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Secondary Calculus and the Covariant Phase Space

L. VITAGLIANO

ABSTRACT. The covariant phase space of a lagrangian field theory is the solution space of the associated Euler-Lagrange equations. It is, in principle, a nice environment for covariant quantization of a lagrangian field theory. Indeed, it is manifestly covariant and possesses a canonical (functional) "presymplectic structure" $\boldsymbol{\omega}$ (as first noticed by Zuckerman in 1986) whose degeneracy (functional) distribution is naturally interpreted as the Lie algebra of gauge transformations. We propose a fully rigorous approach to the covariant phase space in the framework of jet spaces and (A. M. Vinogradov's) secondary calculus. In particular, we describe the degeneracy distribution of $\boldsymbol{\omega}$. As a byproduct we rederive the existence of a Lie bracket among gauge invariant functions on the covariant phase space.

INTRODUCTION

Covariant phase space (CPS) is the solution space of a system of Euler–Lagrange partial differential equations[1] (PDEs). It has been first noticed by Zuckerman in the 1986 [40] (see also [11, 12]) that there is a canonical, closed 2–form $\boldsymbol{\omega}$ on such a functional space generalizes the symplectic form on the phase space of a regular Lagrangian system in mechanics. Moreover, the degeneracy distribution of $\boldsymbol{\omega}$ is naturally interpreted as Lie algebra of gauge transformations [25]. Therefore, the CPS is, in principle, a nice environment to perform a covariant (canonical) quantization of a lagrangian theory. Namely, gauge invariant functions on the CPS possess a well defined Lie bracket induced by $\boldsymbol{\omega}$, which has been proved in [8] to coincide with the so–called Peierls bracket [29].

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[1] Notice that sometimes the name covariant phase space is referred to the quotient of the above mentioned solution space with respect to gauge transformations.

List of Main Symbols

$J^\infty\pi$	space of ∞–jets of local sections of the bundle $\pi : E \longrightarrow M$
$\mathrm{diff}(\pi,\tau)$	module of differential operators from π to τ
$\mathscr{E}_\Phi$	PDE determined by the differential operator Φ
$\mathscr{E}$	∞ prolongation of a PDE
$\mathscr{C}$	Cartan distribution
$\mathscr{C}\mathrm{D}(\mathscr{E})$	module of horizontal vector fields on $\mathscr{E}$
$\mathscr{C}\Lambda(\mathscr{E})$	Cartan ideal of $\mathscr{E}$
$\mathscr{C}^p\Lambda(\mathscr{E})$	pth exterior power of $\mathscr{C}\Lambda(\mathscr{E})$
$\mathscr{C}^\bullet\Lambda(\mathscr{E})$	algebra generated by $\mathscr{C}\Lambda^1(\mathscr{E})$
$\mathscr{C}E(\mathscr{E})$	$\mathscr{C}$–spectral sequence of $\mathscr{E}$
$\overline{\Lambda}(\mathscr{E})$	algebra of horizontal forms on $\mathscr{E}$
$\overline{d}$	horizontal de Rham differential
$\overline{H}(\mathscr{E})$	horizontal de Rham cohomology of $\mathscr{E}$
d^V	vertical de Rham differential
$V\mathrm{D}(\mathscr{E})$	module of vertical vector fields on $\mathscr{E}$
$\mathrm{D}_{\mathscr{C}}(\mathscr{E})$	Lie algebra of symmetries of $(\mathscr{E},\mathscr{C})$
$\mathrm{Sym}(\mathscr{E})$	Lie algebra of non–trivial symmetries of $(\mathscr{E},\mathscr{C})$
$V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$	Lie algebra of vertical symmetries of $(\mathscr{E},\mathscr{C})$
$\varkappa$	module of generating sections of higher symmetries of π
ℓ_Φ	universal linearization of the differential operator Φ
$\boldsymbol{C}^\infty(\boldsymbol{M})^\bullet$	space of secondary functions on the secondary manifold $\boldsymbol{M}$
$\mathbf{D}(\boldsymbol{M})^\bullet$	space of secondary vector fields on $\boldsymbol{M}$
$\boldsymbol{\Lambda}(\boldsymbol{M})^\bullet$	space of secondary differential forms on $\boldsymbol{M}$
$\boldsymbol{d}$	secondary de Rham differential
$\overline{S}$	horizontal Spencer differential
$\mathscr{C}\mathrm{Diff}(P,Q)$	module of horizontal differential operators $P \longrightarrow Q$
$\overline{J}^\infty P$	module of ∞ horizontal jets of elements of P
$\bar{j}_\infty$	∞ horizontal jet prolongation $P \longrightarrow \overline{J}^\infty P$
$h^\infty_\Box$	homomorphism $\overline{J}^\infty P \longrightarrow \overline{J}^\infty Q$ associated to $\Box \in \mathscr{C}\mathrm{Diff}(P,Q)$
η_Φ	natural monomorphism $V\mathrm{D}(\mathscr{E}) \longrightarrow \overline{J}^\infty \varkappa\|_{\mathscr{E}}$
η^*_Φ	natural epimorphism $\mathscr{C}\mathrm{Diff}(\varkappa\|_{\mathscr{E}}, \overline{\Lambda}(\mathscr{E})) \longrightarrow \mathscr{C}\Lambda^1(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E})$
$\int$	natural projection $\overline{\Lambda}^n(\mathscr{E}) \longrightarrow \overline{H}^n(\mathscr{E})$
$\boldsymbol{E}(\mathscr{L})$	left hand side of the Euler–Lagrange equations
$\boldsymbol{P}$	covariant phase space
$\boldsymbol{\omega}$	canonical, closed, secondary 2–form on $\boldsymbol{P}$
Δ_1	compatibility operator for $\ell_{\boldsymbol{E}(\mathscr{L})}$
$\boldsymbol{\Omega}$	linear map $\mathbf{D}(\boldsymbol{M})^\bullet \longrightarrow \boldsymbol{\Lambda}^1(\boldsymbol{M})^\bullet$ associated to $\boldsymbol{\omega}$

In turn, Peierls bracket is at the basis of the global approach to quantum field theory [13].

Despite its conceptual relevance, the CPS is, in general, a complicated functional space, which is difficult to handle with analytic methods. Indeed, most of the literature about it (see [30] and references therein) comes from the physicists community and it is rarely completely rigorous from a mathematical point of view. For instance, it seems to be very hard to rigorously perform, in full generality, a symplectic reduction of the CPS to get rid of gauge (non–physical) degrees of freedom.

On the other hand, A. M. Vinogradov developed a whole theory, the so–called *secondary calculus* (see [37] and references therein, and [38] for a short introduction), which properly formalizes in cohomological terms the idea of a (local) functional differential calculus on the space of solutions of a generic system of PDEs (for this reason, roughly speaking, the word "secondary" in this paper could be considered as a synonym of "functional"). Thus, secondary calculus appears to be a suitable setting to rigorously investigate the CPS and its properties. The aim of the paper is to describe rigorously the CPS, its canonical 2–form and some their properties within secondary calculus. As a byproduct it will become transparent the analogy between the CPS and the phase space of constrained mechanical systems.

The paper is divided into two parts. In order to make it as self–consistent as possible we review, in the first part, those aspects of secondary calculus that are needed for a suitable formalization of the CPS. In Sections 1.1, 1.2 and 1.3 we briefly describe the geometry and the main properties of jet spaces and differential equations, and relevant structures on them. In Section 1.4 we define secondary vector fields and differential forms, and summarize the main formulas of first order secondary calculus. In Sections 1.5 and 1.6 we review the main technical aspects of secondary calculus and how to handle the relevant cohomologies.

The second part of the paper is devoted to the CPS and to original results on the subject. In Section 2.1 we introduce the CPS for a general lagrangian field theory (any number of variable and any order) and rederive the existence of a canonical 2–form $\boldsymbol{\omega}$ on it completing the proof by Zuckerman [40]. In Section 2.2 we propose a "symplectic version" of the first Noether theorem, which makes it evident the analogy with hamiltonian mechanics. In Section 2.3 we describe the degeneracy distribution of $\boldsymbol{\omega}$ and propose, and motivate, a new (and very natural) definition of (infinitesimal) gauge symmetries in field theory. In Section 2.4 we describe gauge invariant secondary functions on the CPS and show that they are endowed with a canonical Lie bracket (such bracket formalizes rigorously the Peierls bracket [29]). In Section 2.5 we outline a possible path through a "secondary symplectic reduction" of the CPS. Applications to concrete lagrangian theories will be presented somewhere else.

Most of the (almost) trivial computations will be performed in some details to emphasize similarities between secondary calculus and standard calculus on manifolds.

Notations and Conventions. In this section we collect notations and conventions about some general constructions in differential geometry that will be used in the following.

Let N be a smooth manifold. We denote by $C^\infty(N)$ the $\mathbb{R}$–algebra of smooth, $\mathbb{R}$–valued functions on N. We will always understand a vector field X on N as a derivation $X : C^\infty(N) \longrightarrow C^\infty(N)$. The value of X at the point $x \in M$ will be denoted by X_x. We denote by $\mathrm{D}(N)$ the $C^\infty(N)$–module of vector fields over N, by $\Lambda(M) = \bigoplus_k \Lambda^k(N)$ the graded $\mathbb{R}$–algebra of differential forms over N and by $d : \Lambda(N) \longrightarrow \Lambda(N)$ the de Rham differential. If $F : N_1 \longrightarrow N$ is a smooth map of manifolds, we denote by $F^* : \Lambda(N) \longrightarrow \Lambda(N_1)$ its pull–back.

Let $\alpha : W \longrightarrow N$ be a vector bundle and $F : N_1 \longrightarrow N$ a smooth map of manifolds. The $C^\infty(N)$–module of smooth sections of α will be denoted by $\Gamma(\alpha)$. For $s \in \Gamma(\alpha)$ and $x \in N$ we put, sometimes, $s_x := s(x)$. The zero section of α will be denoted by $o : N \ni x \longmapsto o_x := 0 \in \alpha^{-1}(x) \subset W$. The vector bundle on N_1 induced by α via F will be denote by $F^\circ(\alpha) : F^\circ(W) \longrightarrow N$:

$$\begin{tikzcd}
F^\circ(W) \arrow[r] \arrow[d, "F^\circ(\alpha)"'] & W \arrow[d, "\alpha"] \\
N_1 \arrow[r, "F"] & N
\end{tikzcd} .$$

For any section $s \in \Gamma(\alpha)$ there exists a unique section, which we denote by $F^\circ(s) \in \Gamma(F^\circ(\alpha))$, such that the diagram

$$\begin{tikzcd}
F^\circ(W) \arrow[r] & W \\
N_1 \arrow[r, "F"] \arrow[u, "F^\circ(s)"] & N \arrow[u, "s"']
\end{tikzcd}$$

commutes. If $i_L : L \hookrightarrow N$ is the embedding of a submanifold then we put $\alpha|_L := i_L^\circ(\alpha)$, $\Gamma(\alpha)|_L := \Gamma(\alpha|_L)$ and for $s \in \Gamma(\alpha)$, $s|_L := i_L^\circ(s)$. $s|_L$ will be referred to as *the restriction to* L of s.

Let $F : N_1 \longrightarrow N$ be as above. A vector field along F is an $\mathbb{R}$–linear map $X : C^\infty(N) \longrightarrow C^\infty(N_1)$ such that the following Leibnitz rule holds: $X(fg) = F^*(f)X(g) + F^*(g)X(f)$, $f, g \in C^\infty(N)$. Vector fields along F identify with sections of the induced bundle $F^\circ(\tau_N) : F^\circ(TN) \longrightarrow N_1$, $\tau_N : TN \longrightarrow N$ being the tangent bundle to N.

Let $\zeta : A \longrightarrow N$ be a fiber bundle. We denote by $\nu\zeta : V\zeta \longrightarrow A$ the vertical (with respect to ζ) tangent bundle to A and by $V_a\zeta := (\nu\zeta)^{-1}(a)$ its fiber over $a \in A$. Notice that $V\zeta \subset TA$, the tangent manifold to A. If $\zeta_1 : A_1 \longrightarrow N_1$ is another fiber bundle, $F : A_1 \longrightarrow A$ a morphism of fiber bundles and $TF : TA_1 \longrightarrow TA$ the associated tangent map, then $(TF)(V\zeta_1) \subset V\zeta$ and, therefore, it is well defined the restriction

$VF : V\zeta_1 \longrightarrow V\zeta$ of TF to $V\zeta_1$ and $V\zeta$, and the diagram

$$\begin{tikzcd} V\zeta_1 \arrow[r, "VF"] \arrow[d, "\nu\zeta_1"'] & V\zeta \arrow[d, "\nu\zeta"] \\ A_1 \arrow[r, "F"] & A \end{tikzcd}$$

commutes.

Let

$$\cdots \longrightarrow K_{l-1} \xrightarrow{\delta_{l-1}} K_l \xrightarrow{\delta_l} K_{l+1} \xrightarrow{\delta_{l+1}} \cdots$$

be a complex. Put $K := \bigoplus_l K_l$ and $\delta := \bigoplus_l \delta_l$. We denote by $H(K,\delta) := \bigoplus_l H^l(K,\delta)$, the cohomology space of (K,δ), $H^l(K,\delta) := \ker \delta_l / \operatorname{im} \delta_{l-1}$. If $\omega \in \ker \delta$, then we denote by $[\omega]$ its cohomology class.

Denote by $\mathbb{N}$ the set of natural numbers and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will always understand the sum over repeated upper-lower (multi-)indexes. Our notations about multi-indexes are the following. Let $n \in \mathbb{N}$, $\mathbb{I}_n = \{1,\ldots,n\}$ and $\mathbb{M}_n$ be the free abelian monoid generated by $\mathbb{I}_n$. Even if $\mathbb{M}_n$ is abelian we keep for it the multiplicative notation. Thus if $I = i_1 \cdots i_l$, $J = j_1 \cdots j_m \in \mathbb{M}_n$ are (equivalence classes of) words, $i_1,\ldots,i_l,j_1,\ldots,j_m \in \mathbb{I}_n$, we denote by $IJ = i_1 \cdots i_l j_1 \cdots j_m$ their composition. If $I = i_1 \cdots i_l \in \mathbb{M}_n$ is a word, $i_1,\ldots,i_l \in \mathbb{I}_n$, denote by $|I| := l$ its length. We denote by O the (equivalence class of the) empty word. An element $I \in \mathbb{M}_n$ is called an *n-multi-index* (or, simply, a multi–index) and $|I|$ the length of the multi-index. For $k \leq \infty$ let $\mathbb{M}_{n,k} \subset \mathbb{M}_n$ be the subset made of multi-indexes of length $\leq k$. If $(x^1,\ldots,x^n)$ are local coordinates on a manifold N, $n = \dim N$, and $I = i_1 \cdots i_k \in \mathbb{M}_n$, we put $\frac{\partial^{|I|}}{\partial x^I} := \frac{\partial^k}{\partial x^{i_1} \cdots \partial x^{i_k}}$. We stress that this notation is different from more popular ones (see, for instance, [3]).

1. Secondary Calculus

1.1. **Jet Spaces and PDEs.** Let $\pi : E \longrightarrow M$ be a fiber bundle, $\dim M = n$, $\dim E = m + n$. For $l \leq k \leq \infty$, we denote by $\pi_k : J^k\pi \longrightarrow M$ the bundle of k-jets of local sections of π, and by $\pi_{k,l} : J^k\pi \longrightarrow J^l\pi$ the canonical projection. For any local section $\boldsymbol{p} : U \longrightarrow E$ of π, $U \subset M$ being an open subset, we denote by $j_k\boldsymbol{p} : U \longrightarrow J^k\pi$ its kth jet prolongation and by $\Gamma^k_{\boldsymbol{p}} := \operatorname{im} j_k\boldsymbol{p}$ its graph. For $x \in U$, put $[\boldsymbol{p}]^k_x := (j_k\boldsymbol{p})(x)$. Any system of adapted to π coordinates $(\ldots,x^i,\ldots,u^\alpha,\ldots)$ on an open subset U of E gives rise to a system of jet coordinates $(\ldots,x^i,\ldots,u^\alpha_I,\ldots)$ on $\pi^{-1}_{k,0}(U) \subset J^k\pi$, $i = 1,\ldots,n$, $\alpha = 1,\ldots,m$, $I \in \mathbb{M}_{n,k}$, where we put $u^\alpha_{\mathsf{O}} := u^\alpha$, $\alpha = 1,\ldots,m$. If a local section $\boldsymbol{p}$ of π is locally given by

$$u^\alpha = p^\alpha(\ldots,x^i,\ldots), \quad \alpha = 1,\ldots,m, \tag{1}$$

then $j_k\boldsymbol{p}$ is locally given by

$$u^\alpha_I = (\tfrac{\partial^{|I|}}{\partial x^I} p^\alpha)(\ldots,x^i,\ldots), \quad \alpha = 1,\ldots,m, \quad I \in \mathbb{M}_{n,k}.$$

Recall that $J^\infty\pi$ is, by definition, an inverse limit of the tower of projections

$$\begin{tikzcd} M & E \arrow[l, "\pi"'] & \cdots \arrow[l] & J^k\pi \arrow[l, "\pi_{k,k-1}"'] & J^{k+1}\pi \arrow[l, "\pi_{k+1,k}"'] & \cdots . \arrow[l] \end{tikzcd} \tag{2}$$

Now, let $k < \infty$, $\tau_0 : T_0 \longrightarrow J^k\pi$ be a vector bundle, $\dim T_0 = \dim J^k\pi + p$, and $(\ldots, x^i, \ldots, u^\alpha_I, \ldots, v^a, \ldots)$ adapted to τ_0, local coordinates on T_0. A (possibly non-linear) *differential operator of order* $\leq k$ 'acting on local sections of π, with values in τ_0' (in short 'from π to τ_0') is a section $\Phi : J^k\pi \longrightarrow T_0$ of τ_0. For any local section $\boldsymbol{p} : U \longrightarrow E$ of π, Φ determines an 'image' section $\Delta_\Phi \boldsymbol{p} := \Phi \circ j_k\boldsymbol{p} : U \longrightarrow T_0$ of the bundle $\underline{\tau}_0 := \pi_k \circ \tau_0 : T_0 \longrightarrow M$. If Φ is locally given by

$$v^a = \Phi^a(\ldots, x^i, \ldots, u^\beta_I, \ldots), \quad a = 1, \ldots, p, \tag{3}$$

and $\boldsymbol{p}$ is locally given by (1), then $\Delta_\Phi\boldsymbol{p}$ is locally given by

$$\left\{ \begin{array}{l} u^\alpha_I = (\frac{\partial^{|I|}}{\partial x^I} p^\alpha)(\ldots, x^i, \ldots) \\ v^a = \Phi^a(\ldots, x^i, \ldots, (\frac{\partial^{|J|}}{\partial x^J} p^\beta)(\ldots, x^j, \ldots), \ldots) \end{array} \right. ,$$

$\alpha = 1, \ldots, m,\ I \in \mathbb{M}_{n,k},\ a = 1, \ldots, p$. This motivates the name 'differential operator' for Φ. Denote by $\mathrm{diff}_k(\pi, \tau_0)$ the set of all differential operators of order $\leq k$ from π to τ_0.

For $\Phi \in \mathrm{diff}_k(\pi, \tau_0)$ and $l \leq \infty$ we define the lth prolongation of Φ as follows. Consider the space $J^l\underline{\tau}_0$ of l-jets of local sections of $\underline{\tau}_0$, and local jet coordinates $(\ldots, x^i, \ldots, u^\alpha_{I,J}, \ldots, v^a_J, \ldots)$ on $J^l\underline{\tau}_0$, $J \in \mathbb{M}_{n,l}$. In $J^l\underline{\tau}_0$ consider the submanifold $T_0^{(l)}$ made of jets of local sections of the form $\Delta_\Psi\boldsymbol{p}$, where $\Psi \in \mathrm{diff}_k(\pi, \tau_0)$ and $\boldsymbol{p}$ is a local section of π. $T_0^{(l)}$ is locally defined by

$$u^\alpha_{I,J} = u^\alpha_{IJ}, \quad \alpha = 1, \ldots, m,\ I \in \mathbb{M}_{n,k},\ J \in \mathbb{M}_{n,l}.$$

Thus $(\ldots, x^i, \ldots, u^\alpha_I, \ldots, v^a_J, \ldots)$, $I \in \mathbb{M}_{n,k+l}$, $J \in \mathbb{M}_{n,l}$, are local coordinates on $T_0^{(l)}$. $T_0^{(l)}$ projects canonically onto $J^{k+l}\pi$ and the projection $\tau_0^{(l)} : T_0^{(l)} \longrightarrow J^{k+l}\pi$ is a vector bundle. Moreover, coordinates $(\ldots, x^i, \ldots, u^\alpha_I, \ldots, v^a_J, \ldots)$ on $T_0^{(l)}$ are adapted to $\tau_0^{(l)}$. Finally, define the lth prolongation $\Phi^{(l)} : J^{k+l}\pi \longrightarrow T_0^{(l)}$ of Φ by putting $\Phi^{(l)}([\boldsymbol{p}]^{k+l}_x) := [\Delta_\Phi\boldsymbol{p}]^l_x \in T_0^{(l)}$, for all local sections $\boldsymbol{p}$ of π and $x \in M$. Then $\Phi^{(l)} \in \mathrm{diff}_{k+l}(\pi, \tau_0^{(l)})$.

For $\Phi \in \mathrm{diff}_k(\pi, \tau_0)$ put $\mathscr{E}_\Phi := \{\theta \in J^k\pi \mid \Phi(\theta) = 0\}$. $\mathscr{E}_\Phi$ is called the *(system of) PDE(s)* determined by Φ. For $l \leq \infty$ put also $\mathscr{E}_\Phi^{(l)} := \mathscr{E}_{\Phi^{(l)}}$. $\mathscr{E}_\Phi^{(l)}$ is locally determined by equations

$$(D_J\Phi^a)(\ldots, x^i, \ldots, u^\alpha_I, \ldots) = 0, \quad a = 1, \ldots, p,\ J \in \mathbb{M}_{n,l}, \tag{4}$$

where $D_{j_1\cdots j_l} := D_{j_1} \circ \cdots \circ D_{j_l}$ and $D_j := \partial/\partial x^j + u^\alpha_{Ij}\partial/\partial u^\alpha_I$ is the *jth total derivative,* $j, j_1, \ldots, j_l = 1, \ldots, m$. In the following we put $\partial^I_\alpha := \partial/\partial u^\alpha_I$ and $\partial_\alpha := \partial/\partial u^\alpha$, $\alpha = 1, \ldots, m$, $I \in \mathbb{M}_n$.

A local section $\boldsymbol{p}$ of π is a *(local) solution of* $\mathscr{E}_\Phi$ iff, by definition, $\Gamma^k_{\boldsymbol{p}} \subset \mathscr{E}_\Phi$ or, which is the same, $\Gamma^{k+l}_{\boldsymbol{p}} \subset \mathscr{E}_\Phi^{(l)}$ for some $l \leq \infty$. Notice that $\mathscr{E}_\Phi^{(\infty)} \subset J^\infty\pi$ is an inverse limit of the tower of maps

$$\begin{tikzcd} M & \mathscr{E}_\Phi \arrow[l, "\pi_k"'] & \cdots \arrow[l] & \mathscr{E}_\Phi^{(l)} \arrow[l, "\pi_{k+l,k+l-1}"'] & \mathscr{E}_\Phi^{(l+1)} \arrow[l, "\pi_{k+l+1,k+l}"'] & \cdots \arrow[l] \end{tikzcd} \tag{5}$$

and consists of "formal solutions" of $\mathscr{E}_\Phi$, i.e., possibly non-converging Taylor series fulfilling (4) for every l. The PDE $\mathscr{E}_\Phi$ is called *formally integrable* iff $\mathscr{E}_\Phi^{(l)} \subset J^{k+l}\pi$ is a (closed) submanifold for any $l < \infty$ and (5) is a sequence of fiber bundles. Let us stress that, basically, all relevant PDEs in Mathematical Physics are formally integrable and, therefore, in the following, we will only consider differential operators determining formally integrable PDEs.

$J^\infty\pi$ and $\mathscr{E}_\Phi^{(\infty)}$ are not finite dimensional smooth manifolds, in general. However, they are *pro-finite dimensional smooth manifolds*. We do not give here a complete definition of a pro-finite dimensional smooth manifold, which would take too much space. Rather, we will just outline it. Basically, a pro-finite dimensional smooth manifold is a(n equivalence class of) set(s) $\mathscr{O}$ together with a sequence of smooth fiber bundles

$$\begin{tikzcd} \mathscr{O}_0 & \mathscr{O}_1 \arrow[l, "\mu_{1,0}"'] & \cdots \arrow[l] & \mathscr{O}_k \arrow[l, "\mu_{k,k-1}"'] & \mathscr{O}_{k+1} \arrow[l, "\mu_{k+1,k}"'] & \cdots \arrow[l] \end{tikzcd} \tag{6}$$

and maps $\mu_{\infty,k} : \mathscr{O} \longrightarrow \mathscr{O}_k$, $0 \leq k < \infty$, such that $\mathscr{O}$ (together with the $\mu_{\infty,k}$'s) is an inverse limit of (6). It is associated to the sequence (6) a filtration of algebras

$$\begin{tikzcd} C^\infty(\mathscr{O}_0) \arrow[r, "\mu^*_{1,0}"] & \cdots \arrow[r] & C^\infty(\mathscr{O}_{k-1}) \arrow[r, "\mu^*_{k,k-1}"] & C^\infty(\mathscr{O}_k) \arrow[r, "\mu^*_{k+1,k}"] & \cdots . \end{tikzcd} \tag{7}$$

We understand the monomorphisms $\mu^*_{l+1,l}$'s and interpret (7) as a sequence of subalgebras. Similarly, we understand the $\mu_{\infty,l}$'s and interpret elements in $C^\infty(\mathscr{O}_k)$ as functions on $\mathscr{O}$. Put $C^\infty(\mathscr{O}) := \bigcup_{l\in\mathbb{N}_0} C^\infty(\mathscr{O}_k)$. $C^\infty(\mathscr{O})$ is interpreted as algebra of *smooth functions on* $\mathscr{O}$. Differential calculus over $\mathscr{O}$ may then be introduced as *filtered differential calculus over* $C^\infty(\mathscr{O})$ [37]. Since the main constructions (smooth maps, vector fields, differential forms, linear jets and differential operators, etc.) of such calculus do not look very different from the analogous ones in finite-dimensional differential geometry we will not insist on this and refer to [37] for the rigorous definitions and the main results (see [31] and [33, 34] for a sketch of alternative approaches).

Here we just recall the definition of finite dimensional vector bundle over $\mathscr{O}$. This is, basically, a vector bundle over $\mathscr{O}_k$ for some $k < \infty$, pull-backed to $\mathscr{O}$ via $\mu_{\infty,k}$. In more details, let $\tau_0 : T_0 \longrightarrow \mathscr{O}_k$ be a (finite dimensional) vector bundle, $k < \infty$. For $l \geq 0$ let $\tau_l := \mu^\circ_{k+l,k}(\tau_0) : T_l := \mu^\circ_{k+l,k}(T_0) \longrightarrow \mathscr{O}_{k+l}$ be the induced (by τ_0 via $\mu_{k+l,k}$) vector bundle and $\nu_{l+1,l} : T_{l+1} \longrightarrow T_l$ the canonical projection. Denote by T the pro-finite dimensional smooth manifold determined by the sequence of fiber bundles

$$\begin{tikzcd} T_0 & T_1 \arrow[l, "\nu_{1,0}"'] & \cdots \arrow[l] & T_l \arrow[l, "\nu_{l,l-1}"'] & T_{l+1} \arrow[l, "\nu_{l+1,l}"'] & \cdots . \arrow[l] \end{tikzcd} \tag{8}$$

The maps $\tau_l : T_l \longrightarrow \mathscr{O}_{l+k}$, $l \geq 0$, determine a smooth map $\tau : T \longrightarrow \mathscr{O}$. Any such map is, by definition, *a (finite-dimensional) vector bundle over* $\mathscr{O}$. Notice that it is associated to the sequence (8) of vector bundle morphisms a filtration of vector spaces

$$\Gamma(\tau_0) \xrightarrow{\mu^\circ_{k+1,k}} \cdots \longrightarrow \Gamma(\tau_{l-1}) \xrightarrow{\mu^\circ_{k+l,k+l-1}} \Gamma(\tau_l) \xrightarrow{\mu^\circ_{k+l+1,k+l}} \cdots .$$

We understand the monomorphisms $\mu^\circ_{k+l+1,k+l}$'s and interpret (7) as a sequence of vector subspaces. Similarly, we understand the $\mu_{\infty,k+l}$'s and interpret elements in $\Gamma(\tau_l)$ as functions $\mathscr{O} \longrightarrow T$. Put $\Gamma(\tau) := \bigcup_{l\in\mathbb{N}_0} \Gamma(\tau_l)$. $\Gamma(\tau)$ is naturally a $C^\infty(\mathscr{O})$-module and it is interpreted as the module of *smooth sections of* τ.

As an example, let $\mathscr{O} = J^\infty\pi$, $\tau_0 : T_0 \longrightarrow J^k\pi$ be a vector bundle for some $k < \infty$ and $\tau := \pi^\circ_{\infty,k}(\tau_0) : T := \pi^\circ_{\infty,k}(T_0) \longrightarrow J^\infty\pi$. Since $\Gamma(\tau_l) = \mathrm{diff}_{k+l}(\pi,\tau_l)$ for any l, we have the filtration $\mathrm{diff}_k(\pi,\tau_0) \subset \mathrm{diff}_{k+1}(\pi,\tau_1) \subset \cdots \subset \mathrm{diff}_{k+l}(\pi,\tau_l) \subset \cdots$. Put $\mathrm{diff}(\pi,\tau) := \bigcup_{l\in\mathbb{N}_0} \mathrm{diff}_{k+l}(\pi,\tau_l) = \Gamma(\tau)$. Elements in $\mathrm{diff}(\pi,\tau)$ are called *differential operators* 'acting on local sections of π, with values in τ_0' (in short 'from π to τ_0'). They are nothing but sections of the vector bundle $\tau : T \longrightarrow J^\infty\pi$.

An important technical advantage of formally integrable PDEs is the following. Let $\mathscr{E} \subset J^\infty(\pi)$ be the ∞th prolongation of a formally integrable PDE, $\tau : T \longrightarrow J^\infty(\pi)$ a vector bundle and $\tau|_{\mathscr{E}} : T|_{\mathscr{E}} \longrightarrow \mathscr{E}$ its restriction to $\mathscr{E}$. Then for any section $s \in \Gamma(\tau|_{\mathscr{E}})$ there exists a section $\widetilde{s} \in \Gamma(\tau)$ such that $s = \tilde{s}|_{\mathscr{E}}$. In the following we will often use this property without further comments.

Finally, let us mention here that a vector field on an pro-finite dimensional manifold does not generate a flow in general (see, for instance, [10]).

1.2. **The Cartan Distribution and the $\mathscr{C}$-Spectral Sequence.** Let $\pi : E \longrightarrow M$ and $\tau : T \longrightarrow J^\infty\pi$ be as in the previous section and $\Phi \in \mathrm{diff}(\pi,\tau)$. In the following we will simply write J^∞ for $J^\infty\pi$ and $\mathscr{E}$ for $\mathscr{E}^{(\infty)}_\Phi$. $i_{\mathscr{E}} : \mathscr{E} \hookrightarrow J^\infty$ will denote the inclusion. Notice that for $\Phi = 0$, $\mathscr{E} = \mathscr{E}^{(\infty)}_\Phi = J^\infty$.

Recall that J^∞ is endowed with the Cartan distribution $\mathscr{C}$ which is defined as follows:

$$\mathscr{C} : J^\infty \ni \theta \longmapsto \mathscr{C}_\theta \subset T_\theta J^\infty,$$

where $\mathscr{C}_\theta := T_\theta \Gamma^\infty_{\boldsymbol{p}}$ for $\theta = [\boldsymbol{p}]^\infty_x$, $x \in M$. Denote by $\mathscr{C}\mathrm{D}(J^\infty) \subset \mathrm{D}(J^\infty)$ the $C^\infty(J^\infty)$-submodule made of vector fields in the Cartan distribution, i.e., vector fields $X \in \mathrm{D}(J^\infty)$ such that $X_\theta \in \mathscr{C}_\theta$ for all $\theta \in J^\infty$. The Cartan distribution is n-dimensional, it is locally spanned by total derivatives $\ldots, D_i, \ldots$ and it is involutive, i.e., $[X,Y] \in \mathscr{C}\mathrm{D}(J^\infty)$ for all $X, Y \in \mathscr{C}\mathrm{D}(J^\infty)$. Moreover, n-dimensional integral submanifolds $L \subset J^\infty$ of $\mathscr{C}$ are of the form $L = \Gamma^\infty_{\boldsymbol{p}}$ for some local section $\boldsymbol{p}$ of π.

Let $\mathscr{E} \subset J^\infty$ be as above. The Cartan distribution $\mathscr{C}$ restricts to $\mathscr{E}$ in the sense that $\mathscr{C}_\theta \subset T_\theta\mathscr{E}$ for any $\theta \in \mathscr{E}$. Abusing the notation we still denote by $\mathscr{C}$ the restricted to $\mathscr{E}$ distribution and call it the *Cartan distribution of* $\mathscr{E}$. Also we denote by $\mathscr{C}\mathrm{D}(\mathscr{E}) \subset \mathrm{D}(\mathscr{E})$ the $C^\infty(\mathscr{E})$-submodule made of vector fields in $\mathscr{C}$. Elements in $\mathscr{C}\mathrm{D}(\mathscr{E})$ are called *horizontal vector fields.* In particular, total derivatives restrict to $\mathscr{E}$, i.e., there are unique local vector fields $\ldots, D^{\mathscr{E}}_i, \ldots$ on $\mathscr{E}$ such that $i^*_{\mathscr{E}} \circ D_i = D^{\mathscr{E}}_i \circ i^*_{\mathscr{E}}$, $i = 1, \ldots, n$. Again $\mathscr{C}$ is locally spanned by vector fields $\ldots, D^{\mathscr{E}}_i, \ldots$, it is involutive and n-dimensional integral submanifolds of it are graphs $\Gamma^\infty_{\boldsymbol{p}}$ of infinite jet prolongations of local solutions $\boldsymbol{p}$ of $\mathscr{E}_\Phi$.

A spectral sequence is naturally associated to an involutive distribution and, in particular, to the Cartan distribution on (the infinite prolongation of) a PDE as follows. Denote by $\mathscr{C}\Lambda(\mathscr{E}) \subset \Lambda(\mathscr{E})$ the subset made of differential forms ω such that

$$\omega(X_1, \ldots, X_k) = 0 \quad \text{for all } X_1, \ldots, X_k \in \mathscr{C}\mathrm{D}(\mathscr{E}),$$

where k is the degree of ω. $\mathscr{C}\Lambda(\mathscr{E})$ is a differential ideal in $\Lambda(\mathscr{E})$. Namely, it is an algebraic ideal and, moreover, it is differentially closed, i.e., $d\omega \in \mathscr{C}\Lambda(\mathscr{E})$ for any $\omega \in \mathscr{C}\Lambda(\mathscr{E})$. $\mathscr{C}\Lambda(\mathscr{E})$ is called the *Cartan ideal* of $\mathscr{E}$. For any $p \in \mathbb{N}$, denote by $\mathscr{C}^p\Lambda(\mathscr{E})$ the pth exterior power of $\mathscr{C}\Lambda(\mathscr{E})$. Thus, the sequence

$$\Lambda(\mathscr{E}) \supset \mathscr{C}\Lambda(\mathscr{E}) \supset \mathscr{C}^2\Lambda(\mathscr{E}) \supset \cdots \supset \mathscr{C}^p\Lambda(\mathscr{E}) \supset \cdots$$

is a filtration of the de Rham complex $(\Lambda(\mathscr{E}), d)$ of $\mathscr{E}$. The associated spectral sequence is denoted by $\mathscr{C}E(\mathscr{E}) = \{(\mathscr{C}E^{p,q}_r(\mathscr{E}), d^{p,q}_r)\}^{p,q}_r$ and called the $\mathscr{C}$-spectral sequence of $\mathscr{E}$ [39]. It is regular and converges to de Rham cohomologies of $\mathscr{E}$.

The first column of the 0th term of $\mathscr{C}E(\mathscr{E})$,

$$0 \longrightarrow \mathscr{C}E^{0,0}_0(\mathscr{E}) \xrightarrow{d^{0,0}_0} \mathscr{C}E^{0,1}_0(\mathscr{E}) \xrightarrow{d^{0,1}_0} \cdots \longrightarrow \mathscr{C}E^{0,q}_0(\mathscr{E}) \xrightarrow{d^{0,q}_0} \cdots,$$

is, by definition, the quotient complex $\Lambda(\mathscr{E})/\mathscr{C}\Lambda(\mathscr{E})$, which is also denoted by

$$0 \longrightarrow C^\infty(\mathscr{E}) \xrightarrow{\overline{d}} \overline{\Lambda}^1(\mathscr{E}) \xrightarrow{\overline{d}} \cdots \longrightarrow \overline{\Lambda}^q(\mathscr{E}) \xrightarrow{\overline{d}} \cdots,$$

and called *the horizontal de Rham complex of* $\mathscr{E}$. Its cohomology algebra $\mathscr{C}E^{0,\bullet}_1(\mathscr{E})$ is denoted by $\overline{H}(\mathscr{E})$, and called *horizontal de Rham cohomology algebra of* $\mathscr{E}$. Recall, in particular, that $\overline{d}$-closed elements in $\overline{\Lambda}^{n-1}(\mathscr{E})$ are called *conserved currents* and cohomology classes in $\overline{H}^{n-1}(\mathscr{E})$ *conservation laws* of the PDE $\mathscr{E}_\Phi$.

In the following we will denote by $\mathscr{C}\Lambda^k(\mathscr{E})$ (resp. $\mathscr{C}^p\Lambda^k(\mathscr{E})$, $\overline{\Lambda}^k(\mathscr{E})$, $\overline{H}^k(\mathscr{E})$) the kth homogeneous component of $\mathscr{C}\Lambda(\mathscr{E})$ (resp. $\mathscr{C}^p\Lambda(\mathscr{E})$, $\overline{\Lambda}(\mathscr{E})$, $\overline{H}(\mathscr{E})$), $k \geq 0$, and by $\mathscr{C}^\bullet\Lambda(\mathscr{E}) := \bigoplus_p \mathscr{C}^p\Lambda^p(\mathscr{E}) \subset \Lambda(\mathscr{E})$ the $C^\infty(\mathscr{E})$-subalgebra generated by $\mathscr{C}\Lambda^1(\mathscr{E})$. Notice that $\mathscr{C}^p\Lambda(\mathscr{E})$ is generated by $\mathscr{C}^p\Lambda^p(\mathscr{E})$ as an ideal, $p > 0$.

The $\mathscr{C}$-spectral sequence $\mathscr{C}E(\mathscr{E})$ contains very relevant "invariants" of the PDE $\mathscr{E}_\Phi$ (see, for instance, [9, 21]). Moreover, it formalizes in a coordinate-free manner variational calculus (on local sections of π) constrained by $\mathscr{E}_\Phi$ [39]. Therefore, it is a most fundamental construction in the geometric theory of differential equations. Finally, it is a very general construction. For instance, it may be defined exactly in the same way when $\mathscr{E}$ is the infinite prolongation of a system of PDEs "imposed on general n-dimensional submanifolds of E". However, in the present case, the fibered structure $\pi_\infty|_{\mathscr{E}} : \mathscr{E} \longrightarrow M$ of $\mathscr{E}$ allows a more simple description (which is, in the general case, valid only locally), *the variational bi-complex* [39], which we briefly recall in the following.

The Cartan distribution and the fibered structure $\pi_\infty|_{\mathscr{E}} : \mathscr{E} \longrightarrow M$ of $\mathscr{E}$ determine a splitting of the tangent bundle $T\mathscr{E} \longrightarrow \mathscr{E}$ into the Cartan or horizontal part $\mathscr{C}$ and the vertical (with respect to π_∞) part $V\pi_\infty|_{\mathscr{E}}$. Accordingly, $\mathrm{D}(\mathscr{E})$ splits into a direct sum: $\mathrm{D}(\mathscr{E}) = \mathscr{C}\mathrm{D}(\mathscr{E}) \oplus V\mathrm{D}(\mathscr{E})$, $V\mathrm{D}(\mathscr{E}) \subset \mathrm{D}(\mathscr{E})$ being the $C^\infty(\mathscr{E})$-submodule made of π_∞-*vertical vector fields*, i.e., vector fields $Y \in \mathrm{D}(\mathscr{E})$ such that $Y \circ \pi_\infty^* = 0$. In particular, $V\mathrm{D}(J^\infty)$ is locally generated by vector fields $\ldots, \partial_a^I, \ldots$. Dually, $\Lambda^1(\mathscr{E})$ splits into the direct sum

$$\Lambda^1(\mathscr{E}) = \mathscr{C}\Lambda^1(\mathscr{E}) \oplus \overline{\Lambda}^1(\mathscr{E}); \tag{9}$$

here and in what follows $\overline{\Lambda}^1(\mathscr{E})$ is identified with the $C^\infty(\mathscr{E})$-submodule in $\Lambda^1(\mathscr{E})$ generated by $\pi_\infty^*(\Lambda^1(M))$. In particular, $\mathscr{C}\Lambda^1(J^\infty)$ is locally generated by forms $\ldots, \omega_I^\alpha := du_I^\alpha - u_{Ii}^\alpha dx^i, \ldots$ and $\overline{\Lambda}^1(J^\infty)$ is locally generated by forms $\ldots, dx^i, \ldots$. Similarly, $\mathscr{C}\Lambda^1(\mathscr{E})$ is locally generated by forms $\ldots, i_{\mathscr{E}}^*(\omega_I^\alpha), \ldots$ and $\overline{\Lambda}^1(\mathscr{E})$ is locally generated by forms $\ldots, i_{\mathscr{E}}^*(dx^i), \ldots$.

In view of splitting (9) $\Lambda(\mathscr{E})$ factorizes as $\Lambda(\mathscr{E}) \simeq \mathscr{C}^\bullet\Lambda(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E})$ (here and in what follows tensor products will be always over $C^\infty(\mathscr{E})$, or $C^\infty(J^\infty)$ for $\Phi = 0$). In particular, there are projections $\mathfrak{p}_{p,q} : \Lambda(\mathscr{E}) \longrightarrow \mathscr{C}^p\Lambda^p(\mathscr{E}) \otimes \overline{\Lambda}^q(\mathscr{E})$ for any $p, q \in \mathbb{N}_0$. Correspondingly, the de Rham complex of $\mathscr{E}$, $(\Lambda(\mathscr{E}), d)$, splits in a bi-complex $(\mathscr{C}^\bullet\Lambda(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d}, d^V)$ (in the following diagram we drop for simplicity the postfix $(\mathscr{E})$),

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 \arrow[r] & \mathscr{C}^{p+1}\Lambda^{p+1} \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \cdots \arrow[r] & \mathscr{C}^{p+1}\Lambda^{p+1} \otimes \overline{\Lambda}^q \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \mathscr{C}^{p+1}\Lambda^{p+1} \otimes \overline{\Lambda}^{q+1} \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \cdots \\
0 \arrow[r] & \mathscr{C}^p\Lambda^p \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \cdots \arrow[r] & \mathscr{C}^p\Lambda^p \otimes \overline{\Lambda}^q \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \mathscr{C}^p\Lambda^p \otimes \overline{\Lambda}^{q+1} \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \cdots \\
 & \cdots \arrow[u] & \cdots & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \\
0 \arrow[r] & C^\infty \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \cdots \arrow[r] & \overline{\Lambda}^q \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \overline{\Lambda}^{q+1} \arrow[u, "d^V"] \arrow[r, "\overline{d}"] & \cdots \\
 & 0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}, \tag{10}$$

defined by

$$\overline{d}(\omega \otimes \overline{\sigma}) := (\mathfrak{p}_{p,q+1} \circ d)(\omega \wedge \overline{\sigma}) \quad \text{and} \quad d^V(\omega \otimes \overline{\sigma}) := (\mathfrak{p}_{p+1,q} \circ d)(\omega \wedge \overline{\sigma}),$$

where $\omega \in \mathscr{C}^p\Lambda^p(\mathscr{E})$ and $\overline{\sigma} \in \overline{\Lambda}^q(\mathscr{E})$, $p, q \in \mathbb{N}_0$. $\overline{d}$ and d^V are called the *horizontal* and the *vertical de Rham differential*, respectively, and (10) is called the *variational bi-complex*. In the following we will often understand isomorphism $\Lambda(\mathscr{E}) \simeq \mathscr{C}^\bullet\Lambda(\mathscr{E})\otimes\overline{\Lambda}(\mathscr{E})$.

As a bi-complex (10) determines two spectral sequences. One of them is the $\mathscr{C}$-spectral sequence while the other is the Leray-Serre spectral sequence of the fibration $\pi_\infty|_{\mathscr{E}} : \mathscr{E} \longrightarrow M$ [27]. In particular, for any p, there is a canonical isomorphisms of complexes

$$(\mathscr{C}E_0^{p,\bullet}(\mathscr{E}), d_0^{p,\bullet}) \simeq (\mathscr{C}^p\Lambda^p(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d}), \tag{11}$$

and the differential $d_1^{p,\bullet} : \mathscr{C}E_1^{p,\bullet}(\mathscr{E}) \longrightarrow \mathscr{C}E_1^{p+1,\bullet}(\mathscr{E})$ is isomorphic to the map induced by d^V in the cohomology $H(\mathscr{C}^p\Lambda^p(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d})$.

Notice that the embedding $i_{\mathscr{E}} : \mathscr{E} \hookrightarrow J^\infty$ of the infinite prolongation $\mathscr{E}$ of a PDE determines via pull-back both a morphism of spectral sequences and a morphism of bi-complexes that, abusing the notation, we denote by the same symbol

$$i_{\mathscr{E}}^* : \{(\mathscr{C}E_r^{\bullet,\bullet}(J^\infty), d_r^{\bullet,\bullet})\} \longrightarrow \{(\mathscr{C}E_r^{\bullet,\bullet}(\mathscr{E}), d_r^{\bullet,\bullet})\},$$

$$i_{\mathscr{E}}^* : (\mathscr{C}^\bullet\Lambda(J^\infty) \otimes \overline{\Lambda}(J^\infty), \overline{d}, d^V) \longrightarrow (\mathscr{C}^\bullet\Lambda(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d}, d^V).$$

1.3. **Higher Symmetries of PDEs.** Denote by $\mathrm{D}_{\mathscr{C}}(\mathscr{E}) \subset \mathrm{D}(\mathscr{E})$ the subset made of vector fields preserving the Cartan distribution, i.e., vector fields X such that $[X, Y] \in \mathscr{C}\mathrm{D}(\mathscr{E})$ for any $Y \in \mathscr{C}\mathrm{D}(\mathscr{E})$. $\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ is clearly a Lie subalgebra in $\mathrm{D}(\mathscr{E})$. Elements in $\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ are called *(infinitesimal) symmetries of* $\mathscr{E}_\Phi$. The theory of infinitesimal symmetries of PDEs is fundamental in many respects [9]. Notice that, since the Cartan distribution is involutive, then $\mathscr{C}\mathrm{D}(\mathscr{E}) \subset \mathrm{D}_{\mathscr{C}}(\mathscr{E})$ and it is an ideal in $\mathrm{D}_{\mathscr{C}}(\mathscr{E})$. Elements in $\mathscr{C}\mathrm{D}(\mathscr{E})$ are called *trivial symmetries of* $\mathscr{E}_\Phi$, in that horizontal vector fields "are symmetries of every PDE". The quotient Lie algebra $\mathrm{Sym}(\mathscr{E}) := \mathrm{D}_{\mathscr{C}}(\mathscr{E})/\mathscr{C}\mathrm{D}(\mathscr{E})$ is called the algebra of *non-trivial higher symmetries of* $\mathscr{E}_\Phi$. Clearly, every equivalence class $\boldsymbol{X} = X + \mathscr{C}\mathrm{D}(\mathscr{E}) \in \mathrm{Sym}(\mathscr{E})$, $X \in \mathrm{D}_{\mathscr{C}}(\mathscr{E})$, has got one and only one vertical representative $X^V \in V\mathrm{D}(\mathscr{E})$. Any vertical element in $\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ is called an *evolutionary vector field*. Thus $\mathrm{Sym}(\mathscr{E})$ is isomorphic to the Lie algebra $V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ of evolutionary vector fields.

In order to effectively describe $V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ and, therefore, $\mathrm{Sym}(\mathscr{E})$ let us first consider the case $\mathscr{E} = J^\infty$. It is easy to prove that any evolutionary vector field $Y \in V\mathrm{D}_{\mathscr{C}}(J^\infty)$ is determined by its restriction to $C^\infty(E) \subset C^\infty(J^\infty)$. Moreover, every vertical vector field $\chi : C^\infty(E) \longrightarrow C^\infty(J^\infty)$ along $\pi_{\infty,0} : J^\infty \longrightarrow E$ (χ is vertical if $\chi \circ \pi^* = 0$) extends to a unique evolutionary vector field $\mathcal{Э}_\chi \in V\mathrm{D}_{\mathscr{C}}(J^\infty)$. We conclude that $V\mathrm{D}_{\mathscr{C}}(J^\infty)$ is in one to one correspondence with the $C^\infty(J^\infty)$-module $\varkappa$ of vector fields along $\pi_{\infty,0}$ or, which is the same, the module of sections of the induced vector bundle $\pi_{\infty,0}^\circ(\nu\pi) : \pi_{\infty,0}^\circ(V\pi) \longrightarrow J^\infty$. Elements in $\varkappa$ are called *generating sections of higher symmetries* of π.

Let us now come to the general case when $\mathscr{E}$ is any. First of all consider the $C^\infty(\mathscr{E})$-module $\varkappa|_{\mathscr{E}}$ of vertical vector fields $\chi : C^\infty(E) \longrightarrow C^\infty(\mathscr{E})$ along $\pi_{\infty,0}|_{\mathscr{E}} : \mathscr{E} \longrightarrow E$ or, which is the same, the module of sections of the induced vector bundle $\pi_{\infty,0}|_{\mathscr{E}}^\circ(\nu\pi)$:

$\pi_{\infty,0}|_{\mathscr{E}}^{\circ}(V\pi) \longrightarrow \mathscr{E}$. Elements in $\varkappa|_{\mathscr{E}}$ are called generating sections of higher symmetries of $\mathscr{E}$. Similarly as to above, a generating section $\chi \in \varkappa|_{\mathscr{E}}$ extends to a unique vertical vector field $\text{Э}_\chi : C^\infty(J^\infty) \longrightarrow C^\infty(\mathscr{E})$ along the inclusion $i_{\mathscr{E}} : \mathscr{E} \hookrightarrow J^\infty$. If χ is locally given by $\chi = \chi^\alpha \partial_\alpha$, where $\ldots, \chi^\alpha, \ldots$ are local functions on $\mathscr{E}$, then Э_χ is locally given by $\text{Э}_\chi = D_I^{\mathscr{E}} \chi^\alpha \partial_\alpha^I|_{\mathscr{E}}$. However, in general Э_χ is not tangent to $\mathscr{E}$ and, therefore, is not in $V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$. Generating sections χ such that $\text{Э}_\chi \in V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ are the ones in the kernel of a suitable differential operator: the so-called *universal linearization of* $\mathscr{E}$, which we now define (notice that to the author's knowledge the following definition never appeared in the literature before in the general form presented here - see also [34]).

Let τ, Φ and $\mathscr{E}$ be as in the previous section, and put $\underline{\tau} := \pi_\infty \circ \tau : T \longrightarrow M$. Since τ is a vector bundle, $V\tau \longrightarrow T$ is naturally isomorphic to the induced bundle $\tau^\circ(\tau) : \tau^\circ(T) \longrightarrow T$, $\tau^\circ(\tau)$ being the (restriction to $\tau^\circ(T) \subset T \times T$ of the) projection $(T \times T \longrightarrow T)$ onto the first factor. Denote by $\rho_2 : \tau^\circ(T) \longrightarrow T$ the projection onto the second factor and by $\rho_2' : V\tau \longrightarrow T$ the map induced by ρ_2 via the isomorphism $V\tau \simeq \tau^\circ(T)$. Consider the vertical tangent map $V\Phi : V\pi_\infty \longrightarrow V\underline{\tau}$. Put $o_{\mathscr{E}} := o \circ i_{\mathscr{E}} : \mathscr{E} \longrightarrow T$ and notice, preliminarily, that $o_{\mathscr{E}} = \Phi \circ i_{\mathscr{E}}$. The short exact sequence of induced bundles $0 \longrightarrow o_{\mathscr{E}}^\circ(V\tau) \longrightarrow o_{\mathscr{E}}^\circ(V\underline{\tau}) \longrightarrow V\pi_\infty|_{\mathscr{E}} \longrightarrow 0$ splits naturally via the map $Vo|_{\mathscr{E}} : V\pi_\infty|_{\mathscr{E}} \longrightarrow o_{\mathscr{E}}^\circ(V\underline{\tau})$ well defined by putting $Vo|_{\mathscr{E}}(\theta, \xi) := (\theta, Vo(\xi))$, $(\theta, \xi) \in V\pi_\infty|_{\mathscr{E}}$. In particular, there is a canonical projection $V_\Phi : o_{\mathscr{E}}^\circ(V\underline{\tau}) \longrightarrow o_{\mathscr{E}}^\circ(V\tau)$. Define a map

$$L_\Phi : V\pi_\infty|_{\mathscr{E}} \longrightarrow T|_{\mathscr{E}}$$

by putting $L_\Phi(\xi) := (\theta, \rho_2'(V))$, where $(\theta, V) := V_\Phi(\theta, V\Phi(\xi)) \in o_{\mathscr{E}}^\circ(V\tau)$, for all $\xi \in V_\theta\pi_\infty$, $\theta \in \mathscr{E}$. L_Φ is a morphism of vector bundles. For any $\chi \in \varkappa|_{\mathscr{E}}$ let $\ell_\Phi\chi \in \Gamma(\tau|_{\mathscr{E}})$ be defined by putting $(\ell_\Phi\chi)_\theta := L_\Phi((\text{Э}_\chi)_\theta)$, $\theta \in \mathscr{E}$. $\ell_\Phi : \varkappa|_{\mathscr{E}} \longrightarrow \Gamma(\tau|_{\mathscr{E}})$ is a well defined linear differential operator called the *universal linearization of* Φ.

Let us describe ℓ_Φ locally. Let $(\ldots, x^i, \ldots, u_I^\alpha, \ldots)$ be local jet coordinates on J^∞, $(\ldots, x^i, \ldots, u_I^\alpha, \ldots, v^a, \ldots)$ adapted to τ local coordinates on T, and $(\ldots, e_a, \ldots)$ the local basis of $\Gamma(\tau|_{\mathscr{E}})$ associated to them. If Φ has local representation (3), $\ldots, \Phi^a, \ldots$ being local functions on J^∞, and $\chi = \chi^\alpha \partial_\alpha$ locally, then

$$\ell_\Phi\chi = e_a(\partial_\alpha^I \Phi^a)|_{\mathscr{E}} D_I^{\mathscr{E}} \chi^\alpha$$

locally.

Now let $\chi \in \varkappa|_{\mathscr{E}}$. It is easy to see that if $\ell_\Phi\chi = 0$ then Э_χ is tangent to $\mathscr{E}$ and, therefore, it is in $V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$. Vice versa, any symmetry $Y \in V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ is of the form Э_χ for a unique $\chi \in \varkappa|_{\mathscr{E}}$ such that $\ell_\Phi\chi = 0$. We conclude that $\mathrm{Sym}(\mathscr{E})$ is in one to one correspondence with $\ker \ell_\Phi$. In particular, $\ker \ell_\Phi$ inherits from $\mathrm{Sym}(\mathscr{E})$ the Lie algebra structure. The corresponding bracket is denoted by $\{\cdot, \cdot\}$ and called the *higher Jacobi bracket* of the equation $\mathscr{E}_\Phi$.

Finally, notice that, for any $\chi \in \ker \ell_\Phi$, the 'insertion of' and the 'Lie derivative along' $\text{Э}_\chi \in V\mathrm{D}(\mathscr{E})$ commute with the horizontal de Rham differential $\overline{d} : \Lambda(\mathscr{E}) \longrightarrow \Lambda(\mathscr{E})$, i.e.,

$$i_{\text{Э}_\chi} \circ \overline{d} + \overline{d} \circ i_{\text{Э}_\chi} = \mathcal{L}_{\text{Э}_\chi} \circ \overline{d} - \overline{d} \circ \mathcal{L}_{\text{Э}_\chi} = 0. \tag{12}$$

In their turn Identities (12) imply

$$i_{\partial_\chi} \circ d^V + d^V \circ i_{\partial_\chi} = \mathcal{L}_{\partial_\chi}, \quad \mathcal{L}_{\partial_\chi} \circ d^V - d^V \circ \mathcal{L}_{\partial_\chi} = 0,$$

$d^V : \Lambda(\mathscr{E}) \longrightarrow \Lambda(\mathscr{E})$ being the vertical de Rham differential.

1.4. **Secondary Differential Forms and Vector Fields.** Let $\mathscr{E}$ be as in the previous section. As noticed above, n-dimensional integral submanifolds of the Cartan distribution $\mathscr{C}$ over $\mathscr{E}$ are in one-to-one correspondence with local solutions of $\mathscr{E}_\Phi$. Thus, informally speaking, the pair $(\mathscr{E}, \mathscr{C})$ encodes all the information about the "functional space $\boldsymbol{M}$ of solutions" of $\mathscr{E}_\Phi$ (in the following we will in fact identify $(\mathscr{E}, \mathscr{C})$ with $\boldsymbol{M}$). For instance, "local functional calculus" over such functional space may be formalized geometrically (and homologically) by using $(\mathscr{E}, \mathscr{C})$ as a starting point and the associated $\mathscr{C}$-spectral sequence as the main structure. Such formalization has been named *secondary calculus* [37] by its discoverer, A. M. Vinogradov, and its simplest constructions will be briefly reviewed in this section.

Suppose temporarily that M is a compact, orientable and oriented manifold without boundary. Then an element $\boldsymbol{S} = [\mathscr{L}] \in \overline{H}^n(\mathscr{E}) = \mathscr{C}E_1^{0,n}(\mathscr{E})$, $\mathscr{L} \in \overline{\Lambda}^n(\mathscr{E})$, identifies with the (local) action functional

$$\boldsymbol{M} \ni \boldsymbol{p} \longmapsto \boldsymbol{S}(\boldsymbol{p}) := \int_M (j^\infty \boldsymbol{p})^*(\mathscr{L}) \in \mathbb{R},$$

and in the following we will denote by

$$\int : \overline{\Lambda}^n(\mathscr{E}) \ni \mathscr{L} \longmapsto \int \mathscr{L} := [\mathscr{L}] \in \overline{H}^n(\mathscr{E})$$

the projection. Thus $\mathscr{L}$ may be interpreted as the lagrangian density of a lagrangian theory constrained by the PDE $\mathscr{E}_\Phi$. As a natural generalization, we interpret $\overline{H}(\mathscr{E})$, not only its n-degree component, as space of local function(al)s on $\boldsymbol{M}$. By considering all less-dimensional cohomologies rather than just top ones we have in mind the possibility of defining functionals by integration on less-dimensional submanifolds of M. Such possibility is crucial in variational calculus with boundary conditions (see [26]).

Similarly, for $p > 0$, $\mathscr{C}E_1^{p,\bullet}(\mathscr{E})$ is naturally interpreted as space of local differential p-forms on $\boldsymbol{M}$. This informal arguments motivate the

Definition 1. *Elements in* $\overline{H}(\mathscr{E}) = \mathscr{C}E_1^{0,\bullet}(\mathscr{E}) =: \boldsymbol{C}^\infty(\boldsymbol{M})^\bullet$ *are called* secondary functions on $\boldsymbol{M}$. *For* $p > 0$, *elements in* $H(\mathscr{C}^p\Lambda^p(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d}) \simeq \mathscr{C}E_1^{p,\bullet}(\mathscr{E}) =: \boldsymbol{\Lambda}^p(\boldsymbol{M})^\bullet$ *are called* secondary differential p-forms on $\boldsymbol{M}$. *We put also* $\boldsymbol{\Lambda}(\boldsymbol{M})^\bullet := \bigoplus_p \boldsymbol{\Lambda}^p(\boldsymbol{M})^\bullet$.

Notice that elements in $\boldsymbol{\Lambda}(\boldsymbol{M})^n$ are sometimes referred to in the literature as *variational forms* [28].

We apply similar arguments to motivate the definition of secondary vector fields. First of all, notice that there exists a complex

$$0 \longrightarrow V\mathrm{D}(\mathscr{E}) \xrightarrow{\overline{S}} \cdots \longrightarrow V\mathrm{D}(\mathscr{E}) \otimes \overline{\Lambda}^q(\mathscr{E}) \xrightarrow{\overline{S}} V\mathrm{D}(\mathscr{E}) \otimes \overline{\Lambda}^{q+1}(\mathscr{E}) \xrightarrow{\overline{S}} \cdots, \qquad (13)$$

somehow "dual" to complex $(\mathscr{C}\Lambda^1(\mathscr{E})\otimes\overline{\Lambda}(\mathscr{E}),\overline{d})\simeq(\mathscr{C}E_0^{1,\bullet}(\mathscr{E}),d_0^{1,\bullet})$, well defined by putting

$$\overline{S}(X\otimes\overline{\omega}):=\overline{S}(X)\wedge\overline{\omega}+X\otimes\overline{d}\overline{\omega},$$

$X\in V\mathrm{D}(\mathscr{E})$, $\overline{\omega}\in\overline{\Lambda}(\mathscr{E})$, where $\overline{S}(X)\in V\mathrm{D}(\mathscr{E})\otimes\overline{\Lambda}^1(\mathscr{E})$ is the $V\mathrm{D}(\mathscr{E})$-valued horizontal 1-form defined by putting $\overline{S}(X)(Y):=[Y,X]^V$, and $[Y,X]^V$ is the vertical component of $[Y,X]$. Complex (13) is called the (horizontal) Spencer complex of $\mathscr{E}$. As we will see later on in more details, 0-cohomology $H^0(V\mathrm{D}(\mathscr{E})\otimes\overline{\Lambda}(\mathscr{E}),\overline{S})$ of the Spencer complex is given by $V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$. Now, let $\boldsymbol{X}\in\mathrm{Sym}(\mathscr{E})$ and $\partial_\chi\in V\mathrm{D}_{\mathscr{C}}(\mathscr{E})$ be the associated evolutionary vector field, $\chi\in\varkappa|_{\mathscr{E}}$ being a generating section such that $\ell_\Phi\chi=0$. Suppose temporarily that ∂_χ generates a flow $\{A_t\}_t$ of local diffeomorphisms of $\mathscr{E}$. Then for any t, A_t preserves the Cartan distribution and therefore the image $A_t(L)$ of an n-dimensional integral submanifold L is an n-dimensional integral submanifold. We conclude that $\boldsymbol{X}$ generates a flow of solutions of $\mathscr{E}_\Phi$ and, therefore, may be interpreted as a (local) vector field on $\boldsymbol{M}$. This makes it rigorous the assertion that *tangent vectors to the solution space of a PDE are solutions of the associated linearized PDE*. As a natural generalization, we interpret the whole $H(V\mathrm{D}(\mathscr{E})\otimes\overline{\Lambda}(\mathscr{E}),\overline{S})$, not only its 0-degree component, as space of vector fields on $\boldsymbol{M}$. This motivates the

Definition 2. *Elements in* $H(V\mathrm{D}(\mathscr{E})\otimes\overline{\Lambda}(\mathscr{E}),\overline{S})=:\mathbf{D}(\boldsymbol{M})^\bullet$ *are called* secondary vector fields *on* $\boldsymbol{M}$.

All standard operations with vector fields and differential forms have their secondary analogue. Namely, let $\boldsymbol{\omega}\in\boldsymbol{\Lambda}^p(\boldsymbol{M})^q$, $\boldsymbol{\omega}_1\in\boldsymbol{\Lambda}^{p_1}(\boldsymbol{M})^{q_1}$, $\boldsymbol{\omega}_2\in\boldsymbol{\Lambda}^{p_2}(\boldsymbol{M})^{q_2}$, $\boldsymbol{X}\in\mathbf{D}(\boldsymbol{M})^r$, $\boldsymbol{X}_1\in\mathbf{D}(\boldsymbol{M})^{r_1}$, $\boldsymbol{X}_2\in\mathbf{D}(\boldsymbol{M})^{r_2}$. Then $\boldsymbol{\omega}=[\omega]$, $\boldsymbol{\omega}_1=[\omega_1]$ and $\boldsymbol{\omega}_2=[\omega_2]$ for some $\omega\in\mathscr{C}^p\Lambda^p(\mathscr{E})\otimes\overline{\Lambda}^q(\mathscr{E})$, $\omega_1\in\mathscr{C}^{p_1}\Lambda^{p_1}(\mathscr{E})\otimes\overline{\Lambda}^{q_1}(\mathscr{E})$ and $\omega_2\in\mathscr{C}^{p_2}\Lambda^{p_2}(\mathscr{E})\otimes\overline{\Lambda}^{q_2}(\mathscr{E})$ such that $\overline{d}\omega=\overline{d}\omega_1=\overline{d}\omega_2=0$. Similarly, $\boldsymbol{X}=[X]$, $\boldsymbol{X}_1=[X_1]$ and $\boldsymbol{X}_2=[X_2]$ for some $X\in V\mathrm{D}(\mathscr{E})\otimes\overline{\Lambda}^r(\mathscr{E})$, $X_1\in V\mathrm{D}(\mathscr{E})\otimes\overline{\Lambda}^{r_1}(\mathscr{E})$ and $X_2\in V\mathrm{D}(\mathscr{E})\otimes\overline{\Lambda}^{r_2}(\mathscr{E})$ such that $\overline{S}(X)=\overline{S}(X_1)=\overline{S}(X_2)=0$. The following operations are well defined:

exterior product of differential forms:

$$\boldsymbol{\omega}_1\wedge\boldsymbol{\omega}_2:=[(-1)^{q_1p_2}\omega_1\wedge\omega_2]\in\boldsymbol{\Lambda}^{p_1+p_2}(\boldsymbol{M})^{q_1+q_2};$$

exterior differential of a differential form:

$$\boldsymbol{d\omega}:=[d^V\omega]\in\boldsymbol{\Lambda}^{p+1}(\boldsymbol{M})^q;$$

commutator of vector fields:

$$[\boldsymbol{X}_1,\boldsymbol{X}_2]:=[[\![X_1,X_2]\!]]\in\mathbf{D}(\boldsymbol{M})^{r_1+r_2};$$

insertion of a vector field into a differential form:

$$i_{\boldsymbol{X}}\boldsymbol{\omega}:=[(-1)^{r(p-1)}i_X\omega]\in\boldsymbol{\Lambda}^{p-1}(\boldsymbol{M})^{q+r};$$

Lie derivative of a differential form along a vector field:

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{\omega}:=(i_{\boldsymbol{X}}\circ\boldsymbol{d}+\boldsymbol{d}\circ i_{\boldsymbol{X}})\boldsymbol{\omega}\in\boldsymbol{\Lambda}^p(\boldsymbol{M})^{q+r};$$

$[\![\cdot , \cdot]\!]$ being the Frölicher-Nijenhuis bracket of form-valued vector fields.

Secondary analogue of the standard relations among the above operations hold. Indeed, let $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2$ be as above. The exterior product endows $\boldsymbol{\Lambda}(\boldsymbol{M})^\bullet = \bigoplus_{p,q} \boldsymbol{\Lambda}^p(\boldsymbol{M})^q$ with the structure of a bi-graded algebra. Namely, $\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 = (-1)^{p_1p_2+q_1q_2}\boldsymbol{\omega}_2 \wedge \boldsymbol{\omega}_1$. The exterior differential is a bi-graded derivation of bi-degree $(1,0)$. Namely, $\boldsymbol{d}(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) = \boldsymbol{d}\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 + (-1)^{p_1}\boldsymbol{\omega}_1 \wedge \boldsymbol{d}\boldsymbol{\omega}_2$. The commutator endows $\mathbf{D}(\boldsymbol{M})^\bullet = \bigoplus_r \mathbf{D}(\boldsymbol{M})^r$ with the structure of a graded Lie algebra, in particular, $[\boldsymbol{X}, [\boldsymbol{X}_1, \boldsymbol{X}_2]] = [[\boldsymbol{X}, \boldsymbol{X}_1], \boldsymbol{X}_2] + (-1)^{rr_1}[\boldsymbol{X}_1, [\boldsymbol{X}, \boldsymbol{X}_2]]$. The 'insertion of' and the 'Lie derivative along' $\boldsymbol{X}$ are bi-graded derivations of bi-degree $(-1, r)$ and $(0, r)$ respectively. Namely, $i_{\boldsymbol{X}}(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) = i_{\boldsymbol{X}}\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 + (-1)^{p_1+rq_1}\boldsymbol{\omega}_1 \wedge i_{\boldsymbol{X}}\boldsymbol{\omega}_2$ and $\mathcal{L}_{\boldsymbol{X}}(\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2) = \mathcal{L}_{\boldsymbol{X}}\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 + (-1)^{rq_1}\boldsymbol{\omega}_1 \wedge \mathcal{L}_{\boldsymbol{X}}\boldsymbol{\omega}_2$. Moreover, $[\boldsymbol{d}, \boldsymbol{d}] = [\boldsymbol{d}, \mathcal{L}_{\boldsymbol{X}}] = [i_{\boldsymbol{X}_1}, i_{\boldsymbol{X}_2}] = 0$, $[\boldsymbol{d}, i_{\boldsymbol{X}}] = \mathcal{L}_{\boldsymbol{X}}$, $[i_{\boldsymbol{X}_1}, \mathcal{L}_{\boldsymbol{X}_2}] = i_{[\boldsymbol{X}_1, \boldsymbol{X}_2]}$, $[\mathcal{L}_{\boldsymbol{X}_1}, \mathcal{L}_{\boldsymbol{X}_2}] = \mathcal{L}_{[\boldsymbol{X}_1, \boldsymbol{X}_2]}$, where $[\cdot , \cdot]$ denotes the bi-graded commutator.

Despite some time has passed since they were introduced [20, 39], to the author knowledge, no general techniques have been developed so far in order to effectively compute secondary differential form and vector field spaces, i.e., cohomologies of complexes (11) and (13), in full generality, other than the one based on the so-called *compatibility complexes* [34, 35] (and, possibly, the Koszul-Tate resolution [36]), which is reviewed in the next two sections.

1.5. **Horizontal Calculus on PDEs.**

The Cartan distribution determines a "horizontal differential calculus" on $\mathscr{E}$. Informally speaking, the horizontal differential calculus is obtained replacing standard partial derivatives with total derivatives. For instance, a horizontal linear differential operator is one which is a linear combination of compositions of total derivatives.

More rigorously, let $\tau : T \longrightarrow \mathscr{E}$ (resp. $\rho : R \longrightarrow \mathscr{E}$) be a finite dimensional vector bundle and $P := \Gamma(\tau)$ (resp. $Q := \Gamma(\rho)$) the $C^\infty(\mathscr{E})$-module of sections of τ (resp. ρ). In the following any such module will be called a *smooth module*. A linear differential operator $\square : P \longrightarrow Q$ is called a *horizontal (linear) differential operator* iff, by definition, for any $\theta \in \mathscr{E}$ and any submanifold $L \subset \mathscr{E}$ such that $\theta \in L$ and $T_\theta L \subset \mathscr{C}_\theta$ there exists a differential operator $\square^L_\theta : P|_L \longrightarrow Q|_L$ such that $(\square p)(\theta) = \square^L_\theta(p|_L)(\theta)$ for all $p \in P$. As examples, notice that horizontal vector fields, the horizontal de Rham differential $\overline{d}$, the Spencer differential $\overline{S}$ and universal linearizations are horizontal differential operators. Indeed, Let $\ldots, e_a, \ldots$ (resp. $\ldots, \varepsilon_A, \ldots$) be a local basis of P (resp. Q). Then a horizontal differential operator $\square : P \longrightarrow Q$ is characterized as being one locally given by

$$\square p = \varepsilon_A \square^{AI}_a D^{\mathscr{E}}_I p^a, \quad \ldots, \square^{AI}_a, \ldots \text{ being local functions on } \mathscr{E}, \tag{14}$$

for all $p = p^a e_a$ local sections of τ, $\ldots, p^a, \ldots$ local functions on $\mathscr{E}$. In particular, if $\mathscr{E} = J^\infty$ and $\mathscr{F} \subset J^\infty$ is the infinite prolongation of a PDE, then any horizontal differential operator $\square : P \longrightarrow Q$ restricts to $\mathscr{F}$, i.e., there exists a unique (horizontal) differential operator $\square^{\mathscr{F}} : P|_{\mathscr{F}} \longrightarrow Q|_{\mathscr{F}}$ such that $\square^{\mathscr{F}}(p|_{\mathscr{F}}) = (\square p)|_{\mathscr{F}}$ for all $p \in P$.

Denote by $\mathscr{C}\mathrm{Diff}(P,Q)$ the set of all horizontal differential operators $\square : P \longrightarrow Q$. Clearly, $\mathscr{C}\mathrm{Diff}(P,Q)$ is a $C^\infty(\mathscr{E})$-module naturally isomorphic to $\mathscr{C}\mathrm{Diff}(P, C^\infty(\mathscr{E})) \otimes Q$ and in what follows we will understand such isomorphism.

Similarly, one may define horizontal jets of sections of vector bundles over $\mathscr{E}$ just replacing partial derivatives with total derivatives in the standard definition. We refer to [36] for the details of the construction. Analogously to the standard case, one may also define (systems of horizontal) PDEs determined by linear horizontal differential operators and, in particular, formally integrable PDEs.

Denote by $\overline{\tau}_\infty : \overline{J}^\infty\tau \longrightarrow \mathscr{E}$ the bundle of horizontal infinite jets of sections of τ and put $\overline{J}^\infty P := \Gamma(\overline{\tau}_\infty)$. For any $p \in P$ denote by $\overline{\jmath}_\infty p \in \overline{J}^\infty P$ its infinite horizontal jet prolongation. There is a canonical monomorphism of $C^\infty(\mathscr{E})$-modules $h : \mathscr{C}\mathrm{Diff}(P,Q) \ni \square \longmapsto h_\square \in \mathrm{Hom}(\overline{J}^\infty P, Q)$, where $h_\square$ is the unique $C^\infty(\mathscr{E})$-linear map such that $h_\square(\overline{\jmath}_\infty p) = \square p$ for all $p \in P$. Moreover $h_\square$ can be uniquely prolonged to a $C^\infty(\mathscr{E})$-linear map $h^\infty_\square : \overline{J}^\infty P \longrightarrow \overline{J}^\infty Q$ such that $h^\infty_\square(\overline{\jmath}_\infty p) = \overline{\jmath}_\infty(\square p)$ for all $p \in P$.

The following remarkable correspondence,

$$\overline{J}^\infty\varkappa \ni \overline{\jmath}_\infty\chi \longmapsto \text{Э}_\chi \in V\mathrm{D}(J^\infty), \tag{15}$$

determines a well defined isomorphism of $C^\infty(J^\infty)$-modules. The dual isomorphism is given by

$$\mathscr{C}\Lambda^1(J^\infty) \ni \omega \longmapsto \square_\omega \in \mathscr{C}\mathrm{Diff}(\varkappa, C^\infty(J^\infty)), \tag{16}$$

where $\square_\omega : \varkappa \longrightarrow C^\infty(J^\infty)$ is defined by putting $\square_\omega\chi := \omega(\text{Э}_\chi)$, $\chi \in \varkappa$. Accordingly, there is a natural embedding $\eta_\Phi : V\mathrm{D}(\mathscr{E}) \hookrightarrow \overline{J}^\infty\varkappa|_{\mathscr{E}}$ given by the composition

$$\begin{tikzcd} V\mathrm{D}(\mathscr{E}) \arrow[r, hook] \arrow[rr, bend right, "\eta_\Phi"'] & V\mathrm{D}(J^\infty)|_{\mathscr{E}} \arrow[r, "\sim"] & \overline{J}^\infty\varkappa|_{\mathscr{E}} \end{tikzcd},$$

and, dually, a natural projection $\eta^*_\Phi : \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, C^\infty(\mathscr{E})) \twoheadrightarrow \mathscr{C}\Lambda^1(\mathscr{E})$ given by the composition

$$\begin{tikzcd} \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, C^\infty(\mathscr{E})) \arrow[r, "\sim"] \arrow[rr, bend right, "\eta^*_\Phi"'] & \mathscr{C}\Lambda^1(J^\infty)|_{\mathscr{E}} \arrow[r, two heads] & \mathscr{C}\Lambda^1(\mathscr{E}) \end{tikzcd},$$

where the arrows "$\overset{\sim}{\longrightarrow}$" are the inverses of restrictions to $\mathscr{E}$ of isomorphisms (15) and (16), respectively. Finally, notice that the sequence

$$0 \longrightarrow V\mathrm{D}(\mathscr{E}) \xrightarrow{\eta_\Phi} \overline{J}^\infty\varkappa|_{\mathscr{E}} \xrightarrow{h^\infty_\Phi} \overline{J}^\infty P\ , \tag{17}$$

where $h_\Phi := h_{\ell_\Phi}$, and its dual

$$\mathscr{C}\mathrm{Diff}(P, C^\infty(\mathscr{E})) \xrightarrow{h^{\infty *}_\Phi} \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, C^\infty(\mathscr{E})) \xrightarrow{\eta^*_\Phi} \mathscr{C}\Lambda^1(\mathscr{E}) \longrightarrow 0\ , \tag{18}$$

where $h^{\infty *}_\Phi(\Delta) := \Delta \circ \ell_\Phi$, $\Delta \in \mathscr{C}\mathrm{Diff}(P, C^\infty(\mathscr{E}))$, are exact.

There exists a horizontal analogue of the concept of adjoint operator to a linear differential operator. Let R be a smooth module (see above). Put $R^\dagger := \mathrm{Hom}(R, \overline{\Lambda}^n(\mathscr{E}))$. $R^\dagger$

is a smooth module as well and it is called the *adjoint module to* R. Obviously, $R^{\dagger\dagger}$ identifies canonically with R. Denote by $R^\dagger \times R \ni (r^\dagger, r) \longmapsto \langle r^\dagger, r\rangle := r^\dagger(r) \in \overline{\Lambda}^n(\mathscr{E})$ the natural bi-linear pairing. For any local basis $\ldots, \kappa_a, \ldots$ of R we denote by $\ldots, \kappa^{\dagger a}, \ldots$ the local basis of $R^\dagger$ such that $\kappa^{\dagger a}$ is the local homomorphism $R \longrightarrow \overline{\Lambda}^n(\mathscr{E})$ defined by putting $\langle \kappa^{\dagger a}, \kappa_b\rangle := \delta^a_b \overline{\mathrm{d}}^n x$ and $\overline{\mathrm{d}}^n x := \overline{d}x^1 \wedge \cdots \wedge \overline{d}x^n$, $a, b = 1, 2, \ldots$.

Proposition 3. *Let* $r \in R$ *(resp.* $r^\dagger \in R^\dagger$*), then* $r = 0$ *(resp.* $r^\dagger = 0$*) iff* $\int \langle r^\dagger, r\rangle = 0$ *for all* $r^\dagger \in R^\dagger$ *(resp.* $r \in R$*).*

Proposition 3 may be referred to as the *cohomological DuBois-Reymond theorem* and will be used later on without further comments.

Now let P, Q be as above and $\square : P \longrightarrow Q$ a horizontal differential operator. It can be proved that there exists a unique differential operator (of the same order as $\square$) $\square^\dagger : Q^\dagger \longrightarrow P^\dagger$ such that

$$\int \langle q^\dagger, \square p\rangle = \int \langle \square^\dagger q^\dagger, p\rangle \tag{19}$$

for all $p \in P$, $q^\dagger \in Q^\dagger$. $\square^\dagger$ is called the *adjoint operator to* $\square$ and (19) is called the (horizontal) *Green formula* [9, 20, 37].

Adjoint operators have the following properties. First, $\square^{\dagger\dagger} = \square$. Second, let $\Delta : Q \longrightarrow R$ be another horizontal differential operator, then $(\Delta \circ \square)^\dagger = \square^\dagger \circ \Delta^\dagger$. If $\square$ is locally given by (14) then $\square^\dagger$ is locally given by

$$\square^\dagger q^\dagger = (-1)^{|I|} e^{\dagger a} D_I (\square^{AI}_a q^\dagger_A),$$

for all $q^\dagger = q^\dagger_A \varepsilon^{\dagger A}$ local elements of $Q^\dagger$, $\ldots, q^\dagger_A, \ldots$ local functions on $\mathscr{E}$. As an example, notice that the adjoint module of $\overline{\Lambda}^q(\mathscr{E})$ is canonically isomorphic to $\overline{\Lambda}^{n-q}(\mathscr{E})$, and that the adjoint operator of the horizontal de Rham differential $\overline{d} : \overline{\Lambda}^q(\mathscr{E}) \longrightarrow \overline{\Lambda}^{q+1}(\mathscr{E})$ is the operator $(-1)^{n-q-1}\overline{d} : \overline{\Lambda}^{n-q-1}(\mathscr{E}) \longrightarrow \overline{\Lambda}^{n-q}(\mathscr{E})$, $q = 0, \ldots, n$.

Notice that the Green formula amounts to say that for any $p \in P$, $q^\dagger \in Q^\dagger$ there exists $\mathscr{K}_{p,q^\dagger} \in \overline{\Lambda}^{n-1}(\mathscr{E})$ such that $\langle q^\dagger, \square p\rangle - \langle \square^\dagger q^\dagger, p\rangle = \overline{d}\mathscr{K}_{p,q^\dagger}$. It can be proved [1] that $\mathscr{K}_{p,q^\dagger}$ can be chosen of the form $\mathscr{K}(p, q^\dagger)$, $\mathscr{K} : P \times Q^\dagger \longrightarrow \overline{\Lambda}^{n-1}(\mathscr{E})$ being a (possibly non unique) horizontal bi-differential operator independent of p and $q^\dagger$. Any such operator $\mathscr{K}$ is called a *Legendre operator for* $\square$ [2]. The Green formula plays a central role in the theory of the $\mathscr{C}$-spectral sequence.

1.6. **Formal Theory of Horizontal PDEs and Secondary Calculus.**

There exists a horizontal analogue of the Goldschmidt-Spencer formal theory of linear differential equations (see [15, 32] for a complete account of the classical theory - see also [16] - and [21, 35] for its horizontal analogue).

Let $\Delta : P \longrightarrow P_1$ be a horizontal differential operator of order $\leq k$ between smooth modules.

Definition 4. *A complex of horizontal differential operators between smooth modules*

$$0 \longrightarrow P \xrightarrow{\Delta} P_1 \xrightarrow{\Delta_1} \cdots \longrightarrow P_q \xrightarrow{\Delta_q} P_{q+1} \xrightarrow{\Delta_{q+1}} \cdots \tag{20}$$

is called a compatibility complex for Δ *iff the sequence of homomorphisms*

$$\begin{tikzcd} 0 \arrow[r] & \overline{J}^\infty P \arrow[r, "h^\infty_\Delta"] & \overline{J}^\infty P_1 \arrow[r, "h^\infty_{\Delta_1}"] & \cdots \arrow[r] & \overline{J}^\infty P_q \arrow[r, "h^\infty_{\Delta_q}"] & \overline{J}^\infty P_{q+1} \arrow[r, "h^\infty_{\Delta_{q+1}}"] & \cdots \end{tikzcd}$$

is exact. Δ_1 *is called a* compatibility operator for Δ*.*

The existence of a non trivial compatibility operator for Δ formalizes the fact that the equation $\Delta p = 0$ is overdetermined [32]. We stress that Definition 4 is slightly different from the one usually found in the literature (see, for instance, [15, 21]). However, it can be shown that, if Δ determines a formally integrable PDE, then the two coincide, and Definition 4 is the most suitable for our purposes.

Theorem 5 (Goldschmidt)**.** *Let Δ be a horizontal differential operator between smooth modules. If Δ determines a formally integrable horizontal PDE, then there exists a (non unique) compatibility complex (20) for Δ, such that Δ_i determines a formally integrable horizontal PDE for any $i = 1, 2, \ldots$.*

Any compatibility complex as in the above theorem will be said *regular*. Let $\Delta : P \longrightarrow P_1$ determine a formally integrable PDE and (20) be a regular compatibility complex for it. Then, the compatibility operator Δ_1 has the following remarkable property.

Proposition 6. *Let $\Box : P_1 \longrightarrow Q$ be a horizontal differential operator such that $\Box \circ \Delta = 0$. Then there exists a horizontal differential operator $\nabla : P_2 \longrightarrow Q$ such that $\Box = \nabla \circ \Delta_1$. If $\Delta_2 = 0$ then ∇ is unique.*

Thus, let $\Delta : P \longrightarrow P_1$ determine a formally integrable PDE and

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "\Delta"] & P_1 \arrow[r, "\Delta_1"] & \cdots \arrow[r] & P_{s-1} \arrow[r, "\Delta_{s-1}"] & P_s \arrow[r] & 0 \end{tikzcd}$$

be a finite length regular compatibility complex. In this situation we say that *the compatibility length of Δ is $\leq s$.*

Now let $\pi : E \longrightarrow M$ be a fiber bundle, $\tau : T \longrightarrow J^\infty$ a vector bundle, $\Phi \in \mathrm{diff}(\pi, \tau)$ and $\mathscr{E} := \mathscr{E}_\Phi^{(\infty)}$. Put $P_1 := \Gamma(\tau)|_{\mathscr{E}}$. Notice that if $\mathscr{E}_\Phi$ is a formally integrable PDE, then $\ell_\Phi : \varkappa|_{\mathscr{E}} \longrightarrow P_1$ determines a formally integrable, linear, horizontal PDE [34].

Theorem 7 (Spencer)**.** *Cohomology $\mathbf{D}(\boldsymbol{M})^\bullet$ of complex $(V\mathrm{D}(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{S})$ is canonically isomorphic to cohomology of any regular compatibility complex*

$$\begin{tikzcd} 0 \arrow[r] & \varkappa|_{\mathscr{E}} \arrow[r, "\ell_\Phi"] & P_1 \arrow[r, "\Delta_1"] & \cdots \arrow[r] & P_q \arrow[r, "\Delta_q"] & P_{q+1} \arrow[r, "\Delta_{q+1}"] & \cdots \end{tikzcd}$$

for ℓ_Φ.

In the following we will only consider regular compatibility complexes.

Isomorphism $\ker \ell_\Phi \simeq \mathbf{D}(\boldsymbol{M})^0$ is given by

$$\ker \ell_\Phi \ni \chi \longmapsto \mathcal{Э}_\chi \in V\mathrm{D}_{\mathscr{C}}(\mathscr{E}) = \mathbf{D}(\boldsymbol{M})^0.$$

We now describe isomorphism $\ker \Delta_1 / \operatorname{im} \ell_\Phi \simeq \mathbf{D}(\boldsymbol{M})^1$ referring to [21] for the remaining homogeneous components. Let $p \in P_1$ be such that $\Delta_1 p = 0$. Consider $\overline{j}^\infty p \in \overline{J}^\infty P_1$. Then $h_{\Delta_1}(\overline{j}^\infty p) = \Delta_1 p = 0$ and, therefore, Δ_1 being a compatibility operator for ℓ_Φ, there exists $j \in \overline{J}^\infty \varkappa$ such that $\overline{j}^\infty p = h_\Phi(j|_{\mathscr{E}})$. Let $X := \eta_0^{-1}(j) \in V\mathrm{D}(J^\infty)$ (here 0 is the trivial differential operator) and $\widetilde{\Omega} := \overline{S}(X) \in V\mathrm{D}(J^\infty) \otimes \overline{\Lambda}^1(J^\infty)$. It is easy to prove, suitably using exactness of sequence (17), that $\widetilde{\Omega}$ restricts to $\mathscr{E}$, i.e., $\Omega := \widetilde{\Omega}|_{\mathscr{E}} \in V\mathrm{D}(\mathscr{E}) \otimes \overline{\Lambda}^1(\mathscr{E}) \subset V\mathrm{D}(J^\infty)|_{\mathscr{E}} \otimes \overline{\Lambda}^1(\mathscr{E})$. Moreover, $\overline{S}(\Omega) = 0$. Finally, the isomorphism $\ker \Delta_1 / \operatorname{im} \ell_\Phi \simeq \mathbf{D}(\boldsymbol{M})^1$ maps $p + \operatorname{im} \ell_\Phi^{\mathscr{E}} \in \ker \Delta_1 / \operatorname{im} \ell_\Phi$ to $[\Omega] \in H^1(V\mathrm{D}(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{S}) = \mathbf{D}(\boldsymbol{M})^1$.

Corollary 8. *Cohomology $\boldsymbol{\Lambda}^1(\boldsymbol{M})^\bullet$ of complex $(\mathscr{C}\Lambda^1(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d})$ is canonically isomorphic to homology of the adjoint complex*

$$\begin{tikzcd} 0 & \varkappa|_{\mathscr{E}}^\dagger \arrow[l] & P_1^\dagger \arrow[l, "\ell_\Phi^\dagger"'] & \cdots \arrow[l] & P_q^\dagger \arrow[l, "\Delta_{q-1}^\dagger"'] & P_{q+1}^\dagger \arrow[l, "\Delta_q^\dagger"'] & \cdots \arrow[l] \end{tikzcd}$$

of any regular compatibility complex for ℓ_Φ.

Isomorphism $\boldsymbol{\Lambda}^1(\boldsymbol{M})^n \simeq \operatorname{coker} \ell_\Phi^\dagger$ is described as follows. Projection $\eta_\Phi^* : \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, C^\infty(\mathscr{E})) \twoheadrightarrow \mathscr{C}\Lambda^1(\mathscr{E})$ gives rise to a projection

$$\eta_\Phi^* \otimes \mathrm{id}_{\overline{\Lambda}(\mathscr{E})} : \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, \overline{\Lambda}(\mathscr{E})) \twoheadrightarrow \mathscr{C}\Lambda^1(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E})$$

which, abusing the notation, we denote again by η_Φ^*. Thus, let $\omega \in \mathscr{C}\Lambda^1(\mathscr{E}) \otimes \overline{\Lambda}^n(\mathscr{E})$ and $\square \in \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, \overline{\Lambda}^n(\mathscr{E}))$ be such that $\eta_\Phi^*(\square) = \omega$. Consider $\square^\dagger : C^\infty(\mathscr{E}) \longrightarrow \varkappa|_{\mathscr{E}}^\dagger$. Isomorphism $\boldsymbol{\Lambda}^1(\boldsymbol{M})^n \simeq \operatorname{coker} \ell_\Phi^\dagger$ maps $[\omega] \in H^n(\mathscr{C}\Lambda^1(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d}) = \boldsymbol{\Lambda}^1(\boldsymbol{M})^n$ to $\square^\dagger 1 + \operatorname{im} \ell_\Phi^\dagger \in \operatorname{coker} \ell_\Phi^\dagger$.

We now describe isomorphism $\boldsymbol{\Lambda}^1(\boldsymbol{M})^{n-1} \simeq \ker \ell_\Phi^\dagger / \operatorname{im} \Delta_1^\dagger$ referring again to [21] for the remaining homogeneous components. Let $\omega \in \mathscr{C}\Lambda^1(\mathscr{E}) \otimes \overline{\Lambda}^{n-1}(\mathscr{E})$ and $\square \in \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, \overline{\Lambda}^n(\mathscr{E}))$ be such that $\overline{d}\omega = 0$ and $\eta_\Phi^*(\square) = \omega$. Then, it follows from exactness of sequence (18) that $\overline{d} \circ \square = \Delta \circ \ell_\Phi$ for some $\Delta \in \mathscr{C}\mathrm{Diff}(P_1, \overline{\Lambda}^n(\mathscr{E}))$. Consider $\Delta^\dagger : C^\infty(\mathscr{E}) \longrightarrow P_1^\dagger$ and put $p^\dagger := \Delta^\dagger 1 \in P_1^\dagger$. We have $\ell_\Phi^\dagger(p^\dagger) = (\ell_\Phi^\dagger \circ \Delta^\dagger)(1) = (\Delta \circ \ell_\Phi)^\dagger(1) = (\overline{d} \circ \square)^\dagger(1) = (\square^\dagger \circ \overline{d}^\dagger)(1) = (\square^\dagger \circ d)(1) = 0$. Thus $p^\dagger \in \ker \ell_\Phi^\dagger$. Isomorphism $\boldsymbol{\Lambda}^1(\boldsymbol{M})^{n-1} \simeq \ker \ell_\Phi^\dagger / \operatorname{im} \Delta_1^\dagger$ maps $[\omega] \in H^{n-1}(\mathscr{C}\Lambda^1(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}), \overline{d}) = \boldsymbol{\Lambda}^1(\boldsymbol{M})^{n-1}$ to $p^\dagger + \operatorname{im} \Delta_1^\dagger \in \ker \ell_\Phi^\dagger / \operatorname{im} \Delta_1^\dagger$.

Notice that the above corollary describes to some extent the 1-st column of the 1-st term of the $\mathscr{C}$-spectral sequence of $\mathscr{E}$. The following theorem due to Verbovetsky [35] (see also [39] for the case $s = 2$) extends it to the remaining columns.

Theorem 9 (s-lines)**.** *Let $\mathscr{E} \subset J^\infty$ be the infinite prolongation of a formally integrable PDE $\mathscr{E}_\Phi$ and let the compatibility length of ℓ_Φ be $\leq s$. Then $\mathscr{C}E_1^{p,q}(\mathscr{E}) = 0$ if $p > 0$ and $q < n - s$.*

Example 10 (empty equation)**.** *If $\Phi = 0$ then $\mathscr{E} = J^\infty$, $\ell_\Phi = 0$ and its compatibility length is* 0*. In this case $\mathbf{D}(\boldsymbol{M})^r = 0$ for $r \neq 0$ and $\mathbf{D}(\boldsymbol{M})^\bullet = \mathbf{D}(\boldsymbol{M})^0 \simeq \varkappa$. The exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{D}(\boldsymbol{M})^0 \arrow[r] & V\mathrm{D}(J^\infty) \arrow[r, "\overline{S}"] \arrow[l, bend left, "\psi"] & V\mathrm{D}(J^\infty) \otimes \overline{\Lambda}^1(J^\infty) \end{tikzcd} \tag{21}$$

splits via the composition

$$\begin{tikzcd} V\mathrm{D}(J^\infty) \arrow[r, "\eta_0"] \arrow[rrr, bend right, "\psi"'] & \overline{J}^\infty(\varkappa) \arrow[r, two heads] & \varkappa \arrow[r, dash, "\sim"] & \mathbf{D}(\boldsymbol{M})^0 \end{tikzcd} . \tag{22}$$

Similarly, $\boldsymbol{\Lambda}^1(\boldsymbol{M})^q = 0$ for $q \neq n$ and $\boldsymbol{\Lambda}^1(\boldsymbol{M})^\bullet = \boldsymbol{\Lambda}^1(\boldsymbol{M})^n \simeq \varkappa^\dagger$. The exact sequence

$$\begin{tikzcd} \mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^{n-1}(J^\infty) \arrow[r, "\overline{d}"] & \mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^n(J^\infty) \arrow[r] & \boldsymbol{\Lambda}^1(\boldsymbol{M})^n \arrow[r] \arrow[l, bend left, "\psi^\dagger"] & 0 \end{tikzcd}$$

splits via the composition

$$\begin{tikzcd} \boldsymbol{\Lambda}^1(\boldsymbol{M})^n \arrow[r, dash, "\sim"] \arrow[rrr, bend right, "\psi^\dagger"'] & \varkappa^\dagger \arrow[r, hook] & \mathscr{C}\mathrm{Diff}(\varkappa, \overline{\Lambda}^n) \arrow[r, "\eta_0^*"] & \mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^n \end{tikzcd} .$$

Let $Y \in V\mathrm{D}(J^\infty)$ and $\varphi \in \varkappa^\dagger$ be locally given by $Y = Y^\alpha_I \partial^I_\alpha$ and $\varphi = \varphi_\alpha \partial^{\dagger\alpha}$, $\ldots, Y^\alpha_I, \ldots, \varphi_\alpha, \ldots$ being local functions on J^∞. Then, locally,

$$\psi(Y) = Y^\alpha_{\mathsf{O}} \partial_\alpha \quad \textit{and} \quad \psi^\dagger(\varphi) = \varphi_\alpha \omega^\alpha_{\mathsf{O}} \otimes \overline{\mathrm{d}}^n x.$$

Finally, notice that both diagrams (21) and (22) restrict to the infinite prolongation of a PDE and such restrictions preserve the exactness.

In the following we will understand the above isomorphisms $\mathbf{D}(\boldsymbol{M})^0 \simeq \varkappa$ and $\boldsymbol{\Lambda}^1(\boldsymbol{M})^n \simeq \varkappa^\dagger$. In order not to make the notation to heavy we will also understand the monomorphism $\psi^\dagger$. According to this convention $\varkappa^\dagger$ is understood as a subset in $\mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^n(J^\infty)$. Moreover, if $\varphi \in \varkappa^\dagger$ and $\chi \in \varkappa$, then $i_{\partial_\chi} \psi^\dagger(\varphi) \in \overline{\Lambda}^n(J^\infty)$ identifies with $\langle \varphi, \chi \rangle$.

Example 11 (irreducible equations)**.** *A non-empty PDE $\mathscr{E}_\Phi$ is called ℓ-*normal *(or, in physical terms,* irreducible*) iff the compatibility length of ℓ_Φ is ≤ 1. In this case Δ_1 may be chosen equal to* 0*, $\mathbf{D}(\boldsymbol{M})^r = 0$ for $r \neq 0, 1$, $\mathbf{D}(\boldsymbol{M})^0 \simeq \ker \ell_\Phi$ as above and*

$$\mathbf{D}(\boldsymbol{M})^1 \simeq \operatorname{coker} \ell_\Phi.$$

Similarly, $\boldsymbol{\Lambda}^1(\boldsymbol{M})^q = 0$ for $q \neq n, n-1$, $\boldsymbol{\Lambda}^1(\boldsymbol{M})^n \simeq \operatorname{coker} \ell^\dagger_\Phi$ as above and

$$\boldsymbol{\Lambda}^1(\boldsymbol{M})^{n-1} \simeq \ker \ell^\dagger_\Phi.$$

2. The Covariant Phase Space

2.1. **Lagrangian Field Theories and the CPS.** The calculus of variations is formalized in a coordinate-free way via the $\mathscr{C}$-spectral sequence.

Definition 12. *A* lagrangian (field) theory *is the datum* $(\pi, \boldsymbol{S})$ *of a fiber bundle* $\pi : E \longrightarrow M$ *and an* action $\boldsymbol{S} \in \overline{H}^n(J^\infty)$*. Any representative* $\mathscr{L} \in \overline{\Lambda}^n(J^\infty)$ *of the cohomology class* $\boldsymbol{S} = \int \mathscr{L}$ *is called a* lagrangian density *of the theory* $(\pi, \boldsymbol{S})$*.*

Recall that the space $\boldsymbol{M}$ of n-dimensional integral submanifolds of the Cartan distribution $\mathscr{C}$ on J^∞ is in one-to-one correspondence (via infinite jet prolongation) with the space of local sections of π. As above we will often identify $\boldsymbol{M}$ with the "store" $(J^\infty, \mathscr{C})$ of its elements. $\boldsymbol{M}$ is known in the Physics literature as the *space of histories* and an action $\boldsymbol{S} \in \overline{H}^n(J^\infty) \subset \boldsymbol{C}^\infty(\boldsymbol{M})^\bullet$ is a secondary function on it.

Within secondary calculus, the Euler-Lagrange equations (whose solutions make it stationary the action) associated to the lagrangian theory $(\pi, \boldsymbol{S})$ are easily obtained by applying to $\boldsymbol{S}$ the secondary de Rham differential $\boldsymbol{d} : \boldsymbol{C}^\infty(\boldsymbol{M})^\bullet \longrightarrow \boldsymbol{\Lambda}^1(\boldsymbol{M})^\bullet$. Indeed, according to the previous section, $\boldsymbol{\Lambda}^1(\boldsymbol{M})^\bullet = \boldsymbol{\Lambda}^1(\boldsymbol{M})^n \simeq \varkappa^\dagger$ and $\boldsymbol{dS}$ identifies with the element $\boldsymbol{E}(\mathscr{L}) := \tilde{\ell}^\dagger_{\mathscr{L}} 1 \in \varkappa^\dagger \subset \mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^n(J^\infty)$, where we put $\tilde{\ell}_{\mathscr{L}} := (\eta_0^*)^{-1}(d^V \mathscr{L}) : \varkappa \longrightarrow \overline{\Lambda}^n(J^\infty)$, $\mathscr{L}$ being any lagrangian density. Locally, $\mathscr{L} = L\, \overline{\mathrm{d}}^n x$ for some local function $L = L(\ldots, x^i, \ldots, u^\alpha_I, \ldots)$ on J^∞ and

$$\boldsymbol{E}(\mathscr{L}) = \tfrac{\delta L}{\delta u^\alpha} \partial^{\dagger\alpha},$$

$\frac{\delta L}{\delta u^\alpha} := (-1)^{|I|} D_I(\partial^I_\alpha L)$ being the so-called *Euler-Lagrange derivatives* of L, $\alpha = 1, \ldots, m$. Thus $\boldsymbol{dS}$ is naturally interpreted as the left hand side of the Euler-Lagrange equations $\mathscr{E}_{\boldsymbol{E}(\mathscr{L})}$ of the theory $(\pi, \boldsymbol{S})$. In the following we will always assume $\mathscr{E}_{\boldsymbol{E}(\mathscr{L})}$ to be a formally integrable PDE.

Let $\mathscr{E} := \mathscr{E}^{(\infty)}_{\boldsymbol{E}(\mathscr{L})}$. The space $\boldsymbol{P}$ of n-dimensional integral submanifolds of the Cartan distribution on $\mathscr{E}$ is in one-to-one correspondence with the space of (local) solutions of $\mathscr{E}_{\boldsymbol{E}(\mathscr{L})}$ and is called, according to Physics literature, the (non-reduced) *CPS* of the theory $(\pi, \boldsymbol{S})$ [8, 4, 11, 12, 25].

By definition

$$d^V \mathscr{L} - \boldsymbol{E}(\mathscr{L}) = \overline{d}\theta \tag{23}$$

for some $\theta \in \mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^{n-1}(J^\infty)$. Any such θ will be called a *Legendre form* [2] (notice that $\mathscr{L} - \theta$ is a so-called *lepagean equivalent* [22, 23] of $\mathscr{L}$). Equation (23) may be interpreted as the *first variation formula* for the action $\boldsymbol{S}$. In this respect, the existence of a global Legendre form was first discussed in [24]. Any two Legendre forms $\theta.\theta'$ differ by a closed, and therefore exact, form $\overline{d}\lambda$, $\lambda \in \mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^{n-2}(J^\infty)$ (see, for instance, [2, 25] for a local description of Legendre forms). Notice that, in view of isomorphism η_0^*, identity (23) may be understood as the Green formula

$$\tilde{\ell}_{\mathscr{L}} - \tilde{\ell}^\dagger_{\mathscr{L}} 1 = (\overline{d} \circ \mathscr{K})(\,\cdot\,, 1)$$

for the horizontal operator $\tilde{\ell}_{\mathscr{L}}$, $\mathscr{K}$ being a Legendre operator for it.

Theorem 13 (Zuckerman)**.** *There is a closed secondary* 2*-form* $\boldsymbol{\omega}$ *on* $\boldsymbol{P}$ *canonically determined by the corresponding lagrangian theory* $(\pi, \boldsymbol{S})$*.*

Proof. Using the language introduced so far we reproduce here the proof in [40] by adding the only missing point, that is the independence of $\boldsymbol{\omega}$ of the choice of a lagrangian density. Thus, let θ be a Legendre form. Put

$$\omega := -i_{\mathscr{E}}^{*}(d^{V}\theta) \in \mathscr{C}\Lambda^{2}(\mathscr{E}) \otimes \overline{\Lambda}^{n-1}(\mathscr{E}).$$

Then

$$\begin{aligned} \overline{d}\omega &= -\overline{d}i_{\mathscr{E}}^{*}(d^{V}\theta) \\ &= i_{\mathscr{E}}^{*}(d^{V}\overline{d}\theta) \\ &= i_{\mathscr{E}}^{*}(d^{V}(d^{V}\mathscr{L} - \boldsymbol{E}(\mathscr{L}))) \\ &= -d^{V}i_{\mathscr{E}}^{*}(\boldsymbol{E}(\mathscr{L})) \\ &= 0. \end{aligned}$$

Since ω is $\overline{d}$-closed we may take its cohomology class $\boldsymbol{\omega} := [\omega] \in \boldsymbol{\Lambda}^{2}(\boldsymbol{P})^{n-1}$. Now, $\boldsymbol{\omega}$ is canonical, as proved in what follows.

1. $\boldsymbol{\omega}$ does not depend on the choice of θ. Indeed, let $\theta' := \theta + \overline{d}\lambda$ be another Legendre form, $\lambda \in \mathscr{C}\Lambda^{1}(J^{\infty}) \otimes \overline{\Lambda}^{n-2}(J^{\infty})$ and $\omega' := -i_{\mathscr{E}}^{*}(d^{V}\theta')$. Then $\omega' = -i_{\mathscr{E}}^{*}(d^{V}\theta + d^{V}\overline{d}\lambda) = \omega + \overline{d}i_{\mathscr{E}}^{*}(d^{V}\lambda)$, so that $[\omega] = [\omega']$.
2. $\boldsymbol{\omega}$ does not depend on the choice of $\mathscr{L}$. Indeed, let $\mathscr{L}$ be a trivial lagrangian density, i.e., $\mathscr{L} = \overline{d}\nu$ for some $\nu \in \overline{\Lambda}^{n-1}(J^{\infty})$. Then $\boldsymbol{S} = 0$, $\boldsymbol{E}(\mathscr{L}) = 0$ and $d^{V}\mathscr{L} - \boldsymbol{E}(\mathscr{L}) = -\overline{d}d^{V}\nu$. This proves that $-d^{V}\nu$ is a Legendre form, so that $\boldsymbol{\omega} = [i_{\mathscr{E}}^{*}(d^{V}d^{V}\nu)] = 0$.

Finally, $\boldsymbol{d\omega} = [d^{V}\omega] = 0$. □

Notice that the above theorem can be generalized to the case of a lagrangian field theory subject to constraints in the form of (the infinite prolongation of) a PDE $\mathscr{F} \subset J^{\infty}$, under suitable cohomological conditions on $\mathscr{F}$. Constrained lagrangian theories will be considered somewhere else.

A general coordinate formula for ω may be found, for instance, in [25]. The expression of ω for specific lagrangian theories may be found, for instance, in [11, 12, 18, 25, 30]. However, we stress that, in general, there is no distinguished representative ω in $\boldsymbol{\omega}$.

2.2. "Symplectic Version" of I Noether Theorem.

Let $(\pi, \boldsymbol{S})$ be a lagrangian field theory and $\chi \in \varkappa$ the generating section of an higher symmetry of π. In view of isomorphism $\varkappa \simeq \mathbf{D}(\boldsymbol{M})^{\bullet}$, χ may be understood as a secondary vector field on $\boldsymbol{M}$. By definition, χ is a *Noether symmetry* of $(\pi, \boldsymbol{S})$ iff $\mathcal{L}_{\chi}\boldsymbol{S} = 0$, or, which is the same, $i_{\chi}\boldsymbol{dS} = 0$. In terms of a lagrangian density $\mathscr{L}$ the last equality reads as $i_{\ni_{\chi}}d^{V}\mathscr{L} = \overline{d}\sigma$ for some $\sigma \in \overline{\Lambda}^{n-1}(J^{\infty})$. Using (23) one gets $i_{\ni_{\chi}}(\boldsymbol{E}(\mathscr{L}) + \overline{d}\theta) = \overline{d}\sigma$. In view of isomorphism η_0^*, this implies $\overline{d}(\sigma - i_{\ni_\chi}\theta) = \langle \boldsymbol{E}(\mathscr{L}), \chi\rangle$ and, pulling-back to $\mathscr{E}$,

$$\overline{d} i^*_{\mathscr{E}}(\sigma - i_{\ni_\chi}\theta) = 0.$$

We have thus shown that $j := i^*_{\mathscr{E}}(\sigma - i_{\ni_\chi}\theta) \in \overline{\Lambda}^{n-1}(\mathscr{E})$ is a conserved current of $\mathscr{E}$ and this is, basically, the content of the *first Noether theorem.* Any such conserved current is called a *Noether current* of $(\pi, \boldsymbol{S})$. The associated conservation law $\boldsymbol{f} := [j] \in \overline{H}^{n-1}(\mathscr{E}) \subset \boldsymbol{C}^\infty(\boldsymbol{P})^\bullet$ is called a *Noether charge.* Notice that nor j nor $\boldsymbol{f}$ are uniquely determined by χ in general.

It is well known that if $\chi \in \varkappa$ is a Noether symmetry of $(\pi, \boldsymbol{S})$, then $\chi|_{\mathscr{E}} \in \varkappa|_{\mathscr{E}}$ is the generating section of a symmetry of $\mathscr{E}$, i.e., $\ell_{\boldsymbol{E}(\mathscr{L})}\chi|_{\mathscr{E}} = 0$. This can be easily proved by means of the following useful

Lemma 14. *Let* $\varphi \in \varkappa^\dagger$, $\mathscr{F} := \mathscr{E}_\varphi^{(\infty)} \subset J^\infty$. *For any* $\chi \in \varkappa$,

$$(\boldsymbol{\mathcal{L}}_{\ni_\chi}\varphi)|_{\mathscr{F}} = \ell_\varphi \overline{\chi}, \quad \overline{\chi} := \chi|_{\mathscr{F}}$$

In particular, $(\boldsymbol{\mathcal{L}}_{\ni_\chi}\varphi)|_{\mathscr{F}} \in \varkappa|^\dagger_{\mathscr{E}}$ *and it does only depend on the values of* χ *on* $\mathscr{F}$.

Proof. For any $\chi_1 \in \varkappa$, put $\overline{\chi}_1 := \chi_1|_{\mathscr{F}}$. Similarly, for a (local) function f on J^∞, put $\overline{f} := f|_{\mathscr{F}}$. Compute

$$\begin{aligned} i_{\ni_{\chi_1}}\boldsymbol{\mathcal{L}}_{\ni_\chi}\varphi &= i_{[\ni_{\chi_1}, \ni_\chi]}\varphi + \boldsymbol{\mathcal{L}}_{\ni_\chi} i_{\ni_{\chi_1}}\varphi \\ &= i_{\ni_{\{\chi,\chi_1\}}}\varphi + \boldsymbol{\mathcal{L}}_{\ni_\chi}\langle \varphi, \chi_1\rangle \\ &= \langle \varphi, \{\chi, \chi_1\}\rangle + \boldsymbol{\mathcal{L}}_{\ni_\chi}\langle \varphi, \chi_1\rangle. \end{aligned}$$

Since $\varphi|_{\mathscr{F}} = 0$, we have

$$(i_{\ni_{\chi_1}}\boldsymbol{\mathcal{L}}_{\ni_\chi}\varphi)|_{\mathscr{F}} = (\boldsymbol{\mathcal{L}}_{\ni_\chi}\langle \varphi, \chi_1\rangle)|_{\mathscr{F}}. \tag{24}$$

Now, let φ, χ and χ_1 be locally given by $\varphi = \varphi_\alpha \partial^{\dagger\alpha}$, $\chi = \chi^\beta \partial_\beta$, $\chi_1 = \chi_1^\gamma \partial_\gamma$, $\ldots, \varphi_\alpha, \ldots, \chi^\beta, \ldots, \chi_1^\gamma, \ldots$ local functions on J^∞. Then locally,

$$\boldsymbol{\mathcal{L}}_{\ni_\chi}\langle \varphi, \chi_1\rangle = D_I\chi^\beta \partial^I_\beta(\varphi_\alpha \chi_1^\alpha)\overline{d}^n x = [D_I\chi^\beta(\partial^I_\beta\varphi_\alpha)\chi_1^\alpha + D_I\chi^\beta(\partial^I_\beta\chi_1^\alpha)\varphi_\alpha]\overline{d}^n x.$$

Since $\varphi_\alpha|_{\mathscr{F}} = 0$, $\alpha = 1, \ldots, m$, we have locally

$$(\boldsymbol{\mathcal{L}}_{\ni_\chi}\langle \varphi, \chi_1\rangle)|_{\mathscr{F}} = D_I^{\mathscr{F}}\overline{\chi}^\beta \overline{(\partial^I_\beta\varphi_\alpha)}\,\overline{\chi}_1^\alpha \overline{d}^n x = \langle \ell_\varphi\overline{\chi}, \overline{\chi}_1\rangle. \tag{25}$$

Using (25) into (24) we get

$$i^{\mathscr{F}}_{\ni_{\chi_1}}(\boldsymbol{\mathcal{L}}_{\ni_\chi}\varphi)|_{\mathscr{F}} = (i_{\ni_{\chi_1}}\boldsymbol{\mathcal{L}}_{\ni_\chi}\varphi)|_{\mathscr{F}} = \langle \ell_\varphi\overline{\chi}, \overline{\chi}_1\rangle = i^{\mathscr{F}}_{\ni_{\chi_1}}\ell_\varphi\overline{\chi},$$

where $i^{\mathscr{F}}_{\ni_{\chi_1}}$ is the restriction to $\mathscr{F}$ of the operator $i_{\ni_{\chi_1}}$ (see Section 1.5). From the arbitrariness of χ_1 the result follows. □

Now, let $\chi \in \varkappa$ be a Noether symmetry of the lagrangian theory $(\pi, \boldsymbol{S})$, and $\mathscr{L}$ a lagrangian density. Then, in view of Lemma 14,

$$\ell_{\boldsymbol{E}(\mathscr{L})}\chi|_{\mathscr{E}} = (\boldsymbol{\mathcal{L}}_{\ni_\chi}\boldsymbol{E}(\mathscr{L}))|_{\mathscr{E}}, \tag{26}$$

and

$$\mathcal{L}_{\ni_\chi}\boldsymbol{E}(\mathscr{L}) = \mathcal{L}_{\ni_\chi}(d^V\mathscr{L} - \overline{d}\theta) = d^V(i_{\ni_\chi}d^V\mathscr{L}) + \overline{d}(\mathcal{L}_{\ni_\chi}\theta) = \overline{d}(\mathcal{L}_{\ni_\chi}\theta - d^V\sigma).$$

This shows that the horizontal cohomology class $[\mathcal{L}_{\ni_\chi}\boldsymbol{E}(\mathscr{L})] \in \boldsymbol{\Lambda}^1(\boldsymbol{M})^n \simeq \varkappa^\dagger$ is zero (and so is its "restriction" to $\mathscr{E}$) and, therefore, $\ell_{\boldsymbol{E}(\mathscr{L})}\chi|_{\mathscr{E}} = 0$ (see the final comment in Example 10).

The above remark proves that if χ is a Noether symmetry, then $\boldsymbol{X} := \chi|_{\mathscr{E}} \in \ker \ell_{\boldsymbol{E}(\mathscr{L})} \simeq \mathbf{D}(\boldsymbol{P})^0$ is a secondary vector field on $\boldsymbol{P}$. Let $\boldsymbol{f} \in \mathbf{C}^\infty(\boldsymbol{P})^{n-1}$ be, as above, a Noether charge associated to χ.

Proposition 15. $\boldsymbol{df} = -i_{\boldsymbol{X}}\boldsymbol{\omega}$ *(see Equation 22 in [25]).*

Proof. Let j, σ, θ and ω be as above. Then

$$\begin{aligned}
\boldsymbol{df} &= [d^V j] \\
&= [d^V i^*_{\mathscr{E}}(\sigma - i_{\ni_\chi}\theta)] \\
&= [i^*_{\mathscr{E}}(d^V\sigma - d^V i_{\ni_\chi}\theta)] \\
&= [i^*_{\mathscr{E}}(d^V\sigma - \mathcal{L}_{\ni_\chi}\theta + i_{\ni_\chi}d^V\theta)] \\
&= [i^*_{\mathscr{E}}(d^V\sigma - \mathcal{L}_{\ni_\chi}\theta) + i_{\ni_{\chi|_{\mathscr{E}}}}i^*_{\mathscr{E}}(d^V\theta)] \\
&= [i^*_{\mathscr{E}}(d^V\sigma - \mathcal{L}_{\ni_\chi}\theta) - i_{\ni_{\chi|_{\mathscr{E}}}}\omega] \\
&= i^*_{\mathscr{E}}[d^V\sigma - \mathcal{L}_{\ni_\chi}\theta] - i_{\boldsymbol{X}}\boldsymbol{\omega},
\end{aligned}$$

where we used that $\overline{d}(d^V\sigma - \mathcal{L}_{\ni_\chi}\theta) = 0$. Now, $d^V\sigma - \mathcal{L}_{\ni_\chi}\theta \in \mathscr{C}\Lambda^1(J^\infty) \otimes \overline{\Lambda}^{n-1}(J^\infty)$ and $[d^V\sigma - \mathcal{L}_{\ni_\chi}\theta] \in \boldsymbol{\Lambda}^1(\boldsymbol{M})^{n-1}$. But, according to Example 10, $\boldsymbol{\Lambda}^1(\boldsymbol{M})^{n-1} = 0$. We conclude that $\boldsymbol{df} = -i_{\boldsymbol{X}}\boldsymbol{\omega}$. □

Notice that Proposition 15 resembles very closely the analogous result in hamiltonian mechanics. Moreover, if $\mathscr{E}_{\boldsymbol{E}(\mathscr{L})}$ is an irreducible equation then $\boldsymbol{d} : \boldsymbol{C}^\infty(\boldsymbol{P})^{n-1} \longrightarrow \boldsymbol{\Lambda}^1(\boldsymbol{M})^{n-1}$ is injective [21, 39] modulo obstructions in $H^{n-1}(E) \subset \overline{H}^{n-1}(\mathscr{E})$. Thus, $\boldsymbol{df}$ determines the "*non-trivial conservation law*" (see [9]) $\boldsymbol{f} + H^{n-1}(E) \in \overline{H}^{n-1}(\mathscr{E})/H^{n-1}(E)$ and is interpreted as the *generating section* of it. Proposition 15 can be then understood as a way to compute the generating section of the non-trivial conservation law associated to a Noether symmetry.

2.3. "Symplectic Version" of (Infinitesimal) II Noether Theorem.

First of all, recall that the operator $\ell_{\boldsymbol{E}(\mathscr{L})} : \varkappa|_{\mathscr{E}} \longrightarrow \varkappa|^\dagger_{\mathscr{E}}$ is self-adjoint, i.e., $\ell_{\boldsymbol{E}(\mathscr{L})} = \ell^\dagger_{\boldsymbol{E}(\mathscr{L})} : \varkappa|_{\mathscr{E}} \longrightarrow \varkappa|^\dagger_{\mathscr{E}}$. This fact is key in the calculus of variations [39] and will be crucial in what follows (for a proof see, for instance, [9, 39] - see also [3] for an alternative approach).

The usual definition of (infinitesimal) gauge symmetries of a lagrangian field theory is the following (see [25]).

Definition 16. *A* Noether gauge *(or* local*)* symmetry *of the lagrangian theory* $(\pi, \boldsymbol{S})$ *is a horizontal linear differential operator* $G : Q \longrightarrow \varkappa$ *such that* $G(\varepsilon)$ *is a Noether symmetry for any* $\varepsilon \in Q$*.*

We added the prefix "Noether" in the above definition of a "gauge symmetry" to distinguish it from an alternative (and, generally, inequivalent) definition that will be proposed below. Physicists say sometimes that G is a *Noether symmetry depending on the arbitrary parameters* ε.

The second Noether theorem states that, in presence of gauge symmetries, there are relations among the Euler-Lagrange equations. Namely, for all $\varepsilon \in Q$,

$$\begin{aligned} 0 &= \mathcal{L}_{G(\varepsilon)} \boldsymbol{S} \\ &= i_{G(\varepsilon)} \boldsymbol{dS} \\ &= \int i_{\ni_{G(\varepsilon)}} d^V \mathscr{L} \\ &= \int \tilde{\ell}_{\mathscr{L}}(G(\varepsilon)) \\ &= \int \langle 1, (\tilde{\ell}_{\mathscr{L}} \circ G)(\varepsilon) \rangle \\ &= \int \langle (\tilde{\ell}_{\mathscr{L}} \circ G)^\dagger(1), \varepsilon \rangle \\ &= \int \langle G^\dagger(\tilde{\ell}^\dagger_{\mathscr{L}} 1), \varepsilon \rangle \\ &= \int \langle G^\dagger(\boldsymbol{E}(\mathscr{L})), \varepsilon \rangle, \end{aligned}$$

and it follows from the arbitrariness of ε that $G^\dagger(\boldsymbol{E}(\mathscr{L})) = 0$. These relations are traditionally called *Noether identities*.

An "infinitesimal version" of the second Noether theorem can be formulated. First of all, notice that, since $G(\varepsilon)$ is a Noether symmetry (so that $G(\varepsilon)|_{\mathscr{E}}$ is the generating section of a symmetry of $\mathscr{E}$) for all ε, one also has $0 = \ell_{\boldsymbol{E}(\mathscr{L})} G(\varepsilon)|_{\mathscr{E}} = (\ell_{\boldsymbol{E}(\mathscr{L})} \circ G^{\mathscr{E}})(\varepsilon|_{\mathscr{E}})$ and, from the arbitrariness of ε,

$$\ell_{\boldsymbol{E}(\mathscr{L})} \circ G^{\mathscr{E}} = 0. \tag{27}$$

Identity (27) may be interpreted by saying that the linearized Euler-Lagrange equations admit "*gauge symmetries*". Indeed, if $\chi \in \varkappa|_{\mathscr{E}}$ is in the kernel of $\ell_{\boldsymbol{E}(\mathscr{L})}$ so is the "*gauge transformed*" element $\chi + G^{\mathscr{E}}(\epsilon)$, for any arbitrary $\epsilon \in Q|_{\mathscr{E}}$. In particular, the linearized Euler-Lagrange equations are, in a sense, "underdetermined".

By passing to the adjoint operators in (27) and using the self-adjointness of $\ell_{\boldsymbol{E}(\mathscr{L})}$ we get

$$(G^{\mathscr{E}})^\dagger \circ \ell_{\boldsymbol{E}(\mathscr{L})} = 0. \tag{28}$$

This shows that there are relations among the linearized Euler-Lagrange equations and that they are, in a sense, "constrained". Thus, "*infinitesimal gauge symmetries correspond to infinitesimal constraints*" via adjunction [25]. Identities (28) (and sometimes the operator $(G^{\mathscr{E}})^{\dagger}$ itself) are called *infinitesimal Noether identities.*

Now let $\Delta_1 : \varkappa|^{\dagger}_{\mathscr{E}} \longrightarrow P_2$ be a compatibility operator for $\ell_{\boldsymbol{E}(\mathscr{L})}$. Consider also the adjoint operator $\Delta_1^{\dagger} : P_2^{\dagger} \longrightarrow \varkappa|^{\dagger}_{\mathscr{E}}$. In particular, $\Delta_1 \circ \ell_{\boldsymbol{E}(\mathscr{L})} = 0$ and (using again the self-adjointness of $\ell_{\boldsymbol{E}(\mathscr{L})}$) $\ell_{\boldsymbol{E}(\mathscr{L})} \circ \Delta_1^{\dagger} = 0$. In view of the last identity, if $\chi \in \varkappa|_{\mathscr{E}}$ is in the kernel of $\ell_{\boldsymbol{E}(\mathscr{L})}$ so is the element $\chi + \Delta_1^{\dagger}\vartheta$, for any arbitrary $\vartheta \in P_2^{\dagger}$. Notice also that, in view of Proposition 6, all infinitesimal Noether identities $(G^{\mathscr{E}})^{\dagger}$ "are generated by Δ_1" in the sense that $(G^{\mathscr{E}})^{\dagger} = \nabla \circ \Delta_1$ for some horizontal differential operator $\nabla : P_2 \longrightarrow Q|_{\mathscr{E}}$. Similarly, by passing to the adjoint operators, we see that all *infinitesimal gauge symmetries* $G^{\mathscr{E}}$ are generated by $\Delta_1^{\dagger}$, i.e., $G^{\mathscr{E}} = \Delta_1^{\dagger} \circ \nabla^{\dagger}$ for some $\nabla^{\dagger} : Q|_{\mathscr{E}} \longrightarrow P_2^{\dagger}$. These simple remarks suggest a more natural definition of infinitesimal gauge symmetries.

Definition 17. *A* gauge symmetry *of the lagrangian theory* $(\pi, \boldsymbol{S})$ *is an element in the image of the adjoint operator* $\Delta_1^{\dagger}$ *of a compatibility operator* Δ_1 *for* $\ell_{\boldsymbol{E}(\mathscr{L})}$.

We will sometimes denote by $\mathfrak{g} := \operatorname{im} \Delta_1^{\dagger}$ the set of gauge symmetries. Notice that, in view of Theorem 7, the above definition is independent of the choice of Δ_1. Moreover, while it is clear that $\operatorname{im} G^{\mathscr{E}} \subset \mathfrak{g}$ for any Noether gauge symmetry G, to the author knowledge it has not been determined yet in full rigour and generality if $\mathfrak{g}$ is generated by the images of Noether gauge symmetries or not. Therefore, we prefer to adopt definition 17. This choice is strengthened even more by the results presented in the remaining part of this section.

Consider the natural $\mathbb{R}$-linear map

$$\boldsymbol{\Omega} : \mathbf{D}(\boldsymbol{P})^{\bullet} \ni \boldsymbol{X} \longmapsto \boldsymbol{\Omega}(\boldsymbol{X}) := i_{\boldsymbol{X}}\boldsymbol{\omega} \in \boldsymbol{\Lambda}^1(\boldsymbol{P})^{\bullet}.$$

Definition 18. *The kernel* $\ker \boldsymbol{\Omega} \subset \mathbf{D}(\boldsymbol{P})^{\bullet}$ *is called the* degeneracy distribution *of* $\boldsymbol{\omega}$ *and will be also denoted by* $\ker \boldsymbol{\omega}$. *The secondary 2-form* $\boldsymbol{\omega}$ *is said to be 1)* weakly symplectic *(or* non-degenerate*) iff* $\ker \boldsymbol{\omega} = 0$*, 2)* strongly symplectic *(or, simply,* symplectic*) iff* $\boldsymbol{\Omega}$ *is an isomorphism.*

In order to better characterize $\boldsymbol{\omega}$ it is desirable to describe its degeneracy distribution.

First of all, notice that, since $\boldsymbol{\omega}$ is closed, $\ker \boldsymbol{\omega}$ *is a secondary involutive distribution*, i.e., it is a graded Lie subalgebra in $\mathbf{D}(\boldsymbol{P})^{\bullet}$. Indeed, let $\boldsymbol{X}, \boldsymbol{Y} \in \ker \boldsymbol{\omega}$ then

$$\boldsymbol{\Omega}([\boldsymbol{X}, \boldsymbol{Y}]) = i_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{\omega} = [i_{\boldsymbol{X}}, \mathcal{L}_{\boldsymbol{Y}}]\boldsymbol{\omega} = [i_{\boldsymbol{X}}, [i_{\boldsymbol{Y}}, \boldsymbol{d}]]\boldsymbol{\omega} = 0,$$

i.e., $[\boldsymbol{X}, \boldsymbol{Y}] \in \ker \boldsymbol{\omega}$.

Denote by $\boldsymbol{\Omega}^r : \mathbf{D}(\boldsymbol{P})^r \longrightarrow \boldsymbol{\Lambda}^1(\boldsymbol{P})^{r+n-1}$ the restriction of $\boldsymbol{\Omega}$ to $\mathbf{D}(\boldsymbol{P})^r$, $r = 0, \ldots, n$. Obviously, $\boldsymbol{\Omega}^r = 0$ for $r > 1$, independently of the lagrangian theory. For this reason, every degree $r > 1$ secondary vector field over $\boldsymbol{P}$ is said to be a *trivial element in* $\ker \boldsymbol{\omega}$. Thus, non-trivial elements in $\ker \boldsymbol{\omega}$ must be searched in $\mathbf{D}(\boldsymbol{P})^0$ and $\mathbf{D}(\boldsymbol{P})^1$. In the following we will "describe" such elements. Put $\operatorname{tker} \boldsymbol{\omega} := \ker \boldsymbol{\omega} \cap \bigoplus_{r>1} \mathbf{D}(\boldsymbol{P})^r$.

Theorem 19. *Diagrams*

$$\begin{tikzcd}
& & \mathbf{D}(\boldsymbol{P})^0 \arrow[r, "\boldsymbol{\Omega}^0"] & \boldsymbol{\Lambda}^1(\boldsymbol{P})^{n-1} \arrow[d, "\wr"] & \\
0 \arrow[r] & \operatorname{im}\Delta_1^\dagger \arrow[r, hook] & \ker \ell_{\boldsymbol{E}(\mathscr{L})} \arrow[u, "\wr"'] \arrow[r, two heads] & \ker \ell_{\boldsymbol{E}(\mathscr{L})} / \operatorname{im}\Delta_1^\dagger \arrow[r] & 0
\end{tikzcd} \tag{29}$$

and

$$\begin{tikzcd}
& & \mathbf{D}(\boldsymbol{P})^1 \arrow[r, "\boldsymbol{\Omega}^1"] & \boldsymbol{\Lambda}^1(\boldsymbol{P})^{n} \arrow[d, "\wr"] & \\
0 \arrow[r] & \ker \Delta_1 / \operatorname{im} \ell_{\boldsymbol{E}(\mathscr{L})} \arrow[r, hook] & \operatorname{coker} \ell_{\boldsymbol{E}(\mathscr{L})} \arrow[u, "\wr"'] \arrow[r, two heads] & \varkappa|^\dagger_{\mathscr{E}} / \ker \Delta_1 \arrow[r] & 0
\end{tikzcd} \tag{30}$$

commute.

Proof. The vertical arrows in Diagram (29) are described in Section 1.6. Thus, let $\boldsymbol{X} = \text{Э}_\chi \in \mathbf{D}(\boldsymbol{P})^0$, $\chi \in \varkappa|_{\mathscr{E}}$, $\ell_{\boldsymbol{E}(\mathscr{L})}\chi = 0$. Let $\tilde{\chi} \in \varkappa$ be such that $\tilde{\chi}|_{\mathscr{E}} = \chi$. Now, $\boldsymbol{\Omega}^0(\boldsymbol{X}) = i_{\boldsymbol{X}}\boldsymbol{\omega} = [i^*_{\mathscr{E}}(-i_{\text{Э}_{\tilde{\chi}}} d^V \theta)]$, θ being a Legendre form. Put $\tilde{\Box} := (\eta_0^*)^{-1}(-i_{\text{Э}_{\tilde{\chi}}} d^V\theta) \in \mathscr{C}\mathrm{Diff}(\varkappa, \overline{\Lambda}^{n-1}(J^\infty))$ and $\Box := \tilde{\Box}^{\mathscr{E}} \in \mathscr{C}\mathrm{Diff}(\varkappa|_{\mathscr{E}}, \overline{\Lambda}^{n-1}(\mathscr{E}))$. Then, obviously, $\eta^*_{\boldsymbol{E}(\mathscr{L})}(\Box) = i^*_{\mathscr{E}}(-i_{\text{Э}_{\tilde{\chi}}} d^V\theta)$. Show that $\overline{d} \circ \Box = \Delta_\chi \circ \ell_{\boldsymbol{E}(\mathscr{L})}$ where $\Delta_\chi \in \mathscr{C}\mathrm{Diff}(\varkappa|^\dagger_{\mathscr{E}}, \overline{\Lambda}^n(\mathscr{E}))$ is defined by putting $\Delta_\chi\varphi := \langle \varphi, \chi\rangle$, $\varphi \in \varkappa|^\dagger_{\mathscr{E}}$ (thus, Δ_χ is actually a $C^\infty(\mathscr{E})$-linear map). Indeed, let $\chi_1 \in \varkappa$ and put $\overline{\chi}_1 := \chi_1|_{\mathscr{E}}$. Compute

$$\begin{aligned}
(\overline{d} \circ \Box)(\overline{\chi}_1) &= \overline{d}(\tilde{\Box}\chi_1)|_{\mathscr{E}} \\
&= \overline{d}(-i_{\text{Э}_{\chi_1}} i_{\text{Э}_{\tilde{\chi}}} d^V\theta)|_{\mathscr{E}} \\
&= (-i_{\text{Э}_{\tilde{\chi}}} i_{\text{Э}_{\chi_1}} d^V \overline{d}\theta)|_{\mathscr{E}} \\
&= (i_{\text{Э}_{\tilde{\chi}}} i_{\text{Э}_{\chi_1}} d^V \boldsymbol{E}(\mathscr{L}))|_{\mathscr{E}} \\
&= (i_{\text{Э}_{\tilde{\chi}}} \boldsymbol{\mathcal{L}}_{\text{Э}_{\chi_1}} \boldsymbol{E}(\mathscr{L}))|_{\mathscr{E}} - (i_{\text{Э}_{\tilde{\chi}}} d^V \langle \boldsymbol{E}(\mathscr{L}), \chi_1\rangle)|_{\mathscr{E}} \\
&= \langle \ell_{\boldsymbol{E}(\mathscr{L})}\overline{\chi}_1, \chi\rangle - (\boldsymbol{\mathcal{L}}_{\text{Э}_{\tilde{\chi}}} \langle \boldsymbol{E}(\mathscr{L}), \chi_1\rangle)|_{\mathscr{E}} \\
&= \langle \ell_{\boldsymbol{E}(\mathscr{L})}\overline{\chi}_1, \chi\rangle - \langle \ell_{\boldsymbol{E}(\mathscr{L})}\chi, \overline{\chi}_1\rangle \\
&= (\Delta_\chi \circ \ell_{\boldsymbol{E}(\mathscr{L})})(\overline{\chi}_1),
\end{aligned}$$

where we used Identities (25) and (26). It follows from the arbitrariness of χ_1 that $\overline{d} \circ \Box = \Delta_\chi \circ \ell_{\boldsymbol{E}(\mathscr{L})}$. Therefore, $i_{\boldsymbol{X}}\boldsymbol{\omega}$ corresponds to $\Delta^\dagger_\chi 1 + \operatorname{im}\Delta^\dagger_1 \in \ker \ell_{\boldsymbol{E}(\mathscr{L})} / \operatorname{im}\Delta^\dagger_1$ via isomorphism $\boldsymbol{\Lambda}^1(\boldsymbol{P})^{n-1} \simeq \ker \ell_{\boldsymbol{E}(\mathscr{L})} / \operatorname{im}\Delta^\dagger_1$. Finally, it is easy to see that $\Delta^\dagger_\chi 1 = \chi$.

Now, consider diagram (30) whose vertical arrows are described in Section 1.6 as well. Let $\varphi \in \varkappa|^\dagger_{\mathscr{E}}$ and $j \in \overline{J}^\infty\varkappa$ be such that $\Delta_1\varphi = 0$ and $\overline{j}_\infty\varphi = h_{\boldsymbol{E}(\mathscr{L})}(j|_{\mathscr{E}}) \in$

$\overline{J}^\infty\varkappa|^\dagger_{\mathscr{E}}$. Since $\overline{J}^\infty\varkappa$ is pro-finitely generated by elements of the form $\overline{j}_\infty\chi$,[2] $\chi\in\varkappa$, then $j=\sum f\overline{j}_\infty\chi$ for some (generally, infinite in number) $\ldots,f,\ldots\in C^\infty(J^\infty)$ and $\ldots,\chi,\ldots\in\varkappa$. Consequently, $\varphi=\sum f|_{\mathscr{E}}\ell_{\boldsymbol{E}(\mathscr{L})}\chi|_{\mathscr{E}}$. Put $Z:=(\overline{S}\circ\eta_0^{-1})(j)=\sum \text{Э}_\chi\otimes\overline{d}f\in VD(J^\infty)\otimes\overline{\Lambda}^1(J^\infty)$ and recall that 1) Z restricts to $\mathscr{E}$ and 2) $\varphi+\operatorname{im}\ell_{\boldsymbol{E}(\mathscr{L})}\in\ker\Delta_1/\operatorname{im}\ell_{\boldsymbol{E}(\mathscr{L})}$ corresponds to $\boldsymbol{Z}:=[\overline{Z}]\in\mathbf{D}(\boldsymbol{P})^1$, $\overline{Z}\in VD(\mathscr{E})\otimes\overline{\Lambda}^1(\mathscr{E})$ being the restriction of Z to $\mathscr{E}$, via isomorphism $\ker\Delta_1/\operatorname{im}\ell_{\boldsymbol{E}(\mathscr{L})}\simeq\mathbf{D}(\boldsymbol{P})^1$. Now, $\boldsymbol{\Omega}^1(\boldsymbol{Z})=i_{\boldsymbol{Z}}\boldsymbol{\omega}=[i_{\overline{Z}}i^*_{\mathscr{E}}(d^V\theta)]=[i^*_{\mathscr{E}}(i_Zd^V\theta)]$. Compute

$$\begin{aligned}
i_Zd^V\theta &= \sum\overline{d}f\wedge i_{\text{Э}_\chi}d^V\theta\\
&=\overline{d}\rho-\sum f\overline{d}i_{\text{Э}_\chi}d^V\theta\\
&=\overline{d}\rho-\sum fi_{\text{Э}_\chi}d^V\overline{d}\theta\\
&=\overline{d}\rho+\sum(f\mathcal{L}_{\text{Э}_\chi}\boldsymbol{E}(\mathscr{L})-fd^V\langle\boldsymbol{E}(\mathscr{L}),\chi\rangle),
\end{aligned}$$

where $\rho=\sum fi_{\text{Э}_\chi}d^V\theta\in\mathscr{C}\Lambda^1(J^\infty)\otimes\overline{\Lambda}^n(J^\infty)$. Therefore,

$$\begin{aligned}
i_{\overline{Z}}i^*_{\mathscr{E}}(d^V\theta)&=\overline{d}i^*_{\mathscr{E}}(\rho)+\sum i^*_{\mathscr{E}}(f\mathcal{L}_{\text{Э}_\chi}\boldsymbol{E}(\mathscr{L})+fd^V\langle\boldsymbol{E}(\mathscr{L}),\chi\rangle)\\
&=\overline{d}i^*_{\mathscr{E}}(\rho)+\sum\eta^*_{\boldsymbol{E}(\mathscr{L})}(f|_{\mathscr{E}}\ell_{\boldsymbol{E}(\mathscr{L})}\chi|_{\mathscr{E}})+\sum fd^Vi^*_{\mathscr{E}}\langle\boldsymbol{E}(\mathscr{L}),\chi\rangle\\
&=\overline{d}i^*_{\mathscr{E}}(\rho)+\eta^*_{\boldsymbol{E}(\mathscr{L})}(\varphi).
\end{aligned}$$

Finally, $\boldsymbol{\Omega}^1(\boldsymbol{Z})=[\eta^*_{\boldsymbol{E}(\mathscr{L})}(\varphi)]$ corresponds to $\varphi^\dagger(1)+\ker\Delta_1\in\varkappa|^\dagger_{\mathscr{E}}\in\operatorname{coker}\,\ell_{\boldsymbol{E}(\mathscr{L})}$ via isomorphism $\boldsymbol{\Lambda}^1(\boldsymbol{P})^n\simeq\operatorname{coker}\,\ell_{\boldsymbol{E}(\mathscr{L})}$. It is easy to prove that $\varphi^\dagger(1)=\varphi$ and this concludes the proof. □

Some corollaries are in order.

Corollary 20. *There is a natural isomorphism* $\ker\boldsymbol{\omega}\simeq\mathfrak{g}\oplus\operatorname{tker}\boldsymbol{\omega}$.

Corollary 21. $\mathfrak{g}\subset\ker\ell_{\boldsymbol{E}(\mathscr{L})}$ *is a Lie subalgebra (see, for instance, [7]).*

Corollary 22. *Let* $G:Q\longrightarrow\varkappa$ *be a Noether gauge symmetry. Then* $\operatorname{im}G^{\mathscr{E}}\subset\ker\boldsymbol{\omega}$ *(see also [25]).*

Corollary 23. *The secondary 2-form* $\boldsymbol{\omega}$ *is weakly symplectic iff it is strongly symplectic iff the Euler-Lagrange equations* $\mathscr{E}_{\boldsymbol{E}(\mathscr{L})}$ *are irreducible.*

Proof. In view of Theorem 19, $\boldsymbol{\Omega}^0$ and $\boldsymbol{\Omega}^1$ are isomorphisms iff $\mathscr{E}_{\boldsymbol{E}(\mathscr{L})}$ is an irreducible PDE (see Example 11). In view of the 2-lines Theorem 9 ($s=2$), irreducibility of $\mathscr{E}_{\boldsymbol{E}(\mathscr{L})}$ implies, in its turn, that $\operatorname{tker}\boldsymbol{\omega}=0$. □

---

[2] This means that an element in $\overline{J}^\infty\varkappa$ may be understood as a(n equivalence class of) formal infinite linear combination(s) of elements of the form $\overline{j}_\infty\chi$, $\chi\in\varkappa$. Notice that, in any case, all the following computations remain still valid.

2.4. **Gauge Invariant Secondary Functions.** Let N be a smooth manifold and $\sigma \in \Lambda^2(N)$ a presymplectic structure on it. There is no Poisson structure on N associated to σ . However, a Poisson bracket may be introduced among "gauge invariant" functions on N, i.e., functions which are constant along the leaves of the degeneracy distribution of σ. This is precisely the Poisson bracket on the symplectic reduction of (N, σ). In this section we describe "gauge invariant secondary functions" on the CPS $\boldsymbol{P}$ and show that, similarly to the standard situation, $\boldsymbol{\omega}$ induces a Lie bracket among them. Thus, the results presented in this section are propaedeutic to a "secondary symplectic reduction" of $(\boldsymbol{P}, \boldsymbol{\omega})$ (see next section).

Definition 24. *A secondary function* $\boldsymbol{f} \in \boldsymbol{C}^\infty(\boldsymbol{P})^\bullet$ *is called* gauge invariant *iff* $\mathcal{L}_{\boldsymbol{Y}} \boldsymbol{f} = 0$ *for all* $\boldsymbol{Y} \in \ker \boldsymbol{\omega}$*.*

Let us describe gauge invariant elements in $\boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$ and $\boldsymbol{C}^\infty(\boldsymbol{P})^n$.

Proposition 25. *Any element in* $\boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$ *is gauge invariant.*

Proof. Recall that the map $\boldsymbol{\Omega}^0 : \mathbf{D}(\boldsymbol{P})^0 \longrightarrow \boldsymbol{\Lambda}^1(\boldsymbol{P})^{n-1}$ is surjective (see Theorem 19). For any $\boldsymbol{f} \in \boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$, let $\boldsymbol{X} \in \mathbf{D}(\boldsymbol{P})^0$ be such that $\boldsymbol{\Omega}(\boldsymbol{X}) = \boldsymbol{df} \in \boldsymbol{\Lambda}^1(\boldsymbol{P})^{n-1}$ and $\boldsymbol{Y} \in \ker \boldsymbol{\omega}$. Then $\mathcal{L}_{\boldsymbol{Y}} \boldsymbol{f} = i_{\boldsymbol{Y}} \boldsymbol{df} = i_{\boldsymbol{Y}} i_{\boldsymbol{X}} \boldsymbol{\omega} = -i_{\boldsymbol{X}} i_{\boldsymbol{Y}} \boldsymbol{\omega} = 0$. □

Now, let $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$ and $\boldsymbol{X}_1, \boldsymbol{X}_2 \in \mathbf{D}(\boldsymbol{P})^0$ be such that $\boldsymbol{\Omega}(\boldsymbol{X}_1) = \boldsymbol{df}_1$ and $\boldsymbol{\Omega}(\boldsymbol{X}_2) = \boldsymbol{df}_2$. Put $\{\boldsymbol{f}_1, \boldsymbol{f}_2\} := -i_{\boldsymbol{X}_1} i_{\boldsymbol{X}_2} \boldsymbol{\omega} \in \boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$.

Corollary 26. $(\boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}, \{\,\cdot\,,\,\cdot\,\})$ *is a well defined Lie algebra.*

Proof. In view of Proposition 25, $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ is well defined for all $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$, i.e., it is independent of the choice of $\boldsymbol{X}_1, \boldsymbol{X}_2$. Skew-symmetry and the Leibnitz rule follow (as in standard presymplectic geometry) from $\boldsymbol{d\omega} = 0$ and the fact that, if $\boldsymbol{\Omega}(\boldsymbol{X}_1) = \boldsymbol{df}_1$ and $\boldsymbol{\Omega}(\boldsymbol{X}_2) = \boldsymbol{df}_2$, then $\boldsymbol{\Omega}([\boldsymbol{X}_1, \boldsymbol{X}_2]) = \boldsymbol{d}\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$. □

Notice that the existence of a natural Lie bracket among conservation laws of an Euler-Lagrange equation was already known and may be also proved by "off shell" methods such as BRST ones (see, for instance, [5, 6]).

Proposition 27. *An element* $\boldsymbol{F} \in \boldsymbol{C}^\infty(\boldsymbol{P})^n$ *is gauge invariant iff* $\boldsymbol{dF} \in \operatorname{im} \boldsymbol{\Omega}$*.*

Proof. If $\boldsymbol{dF} = \boldsymbol{\Omega}(\boldsymbol{Z})$ for some $\boldsymbol{Z} \in \mathbf{D}(\boldsymbol{P})^1$, then $\boldsymbol{F}$ is gauge invariant (see the proof of Proposition 25). Vice versa, suppose $\mathcal{L}_{\boldsymbol{Y}} \boldsymbol{F} = 0$ for all $\boldsymbol{Y} \in \ker \boldsymbol{\omega}$. Let $\boldsymbol{F} = \int \rho$, $\rho \in \overline{\Lambda}^n(\mathscr{E})$. Recall that $\boldsymbol{dF} = [d^V \rho] \in \boldsymbol{\Lambda}^1(\boldsymbol{P})^{n-1}$ corresponds to $\Box^\dagger 1 + \operatorname{im} \ell_{\boldsymbol{E}(\mathscr{L})} \in \operatorname{coker} \ell_{\boldsymbol{E}(\mathscr{L})}$ via the isomorphism $\boldsymbol{\Lambda}^1(\boldsymbol{P})^{n-1} \simeq \operatorname{coker} \ell_{\boldsymbol{E}(\mathscr{L})}$, $\Box : \varkappa|_{\mathscr{E}} \longrightarrow \overline{\Lambda}^n(\mathscr{E})$ being any horizontal differential operator such that $\eta^*_{\mathscr{E}}(\Box) = d^V \rho$. In view of Theorem 19, $\boldsymbol{dF} \in \operatorname{im} \boldsymbol{\Omega}$ if $\Delta_1(\Box^\dagger 1) = 0$. Let $\chi = \Delta_1^\dagger \vartheta \in \varkappa|_{\mathscr{E}}$, $\vartheta \in P_1^\dagger$, and $\boldsymbol{Y} = \text{Э}_\chi$. Then

$$\begin{aligned} 0 &= \mathcal{L}_{\boldsymbol{Y}} \boldsymbol{F} \\ &= \int \mathcal{L}_{\text{Э}_\chi} \rho \end{aligned}$$

$$
\begin{aligned}
&= \int i_{\ni_\chi} d^V \rho \\
&= \int \Box \chi \\
&= \int (\Box \circ \Delta_1^\dagger)(\vartheta) \\
&= \int \langle (\Box \circ \Delta_1^\dagger)(\vartheta), 1 \rangle \\
&= \int \langle \vartheta, (\Delta_1 \circ \Box^\dagger)(1) \rangle .
\end{aligned}
$$

It follows from the arbitrariness of ϑ that $\Delta_1(\Box^\dagger 1) = 0$, and this concludes the proof. □

The Lie algebra $(\boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}, \{\,\cdot\,,\,\cdot\,\})$ acts naturally on gauge invariant elements in $\boldsymbol{C}^\infty(\boldsymbol{P})^n$. Indeed, let $\boldsymbol{F} \in \boldsymbol{C}^\infty(\boldsymbol{P})^n$ be a gauge invariant element and $\boldsymbol{f} \in \boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$. Put $\{\boldsymbol{f}, \boldsymbol{F}\} := \boldsymbol{\mathcal{L}}_{\boldsymbol{X}} \boldsymbol{F} \in \boldsymbol{C}^\infty(\boldsymbol{P})^n$, $\boldsymbol{X} \in \mathrm{D}(\boldsymbol{P})^0$ being any secondary vector field such that $\boldsymbol{\Omega}^0(\boldsymbol{X}) = \boldsymbol{d}\boldsymbol{f}$. Exactly as above, $\{\boldsymbol{f}, \boldsymbol{F}\}$ is well defined. Moreover, it holds the

Proposition 28. *$\{\boldsymbol{f}, \boldsymbol{F}\}$ is gauge invariant.*

Proof. Recall that, in view of Proposition 27, $\boldsymbol{dF} \in \operatorname{im} \boldsymbol{\Omega}^1$, i.e., $\boldsymbol{dF} = i_{\boldsymbol{Z}} \boldsymbol{\omega}$ for some $\boldsymbol{Z} \in \mathrm{D}(\boldsymbol{P})^1$. Show that $\boldsymbol{d}\{\boldsymbol{f}, \boldsymbol{F}\} \in \operatorname{im} \boldsymbol{\Omega}^1$ as well and then apply Proposition 27 again. Indeed,

$$
\begin{aligned}
\boldsymbol{d}\{\boldsymbol{f}, \boldsymbol{F}\} &= \boldsymbol{d} \boldsymbol{\mathcal{L}}_{\boldsymbol{X}} \boldsymbol{F} \\
&= \boldsymbol{\mathcal{L}}_{\boldsymbol{X}} \boldsymbol{dF} \\
&= \boldsymbol{\mathcal{L}}_{\boldsymbol{X}} i_{\boldsymbol{Z}} \boldsymbol{\omega} \\
&= [\boldsymbol{\mathcal{L}}_{\boldsymbol{X}}, i_{\boldsymbol{Z}}] \boldsymbol{\omega} + i_{\boldsymbol{Z}} \boldsymbol{\mathcal{L}}_{\boldsymbol{X}} \boldsymbol{\omega} \\
&= -i_{[\boldsymbol{X}, \boldsymbol{Z}]} \boldsymbol{\omega} + i_{\boldsymbol{Z}} \boldsymbol{d} i_{\boldsymbol{X}} \boldsymbol{\omega} \\
&= \boldsymbol{\Omega}^1(-[\boldsymbol{X}, \boldsymbol{Z}]) + i_{\boldsymbol{Z}} \boldsymbol{ddf} \\
&= \boldsymbol{\Omega}^1([\boldsymbol{Z}, \boldsymbol{X}]).
\end{aligned}
$$

□

It is easy to prove that the action of $\boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}$ on gauge invariant elements in $\boldsymbol{C}^\infty(\boldsymbol{P})^n$ is indeed a Lie-algebra representation.

Remark 29. *Notice that if the Euler-Lagrange equations are irreducible, then $\boldsymbol{\Omega}$ is an isomorphism, $\ker \boldsymbol{\Omega} = 0$ and every element in $\boldsymbol{C}^\infty(\boldsymbol{P})^\bullet$ is trivially gauge invariant. In this case $(\boldsymbol{C}^\infty(\boldsymbol{P})^{n-1}, \{\,\cdot\,,\,\cdot\,\})$ acts on the whole $\boldsymbol{C}^\infty(\boldsymbol{P})^n$.*

In [8] (see also [14]) it has been shown that the bracket described in full rigour in this section coincides with the Peierls bracket [29]. In its turn the Peierls bracket is at the basis of a covariant approach to quantization of field theories [13]. It is likely that the mathematically rigorous picture presented here will help to better understand, deal with and, possibly, generalize this complicated "functional" structure.

2.5. **Perspectives: Secondary Symplectic Reduction.** Most of the remarks in this section will be informal. From the physical point of view, gauge invariant functions on $\boldsymbol{P}$ are the true observables of the lagrangian theory and, therefore, play a special role. We shew in the last section that, basically, a Lie bracket is defined on gauge invariant functions. We may go even further and ask:

(1) are gauge invariant functions secondary functions on some secondary manifold $\widetilde{\boldsymbol{P}}$?
(2) if yes, is $\widetilde{\boldsymbol{P}}$ a symplectic reduction of the secondary "presymplectic manifold" $(\boldsymbol{P}, \boldsymbol{\omega})$?

In some more details, asking the last question amounts to wonder if there is an embedding of algebras $\boldsymbol{\pi}^* : \boldsymbol{\Lambda}(\widetilde{\boldsymbol{P}})^\bullet \hookrightarrow \boldsymbol{\Lambda}(\boldsymbol{P})^\bullet$ and a secondary two form $\widetilde{\boldsymbol{\omega}}$ on $\widetilde{\boldsymbol{P}}$ such that 1) $\ker \widetilde{\boldsymbol{\omega}} = 0$ and 2) $\boldsymbol{\omega} = \boldsymbol{\pi}^*(\widetilde{\boldsymbol{\omega}})$. Finding an answer to the above questions would definitely establish the parallelism between secondary calculus on the CPS and standard theory of constrained (finite-dimensional) hamiltonian systems. Moreover, it would fix the bases of a mathematically rigorous, covariant, symplectic formalism for classical lagrangian field theories. Finally, it would represent a well founded starting point for a covariant quantization of gauge systems [17].

A possible route through the answers to the above questions is described below. First of all, there is a geometric counterpart of the degeneracy distribution of $\boldsymbol{\omega}$. Let

$$\begin{tikzcd} 0 \arrow[r] & \varkappa|_{\mathscr{E}} \arrow[r, "\ell_{\boldsymbol{E}(\mathscr{L})}"] & \varkappa|^\dagger_{\mathscr{E}} \arrow[r, "\Delta_1"] & P_2 \arrow[r, "\Delta_2"] & \cdots \end{tikzcd}$$

be a compatibility complex for $\ell_{\boldsymbol{E}(\mathscr{L})}$ and

$$\begin{tikzcd} 0 & \varkappa|^\dagger_{\mathscr{E}} \arrow[l] & \varkappa|_{\mathscr{E}} \arrow[l, "\ell_{\boldsymbol{E}(\mathscr{L})}"'] & P_2^\dagger \arrow[l, "\Delta_1^\dagger"'] & \cdots \arrow[l] \end{tikzcd}$$

its adjoint complex. There is an associated complex of $C^\infty(\mathscr{E})$-modules:

$$\begin{tikzcd} 0 & \overline{J}^\infty \varkappa|^\dagger_{\mathscr{E}} \arrow[l] & \overline{J}^\infty \varkappa|_{\mathscr{E}} \arrow[l, "h^\infty_{\boldsymbol{E}(\mathscr{L})}"'] & \overline{J}^\infty P_2^\dagger \arrow[l, "h_1^\infty"'] & \cdots \arrow[l] \end{tikzcd} ,$$

where we put $h^\infty_{\boldsymbol{E}(\mathscr{L})} := h^\infty_{\ell_{\boldsymbol{E}(\mathscr{L})}}$, $h^\infty_1 := h^\infty_{\Delta_1^\dagger}$ and so on. As discussed above, $\ker h^\infty_{\boldsymbol{E}(\mathscr{L})} \subset \overline{J}^\infty \varkappa|_{\mathscr{E}}$ identifies with $V\mathrm{D}(\mathscr{E}) \subset V\mathrm{D}(J^\infty)|_{\mathscr{E}}$ via the isomorphism $\overline{J}^\infty \varkappa|_{\mathscr{E}} \simeq V\mathrm{D}(J^\infty)|_{\mathscr{E}}$ that sends $\overline{j}_\infty \chi$ to Э_χ (see Section 1.5). In particular, $\ker h^\infty_{\boldsymbol{E}(\mathscr{L})}$ has got a natural Lie algebra structure. Similarly, $\operatorname{im} h^\infty_1$ identifies with the module of sections of an involutive distribution $\mathscr{G}$ on $\mathscr{E}$ made of vertical vector fields.

Proposition 30. $\operatorname{im} h^\infty_1 \subset \ker h^\infty_{\boldsymbol{E}(\mathscr{L})}$ *is a Lie-subalgebra.*

Proof. Let $j_1, j_2 \in \overline{J}^\infty P_2^\dagger$. Then $j_1 = \sum f_1 \overline{j}_\infty \vartheta_1$ and $j_2 = \sum f_2 \overline{j}_\infty \vartheta_2$ for some $\ldots, f_1, f_2, \ldots \in C^\infty(J^\infty)$ and $\ldots, \vartheta_1, \vartheta_2, \ldots \in P_2^\dagger$ (see Footnote 2.3, Section 2.3, p. 28). Moreover, $h_1^\infty(j_1), h_1^\infty(j_2)$ correspond to vector fields $X_1 := \sum f_1 \text{Э}_{\Delta_1^\dagger \vartheta_1}, X_2 := \sum f_2 \text{Э}_{\Delta_1^\dagger \vartheta_2}$, respectively, via the isomorphism $\ker h^\infty_{\boldsymbol{E}(\mathscr{L})} \simeq V\mathrm{D}(\mathscr{E})$. Compute

$$\begin{aligned}
[X_1, X_2] &= \sum [f_1 \text{Э}_{\Delta_1^\dagger \vartheta_1}, f_2 \text{Э}_{\Delta_1^\dagger \vartheta_2}] \\
&= \sum (f_1 (\text{Э}_{\Delta_1^\dagger \vartheta_1} f_2) \text{Э}_{\Delta_1^\dagger \vartheta_2} - f_2 (\text{Э}_{\Delta_1^\dagger \vartheta_2} f_1) \text{Э}_{\Delta_1^\dagger \vartheta_1} + f_1 f_2 \text{Э}_{\{\Delta_1^\dagger \vartheta_1, \Delta_1^\dagger \vartheta_2\}}).
\end{aligned}$$

Now, recall that $\mathfrak{g} = \operatorname{im} \Delta_1^\dagger \subset \ker \ell_{\boldsymbol{E}(\mathscr{L})}$ is a Lie subalgebra (see Corollary 21) so that $\{\Delta_1^\dagger \vartheta_1, \Delta_1^\dagger \vartheta_2\} = \Delta_1^\dagger \vartheta$ for some $\vartheta \in P_2^\dagger$. Put

$$j := \sum f_1 (\text{Э}_{\Delta_1^\dagger \vartheta_1} f_2) \overline{j}_\infty \vartheta_2 - f_2 (\text{Э}_{\Delta_1^\dagger \vartheta_2} f_1) \overline{j}_\infty \vartheta_1 + f_1 f_2 \overline{j}_\infty \vartheta \in \overline{J}^\infty P_2^\dagger.$$

Then $h_1^\infty(j) \in \operatorname{im} h_1^\infty \subset \ker h^\infty_{\boldsymbol{E}(\mathscr{L})}$ corresponds to $[X_1, X_2] \in V\mathrm{D}(\mathscr{E})$ via the isomorphism $\ker h^\infty_{\boldsymbol{E}(\mathscr{L})} \simeq V\mathrm{D}(\mathscr{E})$. ☐

In the following we will understand isomorphism $\ker h^\infty_{\boldsymbol{E}(\mathscr{L})} \simeq V\mathrm{D}(\mathscr{E})$. In view of Proposition 30, $\mathscr{G}$ is an involutive distribution on $\mathscr{E}$. Namely, $\mathscr{G}$ is the (involutive) distribution generated by evolutionary derivatives with generating sections in $\mathfrak{g}$ (such kinds of distributions have been recently considered in [19]).

Notice that the horizontal Spencer differential $\overline{S} : V\mathrm{D}(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E}) \longrightarrow V\mathrm{D}(\mathscr{E}) \otimes \overline{\Lambda}(\mathscr{E})$ "restricts" to $\operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E})$. Denote by $\overline{s} : \operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E}) \longrightarrow \operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E})$ the restricted differential. Clearly, $\mathfrak{g} \subset \mathfrak{g}_1 := H^0(\operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E}), \overline{s}) = \mathbf{D}(\boldsymbol{P})^0 \cap \operatorname{im} h_1^\infty$. We now describe the quotient $\mathfrak{g}_1/\mathfrak{g}$. Let $\square : \varkappa|_{\mathscr{E}} \longrightarrow Q_2$ be a compatibility operator for $\Delta_1^\dagger : P_2^\dagger \longrightarrow \varkappa|_{\mathscr{E}}$, and put $k := h_\square^\infty : \overline{J}^\infty \varkappa|_{\mathscr{E}} \longrightarrow \overline{J}^\infty Q_2$. Then $\operatorname{im} h_1^\infty = \ker k$, so that

$$\mathfrak{g}_1 = H^0(\operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E}), \overline{s}) = H^0(\ker k \otimes \overline{\Lambda}(\mathscr{E}), \overline{s}) = \ker \square.$$

We conclude that $\mathfrak{g}_1/\mathfrak{g} = \ker \square / \operatorname{im} \Delta_1^\dagger \simeq H^1(\operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E}), \overline{s})$ (see Theorem 7) and there is an exact sequence (of vector spaces)

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g} \arrow[r, hook] & \mathfrak{g}_1 \arrow[r, two heads] & H^1(\operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E}), \overline{s}) \arrow[r] & 0 \end{tikzcd} .$$

Thus $H^1(\operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E}), \overline{s})$ is the obstruction to $\mathfrak{g}$ being isomorphic to 0-cohomology of the complex $(\operatorname{im} h_1^\infty \otimes \overline{\Lambda}(\mathscr{E}), \overline{s})$, that is, in a sense, the obstruction to "the algebraic description and the geometric description of gauge symmetries coinciding".

Despite the possible existence of such an obstruction, define the new distribution on $\mathscr{E}$, $\widetilde{\mathscr{C}} := \mathscr{E} + \mathscr{G}$. $\widetilde{\mathscr{C}}$ is, generally, infinite-dimensional. Moreover, it is an involutive distribution. Roughly speaking, integral submanifolds of $\widetilde{\mathscr{C}}$ identify with "gauge equivalence classes" of solutions of the Euler-Lagrange equations. Therefore, it is natural to put $\widetilde{\boldsymbol{P}} := \{$maximal integral submanifolds of $\widetilde{\mathscr{C}}\}$ and interpret $\widetilde{\boldsymbol{P}}$ as the space of "physical states" of fields of the lagrangian theory $(\pi, \boldsymbol{S})$.

A secondary calculus may be introduced on $\widetilde{\boldsymbol{P}}$, basically via the $\widetilde{\mathscr{C}}$-spectral sequence $\widetilde{\mathscr{C}}E(\mathscr{E})$, so that elements in $\widetilde{\mathscr{C}}E_1(\mathscr{E}) =: \boldsymbol{\Lambda}(\widetilde{\boldsymbol{P}})^\bullet$ are interpreted as (secondary) differential forms on $\widetilde{\boldsymbol{P}}$. The inclusion $\mathscr{C} \subset \widetilde{\mathscr{C}}$ induces a morphism $\widetilde{\mathscr{C}}E(\mathscr{E}) \longrightarrow \mathscr{C}E(\mathscr{E})$ of spectral sequences whose 1-st term we denote by $\boldsymbol{\pi}^* : \boldsymbol{\Lambda}(\widetilde{\boldsymbol{P}})^\bullet \longrightarrow \boldsymbol{\Lambda}(\boldsymbol{P})^\bullet$.

Now, we'd like to interpret $\widetilde{\boldsymbol{P}}$ as a "(symplectically) reduced CPS". In order to be able to do this in a consistent and physically meaningful way at least the following two conditions should be fulfilled:

1. the image of $\boldsymbol{C}^\infty(\widetilde{\boldsymbol{P}})^\bullet := \widetilde{\mathscr{C}}E_1^{0,\bullet}(\mathscr{E})$ under $\boldsymbol{\pi}^*$ should be made of gauge invariant (secondary) functions on $\boldsymbol{P}$,
2. a secondary 2-form $\widetilde{\boldsymbol{\omega}}$ on $\widetilde{\boldsymbol{P}}$ should exist so that $\ker \widetilde{\boldsymbol{\omega}} = 0$ and $\boldsymbol{\pi}^*(\widetilde{\boldsymbol{\omega}}) = \boldsymbol{\omega}$.

If this was the case then, in the author's opinion, $(\widetilde{\boldsymbol{P}}, \widetilde{\boldsymbol{\omega}})$ could be "safely" referred to as the "symplectic reduction of $(\boldsymbol{P}, \boldsymbol{\omega})$" from the mathematical point of view, and as the "reduced CPS" [17, 25, 30] from the physical point of view.

As suggested by the example in this section and by preliminary work by the author, typical homological algebra (and, possibly, homological perturbation theory) techniques seem to be necessary to investigate further in this direction and complete the above sketched program.

Conclusions

We proposed a fully rigorous approach to the geometry of the covariant phase space $\boldsymbol{P}$, and the canonical, closed 2-form $\boldsymbol{\omega}$ on it, in the framework of secondary calculus. In particular, we described the kernel of $\boldsymbol{\omega}$ in terms of the compatibility operator for the linearized Euler-Lagrange equations thus revealing the precise relation between gauge symmetries and constraints in field theory [25]. We also described gauge invariant (secondary) functions on $\boldsymbol{P}$ and their Lie algebra structure. It is likely that such a Lie algebra is at the basis of a covariant canonical quantization of the theory [13]. A step forward in this direction would be to rigorously perform a symplectic reduction of $(\boldsymbol{P}, \boldsymbol{\omega})$. The preliminary analysis presented in Section 2.4 suggests that this is possible, and should be done, within secondary calculus (or a slight generalization of it) and, in any case, by means of cohomological techniques.

We stress that, in this paper, we basically worked "on shell". The relationship with "off shell" methods (Koszul-Tate resolution and BRST complex [5, 6, 17] - see also [36]) should be carefully analyzed.

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