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On geometry of second order parabolic differential equations in two independent variables

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Abstract. In this note we announce some new results concerning second order differential parabolic equation in two independent variables. Equations of Monge-Ampere type are distinguished among them. They are characterized by the fact that the associated subsidiary equations describing singularities of their multivalued solutions has, in a sense, the simplest form. The structure of these subsidiary equations allows to subdivide parabolic equations into four classes. Each of them can be described as a special geometrical structure on 4-dimensional manifolds introduced below. Beside other, this leads to a complete classification of considered parabolic equations with respect to the group of contact transformation.

Our approach differs from the traditional one (see, for instance, [5]) by the fact that we focus on the corresponding subsidiary, or characteristic, equations rather than on original ones. This leads to a noteworthy simplification. In [3, 4] this approach was used in construction of scalar differential invariants of hyperbolic Monge-Ampere equations.

1. Parabolic and Monge-Ampere equations of second order

Let E be a 3-dimensional manifold. The manifold of k-th order jets, $k \geq 0$, of 2-dimensional submanifolds of E will be denoted by $J^k(E,2)$ and $\pi_{k,l} : J^k(E,2) \longrightarrow J^l(E,2)$, $k \geq l$, denotes the canonical projection. If E is fibered by a map $\pi : E \to M$ over a 2-dimensional manifold M, the $J^k\pi$ stands for the k-th order jet manifold of local sections of π. $J^k\pi$ is an open subset of $J^k(E,2)$. A k-th order differential equation on one unknown function in two independent variables is a hyper-surface $\mathcal{E} \subset J^k(E,2)$. In the sequel we deal with second order equations of this kind. A chart (x, y, u) in E where (x, y) are interpreted as independent variables and u as the dependent one extends to the chart $(x, y, u, u_x = p, u_y = q, u_{xx} = r, u_{xy} = s, u_{yy} = t)$ in $J^2(3,2)$, in terms of which a local description of $\mathcal{E}$ looks as

$$F(x, y, u, p, q, r, s, t) = 0 \qquad (1)$$

Equation (1) is called elliptic (resp., parabolic, or hyperbolic) if $4F_pF_t - F_s^2 > 0$ (resp., $=0$, or < 0) in all points of $\mathcal{E}$. Intrinsically these three types of equations are distinguished one from other by the character of singularities that their multi-valued solutions admit (see [1]). These singularities are described by subsidiary equations. Equations for which these subsidiary equations are in a sense simplest, form the class of Monge-Ampere (MA) equations. We have no possibility to discuss here details of this conceptual characterization of MA-equations and shall refer to the traditional descriptive definition of MA-equations as equations of the form

$$N(rt - s^2) + Ar + Bs + Ct + D = 0 \tag{2}$$

where N, A, B, C, and D are functions of variables x, y, u, p and q. For a coordinate-free but nevertheless descriptive definition of (MA) equations see, for instance, [Mor][Lych]. The subject of this note, parabolic MA-equations, are those for which

$$AC - B^2 - 4ND = 0 \tag{3}$$

The first result concerning parabolic equations (1) is somehow surprising.

Theorem 1. *Characteristic cones of parabolic equations (1) are bidimensional.*

Characteristic cones of parabolic Monge-Ampere equations are planes, i.e., geometrically simplest ones. This property distinguishes parabolic Monge-Ampere equations from other parabolic equations.

2. Geometrical interpretation of parabolic Monge-Ampere equations.

Recall that $J^1(E, 2)$ is canonically supplied with a contact distribution C given locally by the Pfaff equation $du - pdx - qdy = 0$. Vector fields X, Y belonging to C are called C-orthogonal if $[X, Y]$ belongs to C as well. C-orthogonality is, obviously, a $C^\infty(J^1(E, 2))$-linear condition. If C is locally given by a Pfaff equation $\omega = 0$, ω being a 1-form, then X, Y are C-orthogonal iff $d\omega(X, Y) = 0$. A bidimensional subdistribution $D \subset C$ is called lagrangian if any two belonging to it vector fields are C-orthogonal.

Let D be lagrangian. Denote by $L_{(1)} \subset J^1(E, 2)$ the first jet prolongation of a bidimensional submanifold $L \subset E$. The condition

$$\dim\{T_\theta(L_{(1)}) \bigcap D_\theta\} > 0, \ \forall\, \theta \in L_{(1)} \tag{4}$$

determines a second order differential equation imposed on bidimensional sudmanifolds of E. Denote it by $\mathcal{E}_D \subset J^2(E, 2)$. So, by definition, $L \subset E$ is a solution of $\mathcal{E}_D$ iff (4) holds.

Proposition 2. *The correspondence $D \longmapsto \mathcal{E}_D$ between lagrangian distributions on $J^1(E, 2)$ and parabolic Monge-Ampere equations subjecting 2-dimensional submanifolds of E is biunique.*

We shall use $D_\mathcal{E}$ the lagrangian distribution corresponding to a parabolic Monge-Ampere equations $\mathcal{E} \subset J^2(E, 2)$ and $< X, Y >$ for the bidimensional distribution generated by vector fields X, Y. Then for equation (2) we have

$$D_\mathcal{E} = \langle \partial_x + p\partial_u - \frac{C}{N}\partial_p + \frac{B}{2N}\partial_q \ , \ \partial_y + q\partial_u + \frac{B}{2N}\partial_p - \frac{A}{N}\partial_q \rangle$$

assuming that $N \neq 0$. If $N = 0$, i.e., (2) is quasi-linear, then

$$D_{\mathcal{E}} = \langle \partial_x + \frac{B}{2A}\partial_y + (p + \frac{B}{2A}q)\partial_u - \frac{D}{A}\partial_p \ , \ \frac{B}{2A}\partial_p - \partial_q \rangle.$$

Proposition (2) suggests the idea to define *generalized* $PAMs$ as triples of the form $\mathcal{E} = (M, C, D)$ with C being a contact distribution on a 5-fold M and D being a lagrangian subdistribution of C. A solution of $\mathcal{E}$ is defined to be a legendrian submanifold S of M such that

$$\dim\{T_\theta(S) \cap D_\theta\} > 0, \ \forall\, \theta \in S.$$

In what follows the term parabolic Monge-Ampere equation will refer to such a triple.

3. Directing distribution

A PAM $\mathcal{E} = (M, C, D)$ will be called *integrable* if D is integrable.

Theorem 3. *All integrable $PAMs$ are locally contact equivalent one to other and, in particular, to the equation $u_{xx} = 0$.*

So, further on we shall concentrate on nonintegrable PMAs. In this case the first prolongation $D_{(1)}$ of D, i.e., the span of vector fields belonging to D and their commutators, is 3-dimensional and belongs to C. The C-orthogonal complement R of $D_{(1)}$ is 1-dimensional and belongs to D. This way one gets the following flag of distributions

$$R \subset D \subset D_{(1)} \subset C.$$

R is called *the directly distribution* of D (alternatively, of $\mathcal{E}$).

Obviously, the distribution $D' = \{X \in D_{(1)} \mid [X, R] \in D_{(1)}$ contains D. Since $D' \subset D_{(1)}$ there are two possibilities (except eventual singular points): either $D' = D$, or $D' = D_{(1)}$.

A PMA will be called *generic* if $D' = D$ and *special* if $D' = D_{(1)}$. The distribution $D_{(1)}$ is the C-orthogonal compliement of R in C and as is uniquely determined by R. So, D' is uniquely defined by R as well. This shows that a generic PMA is uniquely determined by its directing distribution. On the contrary, it is no longer so for special PMAs. In this case, R is the characteristic distribution of $D_{(1)}$ and any transversal to R 1-dimensional distribution $R' \in D_{(1)}$ defines a special PMA $D = R \bigoplus R'$ for which R is the directing distribution.

4. Projective curve bundles and the associated PMA's.

Let N be a 4-dimensional manifold and $p\tau^*\colon PT^*N \longrightarrow N$ be the *projectivization* of the cotangent bundle $T^*N \to N$. By definition the fiber of $p\tau^*$ over a point $y \in N$ is the 3-dimensional projective space PT^*_yN of 1-dimensional subspaces of T^*_yN. A projective curve bundle (PCB) over N is a 1-dimensional subbundle $\pi : K \to N$ of $p\tau^*$. Its fiber $F_y = \pi^{-1}(y)$ is a (smooth) curve in the projective space PT^*_yN. If the curve F_y is not projectively flat at a point $\theta \in F_y$, then θ is called *regular*. A PCB π is *regular* if all points of composing it curves are regular. A diffeomorphism $\Phi : N \to N'$ lifts canonically to a fibered diffeomorphism $PT^*N \to PT^*N'$ which sends a PCB over N to a PCB over N'. Such two PCBs are called *equivalent* (via Φ).

Let $\theta \in PT_y^*N$ and $\theta = \langle \rho \rangle$ with $\rho \in T_y^*N$. Then $W_\theta = \{\xi \in T_yN \mid \rho(\xi) = 0\}$ is a 3-dimensional subspace of T_yN. Put

$$V_\theta = \{\eta \in T_\theta K \mid d_\theta \pi(\eta) \in W_\theta\} \subset T_\theta K.$$

Then $C_\pi : \theta \mapsto V_\theta$ is a 4-dimensional distribution on K containing the distribution $\mathrm{vert}(\pi)$ of tangent to fibers of π lines.

Proposition 4. *If π is a regular PCB, then the distribution C_π is a contact structure on K.*

In view of this proposition the C_π-orthogonal to $\mathrm{vert}(\pi)$ subdistribution of C_π, denoted $\mathrm{vert}^\perp(\pi)$, is well-defined and we put

$$D_\pi = \{X \in \mathrm{vert}^\perp(\pi) \mid [X, \mathrm{vert}(\pi)] \subset \mathrm{vert}^\perp(\pi)\}.$$

Theorem 5. *If π is a regular PCB, then D_π is a lagrangian with respect to C_π distribution and hence (K, C_π, D_π) is a generic parabolic Monge-Ampere equation whose directing distribution is vert(π). Conversely, any generic parabolic Monge-Ampere equation is locally equivalent to a such one.*

Corollary 6. *The problem of local contact classification of generic PMAs is equivalent to the problem of local classification of regular PCBs with respect to diffeomorphisms of base manifolds.*

5. Special PMA's and fringes

Let N be a 4-dimensional manifold supplied with a 2-dimensional distribution Q. The associated with Q subbundle $\varrho = \varrho_Q \,:\, N_Q \to N$ of $p\tau^*$ is defined as

$$N_Q = \{\theta \in PT^*N \mid W_\theta \supset Q_{p\tau^*(\theta)}\} \subset PT^*N, \quad \varrho = p\tau^*|_{N_Q}.$$

Note that fibers of ϱ are projective lines in PT^*N and $\dim N_Q = 5$.

The projectivization $p\tau : PTN \to N$ of the tangent bundle of N contains a 1-dimensional subbundle $\iota = \iota_Q : N^Q \to N$ composed of all tangent to N lines belonging to Q. A *fringe* over Q is a map $\Psi : N_Q \to N^Q$ such that $\Psi(\varrho^{-1}(y)) \subset \iota^{-1}(y)$, $\forall y \in N$. A 2-dimensional distribution on N_Q is naturally associated with a fringe Ψ:

$$D_\Psi \;:\; \theta \mapsto \{\xi \in T_\theta(N_Q) \mid d_\theta\varrho(\xi) \in \Psi(\theta)\}.$$

Consider the 4-dimensional distribution C_ϱ on N_Q associated with PCB ϱ (see the previous secttion).

Proposition 7. *If Q is not integrable, then the distribution C_ϱ is a contact structure on N_Q with respect to which D_Ψ is lagrangian for any fringe Ψ over Q and $\mathcal{E}_\Psi = (N_Q, C_\varrho, D_\Psi)$ is a special PMA with directing distribution vert(ϱ).*

Describe now two model nonintegrable 2-dimensional distributions on 4-dimensional manifolds. Let C_α^k be the Cartan distribution on the k-th order jet bundle $J^k(\alpha)$ of the trivial bundle $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. It is bidimensional. The 4-dimensional manifold $J^1(\alpha) \times \mathbb{R}$ possesses a natural 2-dimensional distribution $C_\alpha^1 \times 0$, which is the direct product of the contact distribution C_α^1 on $J^1(\alpha)$ and the zero distribution on $\mathbb{R}$. This is the first model. The second is C_α^2 on $J^2(\alpha)$.

Theorem 8. *Any special PMA is locally contact equivalent to $\mathcal{E}_\Psi$ with Ψ being a fringe either over $C_\alpha^1 \times 0$, or over C_α^2 and vice versa.*

Corollary 9. *The problem of local contact classification of special PMAs is equivalent to that of local classification of fringes over model distributions $J^1(\alpha) \times \mathbb{R}$ and C_α^2 with respect to diffeomorphisms preserving these distributions.*

It is not difficult to see that these diffeomorphisms are either fiberwise contact diffeomorphisms of the bundle $J^1(\alpha) \times \mathbb{R} \to \mathbb{R}$ in the first case, or lifted to $J^2(\alpha)$ contact diffeomorhisms of the contact manifold $(J^1(\alpha), C_\alpha^1)$ in the second.

Corollary 10. *Any special PMA is locally contact equivalent to a quasi-linear one.*

6. Differential invariants and contact classification.

It follows from theorems 5 and 8 that nonitegrable PMAs are subdivided into 3 classes: first, generic equations, then special ones, associated with fringes overC_α^2, and, finally, special equations, associated with fringes over $C_\alpha^1 \times 0$. We shall refer to them as G, SG and SI, respectively. Canonical models of PMAs described in these theorems immediately suggest a construction of scalar differential invariants which turns out to be sufficient for a complete classification of PMAs on the basis of the "principle of n-invariants" (see [10]). A general idea on how it can be done in each of these three cases is as follow.

I. Type G. In this case we look for (scalar) differential invariants of PCBs with respect to diffeomorphisms of base manifolds. Let $\mathcal{I}$ be a scalar projective differential invariant of curves in $\mathbb{R}P^3$, say, the *projective curvature* (see [8, 9]), and $\Theta \in K,\ y = \pi(\Theta)$ (see n.4). The value of this invariant for the curve $\pi^{-1}(y)$ in PT_y^* is a function on it. Denote it by $\mathcal{I}_{\pi,y}$ and put $\mathcal{I}_\pi(\Theta) = \mathcal{I}_{\pi,y}(\Theta)$. Then $\mathcal{I}_\pi \in C^\infty(K)$ is a differential invariant of the PCB π and, as such, of the PMA associated with π.

II. Type SG. In this case we are interested in differential invariants of fringes with respect to the group of diffeomorphisms of $J^2(\alpha)$ preserving the distribution C_α^2. Let $\mathcal{I}$ be a (scalar) differential invariant of maps $\mathbb{R}P^1 \to \mathbb{R}P^1$ with respect to a natural action of the group $SL(2) \times SL(2)$ on them. Denote by $\Psi_y : \varrho^{-1}(y) \to \iota^{-1}(y)$ the restriction of the fringing $\Psi : N_Q \to N^Q$ over Q (see n.5) to $\varrho^{-1}(y),\ y \in J^2(\alpha)$. Ψ_y is a map of one projective line to another. Put $\mathcal{I}_\Psi(\Theta) = \mathcal{I}_{\Psi,y}(\Theta)$ with $\mathcal{I}_{\Psi,y}$ being the value of the invariant $\mathcal{I}$ for Ψ_y. Then $\mathcal{I}_\Psi \in C^\infty(N_Q)$ is a differential invariant of Ψ with respect to contact transformations of $J^2(\alpha)$.

III. Type Sl. Construction of invariants of the type $\mathcal{I}_\Psi$ in this case is identical to the preceding.

Theorem 11. *The differential invariants of one of the forms $\mathcal{I}_\Psi, \mathcal{I}_\Psi$ are sufficient for a complete classification of generic and special PMAs, respectively, on the basis of the "principle of n-invariants".*

Concerning "principle of n-invariants" we refer to [10, 11]. A detailed description of these and some more delicate invariants constructed on the basis of the proposed here geometrical interpretation of PMAs will be done in a joint paper by D. Catalano Ferraioli and the author in preparation. This interpretation has a number of other applications to the theory of PMAs which will be discussed separately.

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