].

7. Differential invariants of generic PCBs

Let $\pi : K \to N$ be a regular PCB and, as before, $\mathcal{R}_\pi = < Z >$, $\mathcal{D}_\pi = < Z, X >$ and $\mathcal{C}_\pi = Ann(U)$. Here the vector field Z and the 1-form U are unique up to a functional nowhere vanishing factor, while X is unique up to a transformation $X \longmapsto gX + \varphi Z$, $g, \varphi \in C^\infty(K)$ with nowhere vanishing g.

Since the considered PCB is regular we have the following flag of distributions

$$\mathcal{R}_\pi \subset \mathcal{D}_\pi \subset \mathcal{R}_\pi^\perp \subset \mathcal{C}_\pi \subset \mathcal{D}(K)$$

of dimensions increasing from 1 to 5, respectively. In terms of X, Z and U they are described as follows:

Proposition 9. *Locally, with the exception of a thin set of singular points we have:*

(1) $\mathcal{R}_\pi = Ann(Z^i(U) : i = 0, 1, 2, 3) = \langle Z \rangle$;
(2) $\mathcal{D}_\pi = Ann(U, Z(U), Z^2(U)) = \langle Z, X \rangle$;
(3) $\mathcal{R}_\pi^\perp = Ann(U, Z(U)) = < Z, X, Z(X) >$;
(4) $\mathcal{C}_\pi = Ann(U) = < Z, Z^i(X) : i = 0, 1, 2 >$;
(5) $\{Z, X, Z(X), Z^2(X), Z^3(X)\}$ *is a base of the* $C^\infty(K)$*-module* $\mathcal{D}(K)$*;*
(6) *for some functions* $r_i \in C^\infty(K)$

$$Z^4(U) + r_1 Z^3(U) + r_2 Z^2(U) + r_3 Z(U) + r_4 U = 0 \tag{14}$$

Proof. Assertions (1)-(3) are direct consequences of Lemma 4, Proposition 6 and definitions. Assertion (4) follows from independence of $Z^2(X)$ from $Z, X, Z(X)$. This is so because otherwise Z would be a characteristic of $\mathcal{R}_\pi^\perp = < Z, X, Z(X) >$ in contradiction with the fact that $\mathcal{R}_\pi$ is of rank 3. Similarly, $Z^3(X)$ is independent of $Z, X, Z(X), Z^2(X)$. Indeed, otherwise Z would be a characteristic of $\mathcal{C}_\pi$. This proves (5).

Finally, forms $Z^s(U)$, $s \geq 0$, are annihilated by Z. Since $\dim \mathcal{M} = 5$ this implies that $Z^4(U)$ depends on $Z^s(U)$, $s \leq 3$. But this is equivalent to (14). □

Corollary 5. *If* $0 \leq k, l \leq 3$ *and* $k + l = 3$*, then*

$$Z^k(X) \lrcorner Z^l(U) = (-1)^{k+1} Z^3(X) \lrcorner U \neq 0.$$

Proof. Assertions (5) and (6) of the above Proposition show that $Z^3(X)$ completes a basis of $\mathcal{C}$ to a basis of $D(K)$. So, $Z^3(X) \lrcorner U$ is a nowhere vanishing function.

It follows from the standard formula $[i_X, L_Y] = i_{[X,Y]}$ that

$$\begin{aligned} Z^k(X) \lrcorner Z^l(U) &= \left[i_{Z^k(X)}, L_Z\right](Z^{l-1}(U)) - L_Z(Z^k(X) \lrcorner Z^{l-1}(U)) = \\ &- Z^{k+1}(X) \lrcorner Z^{l-1}(U) - L_Z(Z^k(X) \lrcorner Z^{l-1}(U)) \end{aligned} \tag{15}$$

According to Proposition 9, (1)-(3), $Z^r(X) \lrcorner Z^s(U) = 0$ if $r + s = 2$. So, for $k + l = 3$ relation (15) becomes

$$Z^k(X) \lrcorner Z^l(U) = -Z^{k+1}(X) \lrcorner Z^{l-1}(U)$$

□

By decomposing $Z^4(X)$ with respect to the base in (5) of Proposition 9 we get

$$Z^4(X) + \rho_1 Z^3(X) + \rho_2 Z^2(X) + \rho_3 Z(X) + \rho_4 X + \rho_5 Z = 0 \tag{16}$$

for some functions $\rho_i \in C^\infty(K)$ and hence

$$Z^4(X \wedge Z) + \rho_1 Z^3(X \wedge Z) + \rho_2 Z^2(X \wedge Z) + \rho_3 Z(X \wedge Z) + \rho_4 X \wedge Z = 0. \tag{17}$$

The last is a relation binding the bivector $X \wedge Z$, which generates the distribution $\mathcal{D}_\pi$. This bivector is unique up to a functional factor.

Proposition 10. *Functions r_i's in decomposition (14) are expressed in terms of iterated Lie derivatives $Z^i(U), Z^j(X)$ as follows:*

$$r_1 = -\frac{X \lrcorner Z^4(U)}{X \lrcorner Z^3(U)}, \qquad r_2 = -\frac{Z(X) \lrcorner Z^4(U) + r_1 Z(X) \lrcorner Z^3(U)}{Z(X) \lrcorner Z^2(U)},$$

$$r_3 = -\frac{Z^2(U) \lrcorner Z^4(U) + r_1 Z^2(X) \lrcorner Z^3(U) + r_2 Z^2(X) \lrcorner Z^2(U)}{Z^2(X) \lrcorner Z(U)},$$

$$r_4 = -\frac{Z^3(X) \lrcorner Z^4(U) + r_1 Z^3(X) \lrcorner Z^3(U) + r_2 Z^3(X) \lrcorner Z^2(U) + r_3 Z^3(X) \lrcorner Z(U)}{Z^3(X) \lrcorner U}.$$

Proof. By subsequently inserting fields $Z^s(X)$, $0 \le s \le 3$, in (14) one easily gets the result by taking into account Proposition 9 and Corollary 5. □

Similarly, we have

Proposition 11. *Functions ρ_i in decomposition (17) are expressed in terms of iterated Lie derivatives $Z^i(U), Z^j(X)$ as follows:*

$$\rho_1 = -\frac{Z^4(X) \lrcorner U}{Z^3(X) \lrcorner U}, \qquad \rho_2 = -\frac{Z^4(X) \lrcorner Z(U) + \rho_1 Z^3(X) \lrcorner Z(U)}{Z^2(X) \lrcorner Z(U)},$$

$$\rho_3 = -\frac{Z^4(U) \lrcorner Z^2(U) + \rho_1 Z^3(X) \lrcorner Z^2(U) + \rho_2 Z^2(X) \lrcorner Z^2(U)}{Z(X) \lrcorner Z^2(U)},$$

$$\rho_4 = -\frac{Z^4(X) \lrcorner Z^3(U) + \rho_1 Z^3(X) \lrcorner Z^3(U) + \rho_2 Z^2(X) \lrcorner Z^3(U) + \rho_3 Z(X) \lrcorner Z^3(U)}{X \lrcorner Z^3(U)}.$$

Proof. As before we get the desired result by subsequently inserting the vector field in the left hand side of (16) into 1-forms $Z^s(U)$, $0 \leq s \leq 3$. □

Remark 1. *By introducing functions* $\alpha_{kl} = Z^k(X) \lrcorner Z^l(U)$ *we see that* r_i*'s and* ρ_i*'s are rational functions of* α_{kl}*'s:*

$$r_1 = \frac{\alpha_{04}}{\alpha_{03}}, \qquad r_2 = \frac{\alpha_{03}\alpha_{14} + \alpha_{04}\alpha_{13}}{\alpha_{03}\alpha_{12}}, \qquad etc.$$

Differential invariants we are going to construct are projective differential invariants of curves composing the considered PCB. Clearly, it is not possible to describe explicitly these curves. So, the problem is how to express these invariants in terms of the data at our disposal, i.e., X, Z and U. In what follows this problem is solved on the basis of a rather transparent analogy. For instance, the field Z restricted to one of these curves may be thought as the derivation with respect to a parameter along this curve, and so on. So, with similar interpretations in mind it is sufficient just to mimic a known construction of projective differential invariants for curves in order to obtain the desired result. In doing that we follows classical Wilczynski's book [11]. By stressing the used analogy we pass to Wilczynski's p_i's and q_j's instead of above r_i's and ρ_j's:

$$r_1 = 4p_1, \qquad r_2 = 6p_2, \qquad r_3 = 4p_3, \qquad r_4 = p_4 \tag{18}$$

and

$$\rho_1 = 4q_1, \qquad \rho_2 = 6q_2, \qquad \rho_3 = 4q_3, \qquad \rho_4 = q_4. \tag{19}$$

In terms of these functions relations (14) and (17) read

$$Z^4(U) + 4p_1 Z^3(U) + 6p_2 Z^2(U) + 4p_3 Z(U) + p_4 U = 0, \tag{20}$$

$$Z^4(X \wedge Z) + 4q_1 Z^3(X \wedge Z) + 6q_2 Z^2(X \wedge Z) + 4q_3 Z(X \wedge Z) + q_4 X \wedge Z = 0. \tag{21}$$

These relations are identical to Wilczynski's formulas (see equation (1), page 238 of [11]).

Z and U are unique up to a "gauge" transformation $(Z, U) \longmapsto (\overline{Z}, \overline{U})$

$$Z = f\overline{Z}, \qquad U = h\overline{U} \tag{22}$$

with nowhere vanishing $f, h \in C^\infty(K)$. The corresponding transformation of coefficients $\{p_i\} \longmapsto \{\bar{p}_i\}$ can be easily obtained from (20) by a direct computation:

$$
\begin{aligned}
p_1 \longmapsto \bar{p}_1 &= \frac{Z(h)}{h} + \frac{p_1}{f} + \frac{3Z(f)}{2f}, \\
p_2 \longmapsto \bar{p}_2 &= \frac{p_2}{f^2} + \frac{2p_1\, Z(f)}{f^2} + \frac{2p_1\, Z(h)}{3hf} + 3\,\frac{Z(h)Z(f)}{hf} + \frac{7Z(f)^2}{6f^2} \\
&\quad + \frac{2Z^2(f)}{3f} + \frac{Z^2(h)}{h}, \\
p_3 \longmapsto \bar{p}_3 &= \frac{p_3}{f^3} + \frac{3p_2 Z(h)}{hf^2} + \frac{3p_2 Z(f)}{2f^3} + \frac{p_1\, Z(f)^2}{f^3} + \frac{6p_1 Z(h)Z(f)}{hf^2} \\
&\quad + \frac{3p_1 Z^2(h)}{hf} + \frac{p_1 Z^2(f)}{f^2} + \frac{Z^3(f)}{4f} + \frac{9Z(f)Z^2(h)}{2hf} + \frac{Z(f)^3}{4f^3} \\
&\quad + \frac{7Z(h)Z(f)^2}{2hf^2} + \frac{2Z(h)Z^2(f)}{hf} + \frac{Z^3(h)}{h} + \frac{Z^2(f)Z(f)}{f^2}, \\
p_4 \longmapsto \bar{p}_4 &= \frac{p_4}{f^4} + \frac{4p_3 Z(h)}{hf^3} + \frac{6p_2 Z^2(f)}{hf^2} + \frac{6p_2 Z(h)Z(f)}{hf^3} + \frac{4p_1\, Z^3(h)}{hf} \\
&\quad + \frac{4p_1 Z(h)Z^2(f)}{hf^2} + \frac{4p_1 Z(h)Z(f)^2}{hf^3} + \frac{12p_1 Z(f)Z^2(h)}{hf^2} \\
&\quad + \frac{6Z(f)Z^3(h)}{hf} + \frac{Z^4(h)}{h} + \frac{7Z(f)^2Z^2(h)}{hf^2} + \frac{Z(h)Z^3(f)}{hf} \\
&\quad + \frac{4Z(h)Z^2(f)Z(f)}{hf^2} + \frac{Z(f)^3Z(h)}{hf^3} + \frac{4Z^2(h)Z^2(f)}{hf}.
\end{aligned}
\tag{23}
$$

Now the problem is to combine p_i's in a way to obtain expressions which are invariant with respect to transformations (23). To this end we first normalize (Z, U) by the condition $p_1 = 0$. This can be easily done with $f = 1$ and a solution h of the equation

$$Z(h) + p_1 h = 0.$$

After this normalization, equation (20) takes a simpler form

$$Z^4(\overline{U}) + 6P_2 Z^2(\overline{U}) + 4P_3 Z(\overline{U}) + P_4\overline{U} = 0 \tag{24}$$

with

$$
\begin{aligned}
P_2 &:= p_2 - Z(p_1) - Z(p_1)^2, \\
P_3 &:= p_3 - Z(Z(p_1)) - 3p_1p_2 + 2p_1^3, \\
P_4 &:= p_4 - 4p_1p_3 - 3p_1^4 - Z(Z(Z(p_1))) + 3Z(p_1)^2 \\
&\quad + 6p_1^2 Z(p_1) + 6p_1^2 p_2 - 6Z(p_1)p_2.
\end{aligned}
\tag{25}
$$

Proposition 12. *Transformations (22) preserving the normalization $p_1 = 0$ are subject to the condition*

$$Z(h) + \frac{3}{2} Z(\ln(f))h = 0. \tag{26}$$

Proof. A direct computation. □

Now the problem reduces to finding invariant combinations of P_2, P_3 and P_4 with respect to normalized, i.e., respecting condition (26), transformations (22). This can be done, for instance, by mimicking the construction of projective curvature and torsion in [11]. Namely, introduce first the functions

$$\begin{aligned}\Theta_3 &= P_3 - \tfrac{3}{2}Z(P_2),\\ \Theta_4 &= P_4 - 2Z(P_3) + \tfrac{6}{5}Z(Z(P_2)) - \tfrac{81}{25}P_2^2,\\ \Theta_{3\cdot 1} &= 6\Theta_3 Z(Z(\Theta_3)) - 7Z(\Theta_3)^2 - \tfrac{108}{5}P_2\Theta_3^2\end{aligned} \tag{27}$$

They are *semi-invariant* with respect to normalized transformations (22), i.e., they are transformed according to formulas

$$\overline{\Theta}_3 = \frac{\Theta_3}{f^3}, \qquad \overline{\Theta}_4 = \frac{\Theta_4}{f^4}, \qquad \overline{\Theta}_{3\cdot 1} = \frac{\Theta_{3\cdot 1}}{f^8}. \tag{28}$$

Obviously, the following combinations of the Θ_i's

$$\kappa_1 = \frac{\Theta_4}{\Theta_3^{4/3}}, \qquad \kappa_2 = \frac{\Theta_{3\cdot 1}}{\Theta_3^{8/3}} \tag{29}$$

are invariant with respect to normalized transformations (22).

Thus we have

Proposition 13. *κ_1 and κ_2 are scalar differential invariants of parabolic Monge-Ampere equations (10) with respect to contact transformations.*

Explicit expressions of κ_1 and κ_2 in terms of coefficients of PMA (10) can be straightforwardly obtained from those of Z and U. However, they are not very instructive and too cumbersome to be reported here.

Another invariant, which can be readily extracted from (28), is the *invariant vector field*

$$N_1 = \Theta_3^{-1/3} Z. \tag{30}$$

Another set of scalar differential invariants can be constructed in a similar manner by starting from equation (17). Indeed, the vector field Z and the bivector field $X \wedge Z$ generating distributions $\mathcal{R}$ and $\mathcal{D}$, respectively, are unique up to transformations

$$Z = f\overline{Z}, \qquad X \wedge Z = g\overline{X} \wedge \overline{Z} \tag{31}$$

with nowhere vanishing $f, g \in C^\infty(K)$.

It is easy to check that coefficients q_i's are transformed according to formulas (23) and one can repeat what was already done previously in the case of equation (14). In particular, equation (17) can be normalized as

$$Z^4(\overline{X} \wedge \overline{Z}) + 6Q_2 Z^2(\overline{X} \wedge \overline{Z}) + 4Q_3 Z(\overline{X} \wedge \overline{Z}) + Q_4 \overline{X} \wedge \overline{Z} = 0 \tag{32}$$

with

$$
\begin{aligned}
Q_2 &= q_2 - Z(q_1) - Z(q_1)^2, \\
Q_3 &= q_3 - Z(Z(q_1)) - 3q_1q_2 + 2q_1^3, \\
Q_4 &= q_4 - 4q_1q_3 - 3q_1^4 - Z(Z(Z(q_1))) + 3Z(q_1)^2 \\
&\quad +6q_1^2Z(q_1) + 6q_1^2q_2 - 6Z(q_1)q_2.
\end{aligned} \tag{33}
$$

This way we obtain the following scalar differential invariants of PMAs

$$
\tau_1 := \frac{\Lambda_4}{\Lambda_3^{4/3}}, \qquad \tau_2 := \frac{\Lambda_{3\cdot 1}}{\Lambda_3^{8/3}} \tag{34}
$$

with *semi-invariants* Λ_i's defined by

$$
\begin{aligned}
\Lambda_3 &= Q_3 - \tfrac{3}{2}Z(Q_2), \\
\Lambda_4 &= Q_4 - 2Z(Q_3) + \tfrac{6}{5}Z(Z(Q_2)) - \tfrac{81}{25}Q_2^2, \\
\Lambda_{3\cdot 1} &= 6\Lambda_3 Z(Z(\Lambda_3)) - 7Z(\Lambda_3)^2 - \tfrac{108}{5}Q_2\Lambda_3^2.
\end{aligned} \tag{35}
$$

Similarly,

$$
N_2 = \Lambda_3^{-1/3} Z. \tag{36}
$$

is an invariant vector field.

Remark 2. *The observed parallelism in construction of two sets of differential invariants is explained by the fact that in both cases we compute the same invariants in two different PCB, namely, the first and the second ones, associated with the considered PMA.*

Since vector fields N_1 and N_2 are invariant and $N_1 = \lambda N_2$ the factor $\lambda = \Theta_3^{-1/3}\Lambda_3^{1/3}$ is a scalar differential invariant. So,

$$
\gamma_3 := \lambda^3 = \frac{\Lambda_3}{\Theta_3}
$$

is a scalar differential invariant of a new "mixed" kind as well as ratios

$$
\gamma_4 := \frac{\Lambda_4}{\Theta_4}, \qquad \lambda_{3\cdot 1} := \frac{\Lambda_{3\cdot 1}}{\Theta_{3\cdot 1}}
$$

By applying to already constructed scalar differential invariants

$$
\kappa_1, \quad \kappa_2, \quad \tau_1, \quad \tau_2, \quad \gamma_3, \quad \gamma_4, \quad \gamma_{3\cdot 1}
$$

various algebraic operations and arbitrary compositions of invariant vector fields N_1 and N_2 one can construct many other scalar differential invariants of PMAs. These does not exhaust all invariants. Nevertheless, various quintuples of (functionally) independent invariants can be composed from them, and this is the only one need in order to apply the "principle of n invariants". For instance, we have

Theorem 5. *Quintuples* $(\kappa_1, \kappa_2, \tau_1, \tau_2, \gamma_3)$ *and* $(\gamma_3, N_1(\gamma_3), N_1^2(\gamma_3), \kappa_1, \kappa_2)$ *are composed of independent invariants.*

Proof. It is sufficient to exhibit an example for which invariants composing each of these two quintuples are independent. For instance, for both quintuples such is the equation determined by the directing distribution $\mathcal{R}$ generated by

$$Z = q\partial_x + y\partial_y + (qp + yq)\partial_z + x\partial_p - xz\partial_q.$$

In this case the corresponding Lagrangian distribution $\mathcal{D}$ is generated by

$$X_1 = a_1(\partial_x + p\partial_u) + a_2\partial_p + a_3\partial_q, \qquad X_2 = a_1(\partial_y + q\partial_u) + a_3\partial_p + a_4\partial_q$$

with

$$a_1 = xyu - (x - y)q, \qquad a_2 = -(x - y)x - (u + xp)y^2,$$
$$a_3 = x^2u + (u + xp)yq, \qquad a_4 = -(q + xu)xu - (u + xp)q^2,$$

and the corresponding PMA is

$$a_1^2(rt - s^2) + a_1a_4r + 2a_1a_3s - a_1a_2t + xa_1(xyup - xu - xpq + yu^2 - uq) = 0. \tag{37}$$

Explicit expressions of invariants $(\kappa_1, \kappa_2, \tau_1, \tau_2, \gamma_3)$ and $(\gamma_3, N_1(\gamma_3), N_1^2(\gamma_3), \kappa_1, \kappa_2)$ for equation (37) are too cumbersome to be reported here. A direct check shows that they are functionally independent in each of above two quintuples. □

Thus, according to the "principle of n-invariants" (see [1, 10]), the proven existence of five independent scalar differential invariants solves in principle the equivalence problem for generic PMA equations. It should be stressed, however, that a practical implementation of this result could meet some boring computational problems.

8. Concluding remarks

Representation of a PMA equation $\mathcal{E}$ by means of the associated PCB makes clearly visible the nature of its nonlinearities. For example, if all curves of this bundle are projectively nonequivalent each other, then $\mathcal{E}$ does not admit contact symmetries, etc. The instance of this can be detected by means of invariants constructed in the previous section. On the contrary, it may happen that all curves composing a PCB are projectively equivalent, i.e., nonlinearities of the corresponding PMA $\mathcal{E}$ are "homogeneous". The above constructed invariants are not sufficient to distinguish one homogeneous in this sense PMA from another, and a need of new finer invariants arises. It is remarkable that in similar situations PCBs themselves give an idea of how such invariants can be constructed. For instance, in the above homogeneous case one can observe that the bundle $PT^*N \to N$ is naturally supplied with a full parallelism structure which immediately furnishes the required new invariants. It is not difficult to imagine various intermediate situations, which demonstrate the diversity and complexity of the world of parabolic Monge-Ampere equations. In particular, the problem of describing all strata of the characteristic diffiety (see [10]) for parabolic Monge-Ampere equations is a task of a rather large scale. Further results in this direction will appear in a series of forthcoming publications.

References

[1] D. V. Alekseevsky, V. V.Lychagin, A. M. Vinogradov, Basic ideas and concepts of differential geometry, Encyclopedia of Math. Sciences, **28** (Springer-Verlag, Berlin, 1991).

[2] Su Buchin, The General Projective Theory of Curves (Science Press, Peking, 1973).

[3] I. S. Krasil'shchik, V. V. Lychagin and A. M. Vinogradov, Geometry of Jet spaces and Nonlinear Partial Differential Equations, Gordon and Breach, New York, 1986

[4] Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, Ed. by I. S. Krasil'shchik and A. M. Vinogradov, "Translation of Mathematical Monographs Series "**182** (Amer.Math.Soc., Providence, RI, 1999).

[5] A. Kushner, V. Lychagin, V. Rubtsov, Contact Geometry and Nonlinear Differential Equations (Cambridge Univ. Press, Cambridge, 2007).

[6] T. Morimoto, in *Symplectic Singularities and Geometry of Gauge Fields*, Banach Center Publ. **39**, 105-121 (1997).

[7] J. Nestruev, Smooth Manifolds and Observables (Springer-Verlag, Berlin - New-York, 2002).

[8] A. M. Vinogradov, *On geometry of Second Order Parabolic Differential Equations in Two Indipendent Variables*, Dokl.Math. **423**(5), (2008).

[9] A. M. Vinogradov, *Geometric Singularities of Solutions of Nonlinear Partial Differential Equations*, in Proceedings of Conference "Differential Geometry and Applications", Brno, 1986.

[10] A. M. Vinogradov, in *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, Ed. by M. Francaviglia (North Holland, North Holland, 1991).

[11] E. J. Wilczinsky, Projective Differential Geometry of Curves and Ruled Surfaces (Chealsea, 1961).

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