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**Iterated Differential Forms VI: Differential
Equations**

by

A. M. Vinogradov and L. Vitagliano

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Iterated Differential Forms VI: Differential Equations

A. M. VINOGRADOV AND L. VITAGLIANO

ABSTRACT. We describe the first term of the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence [1] of the diffiety $(\mathcal{E}, \mathcal{C})$, \mathcal{E} being the infinite prolongation of an ℓ -normal system of partial differential equations, and \mathcal{C} the Cartan distribution on it.

According to [1] secondary k -times iterated differential forms and, in particular, secondary k -th degree covariant tensors on a generic diffiety $(\mathcal{O}, \mathcal{C})$ are elements of the first term of the associated with it $\Lambda_{k-1}\mathcal{C}$ -spectral sequence. In this note we report results of computation of this term for diffieties that are infinite prolongations of normal systems of partial differential equations. To simplify the exposition the equations we deal with are assumed to be imposed on sections of a vector bundle $\pi : E \rightarrow M$, i.e., the diffieties in consideration are of the form $(\mathcal{E}, \mathcal{C})$ with $\mathcal{E} \subset J^\infty(\pi)$ being the infinite prolongation of a normal system of partial differential equations $\mathcal{Y} \subset J^s(\pi)$ and \mathcal{C} being the Cartan distribution on \mathcal{E} .

1. PRELIMINARIES

Here we collect all necessary preliminaries concerning geometry of infinitely prolonged PDEs, iterated differential forms (shortly, IDFs) and secondary IDFs by following [2], [3] and [4], respectively. Accordingly, we use the notation of these works. As in [4] we sometimes shorten the notation by using, for instance, J^k instead of $J^k(\pi)$ for the k -th jet manifold of a fiber bundle π , etc. As usually x^1, \dots, x^n denote a local chart of M .

Let $\xi : V \rightarrow M$ be a vector bundle, $r = \dim \xi$. Denote by v^1, \dots, v^r a system of local linear fiber coordinates extending x^1, \dots, x^n . Any s -th order non-linear differential operator which sends sections of π to sections of ξ may be interpreted as a morphism of bundles $F : J^s \rightarrow V$. Locally F is represented in the form

$$v^a = F^a(\dots, x^\mu, \dots, u_\sigma^j, \dots), \quad F^a \in C^\infty(J^k), \quad a = 1, \dots, r, \quad |\sigma| \leq s$$

and defines the differential equation $\mathcal{E}_F = \{\theta \in J^s \mid F(\theta) = 0\} \subset J^s$. Further on \mathcal{E}_F is assumed to be regular (see [2]). Denote by $\mathcal{E} = \mathcal{E}_F^\infty$ the infinite prolongation of \mathcal{E}_F

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and by $i_{\mathcal{E}} : \mathcal{E} \hookrightarrow J^{\infty}$ the corresponding natural embedding. Put $P = \mathcal{F} \otimes_{C^{\infty}(M)} \Gamma(\xi)$ and recall that there is a unique morphism of diffeities $\widehat{F} : J^{\infty} \longrightarrow J^{\infty}(\xi)$ such that $\xi_{\infty,0} \circ \widehat{F} = F \circ \pi_{\infty,k}$.

Let $\Lambda_{k-1}(\xi)$ be the algebra of $(k-1)$ -times IDF's on $J^{\infty}(\xi)$ and $\mathcal{C}_o\Lambda_{k-1}(\xi) \subset \Lambda_{k-1}(\xi)$ be the $C^{\infty}(V)$ -subalgebra generated by elements in the form $d_K^{\vee}g$, $g \in C^{\infty}(V) \subset C^{\infty}(J^{\infty}(\xi))$, $K \subset \{1, \dots, k-1\}$. The pull-back homomorphism $\widehat{F}^* : \Lambda_{k-1}(\xi) \longrightarrow \Lambda_{k-1}$ is locally given by

$$\widehat{F}^*(d_L x^{\mu}) = d_L x^{\mu}, \quad \widehat{F}^*(d_K^{\vee} v_{\sigma}^a) = (d_K^{\vee} \circ D_{\sigma})(F^a),$$

with $L, K \subset \{1, \dots, k-1\}$, $\mu = 1, \dots, n = \dim M$, $a = 1, \dots, r$ and σ being a multi-index. Obviously, \widehat{F}^* sends $\mathcal{C}_o\Lambda_{k-1}(\xi)$ to $\mathcal{C}_{\star}\Lambda_{k-1}$ (see [4]) and this way $\mathcal{C}_{\star}\Lambda_{k-1}$ acquires a structure of a graded $\mathcal{C}_o\Lambda_{k-1}(\xi)$ -module. By this reason the notion of a $\mathcal{C}_{\star}\Lambda_{k-1}$ -valued derivation of the algebra $\mathcal{C}_o\Lambda_{k-1}(\xi)$ is well-defined. By noticing that $C^{\infty}(M) \subset \mathcal{C}_o\Lambda_{k-1}(\xi)$ we call a derivation Y of $\mathcal{C}_o\Lambda_{k-1}(\xi)$ *vertical* if $Y(C^{\infty}(M)) = 0$. Denote by $\Lambda_{k-1}P$ the $\mathcal{C}_{\star}\Lambda_{k-1}$ -module of vertical $\mathcal{C}_{\star}\Lambda_{k-1}$ -valued derivations of $\mathcal{C}_o\Lambda_{k-1}(\xi)$ and note that $\Lambda_0P \simeq P$.

Let $W_a^K \in D(\Lambda_{k-1}(\xi), \Lambda_{k-1}(\xi))$, $a = 1, \dots, r$, $K \subset \{1, \dots, k-1\}$, be the derivations uniquely defined by relations

$$W_a^K(d_L^{\vee} v_{\sigma}^b) = \begin{cases} 1 & \text{if } a = b, \sigma = \emptyset \text{ and } K = L \\ 0 & \text{otherwise} \end{cases},$$

$$W_a^K(d_L x^{\mu}) = 0,$$

with $b = 1, \dots, r$, $\mu = 1, \dots, n$, $L \subset \{1, \dots, k-1\}$. Then $W_a^K(\mathcal{C}_o\Lambda_{k-1}(\xi)) \subset \mathcal{C}_o\Lambda_{k-1}(\xi)$, $a = 1, \dots, r$, $K \subset \{1, \dots, k-1\}$. Observe that the module $\Lambda_{k-1}P$ is locally generated by elements $(W_a^K)_F = \widehat{F}^* \circ W_a^K$ and is horizontal (see [4]). Indeed, the $C^{\infty}(M)$ -module $W_{k-1}(\xi) \subset \Lambda_{k-1}P$ composed of derivations that are locally of the form $h_K^a (W_a^K)_F$, $h_K^a \in C^{\infty}(M)$, $a = 1, \dots, r$, $K \subset \{1, \dots, k-1\}$ is well defined and

$$\Lambda_{k-1}P \simeq \mathcal{C}_{\star}\Lambda_{k-1} \otimes_{C^{\infty}(M)} W_{k-1}(\xi).$$

Let $\mathcal{L} \in D(V)$ be the standard Liouville vector field on V , i.e., $\mathcal{L} = v^a \frac{\partial}{\partial v^a}$. It naturally extends to a vertical derivation \mathcal{L}^{\vee} of $\mathcal{C}_o\Lambda_{k-1}(\xi)$, which locally looks as

$$\mathcal{L}^{\vee} \stackrel{\text{def}}{=} d_K^{\vee} v^a W_a^K.$$

If $\Lambda_{k-1}F \stackrel{\text{def}}{=} \widehat{F}^* \circ \mathcal{L}^{\vee} \in \Lambda_{k-1}P$, then $\Lambda_{k-1}F$ has the following local expression

$$\Lambda_{k-1}F = d_K^{\vee} F^a (W_a^K)_F.$$

Put $\Lambda_{k-1}\mu_{\mathcal{E}} \stackrel{\text{def}}{=} \ker i_{\mathcal{E}}^* \subset \Lambda_{k-1}$, $i_{\mathcal{E}}^* : \Lambda_{k-1} \longrightarrow \Lambda_{k-1}(\mathcal{E})$, $\mathcal{C}_{\star}\Lambda_{k-1}\mu_{\mathcal{E}} \stackrel{\text{def}}{=} \Lambda_{k-1}\mu_{\mathcal{E}} \cap \mathcal{C}_{\star}\Lambda_{k-1}$ and $\mathcal{C}_{\star}\Lambda_{k-1}(\mathcal{E}) \stackrel{\text{def}}{=} i_{\mathcal{E}}^*(\mathcal{C}_{\star}\Lambda_{k-1}) \simeq \mathcal{C}_{\star}\Lambda_{k-1}/\mathcal{C}_{\star}\Lambda_{k-1}\mu_{\mathcal{E}}$. The restriction of a $\mathcal{C}_{\star}\Lambda_{k-1}$ -module Q to \mathcal{E} is defined to be $Q|_{\mathcal{E}} \stackrel{\text{def}}{=} Q/\mathcal{C}_{\star}\Lambda_{k-1}\mu_{\mathcal{E}} \cdot Q$ and $R_{\mathcal{E}}^Q : Q \ni q \longmapsto [q] \in Q|_{\mathcal{E}}$ is called the *restriction homomorphism*.

2. $\Lambda_{k-1}\mathcal{C}$ -MODULES

Total derivatives D_μ , $\mu = 1, \dots, n$, restricted to \mathcal{E} extend canonically to derivations of the algebra $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$. These extensions will be still denoted by D_μ 's.

In what follows every $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -module is assumed to be graded, locally free and of finite rank. Let Q_1, Q_2 be such ones.

Definition 1. A linear differential operator $\Delta : Q_1 \rightarrow Q_2$ is called \mathcal{C} -differential if for any local basis $\{e_1, \dots, e_t\}$ of Q_1 , Δ is locally of the form $\Delta(p) = (-1)^{|\mu|+|\alpha|} \Delta_\alpha^\sigma D_\sigma(p^\alpha)$ with $\Delta_\alpha^\sigma \in Q_2$, $p = p^\alpha e_\alpha \in Q_1$, $|\alpha| = |e_\alpha|$ and $p^\alpha \in \mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$, $\alpha = 1, \dots, t$.

The left multiplication transforms the totality of all \mathcal{C} -differential operators $\Delta : Q_1 \rightarrow Q_2$ into a $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -module denoted by $\mathcal{C}\text{Diff}(Q_1, Q_2)$. Similarly, the $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -module of multiplicity l , Q_2 -valued, multi- \mathcal{C} -differential operators on Q_1 , denoted by $\mathcal{C}\text{Diff}_{(l)}(Q_1; Q_2)$, is defined. For $l = 0$ we put $\mathcal{C}\text{Diff}_{(0)}(Q_1; Q_2) = Q_2$. The sub-module of $\mathcal{C}\text{Diff}_{(l)}(Q_1; Q_2)$, composed of skew-symmetric operators, is denoted by $\mathcal{C}\text{Diff}_{(l)}^{\text{alt}}(Q_1; Q_2)$ and

$$\text{alt}_l : \mathcal{C}\text{Diff}_{(l)}(Q_1, Q_2) \rightarrow \mathcal{C}\text{Diff}_{(l)}^{\text{alt}}(Q_1, Q_2).$$

stands for the alternation operator.

Let P_1, P_2 be $\mathcal{C}_*\Lambda_{k-1}$ -modules and $\square : P_1 \rightarrow P_2$ a \mathcal{C} -differential operator.

Proposition 2. \square restricts to \mathcal{E} , i.e., there exists a unique $\square|_{\mathcal{E}} \in \mathcal{C}\text{Diff}(P_1|_{\mathcal{E}}, P_2|_{\mathcal{E}})$ such that $R_{\mathcal{E}}^{P_2} \circ \square = \square|_{\mathcal{E}} \circ R_{\mathcal{E}}^{P_1}$.

The module $\mathcal{C}\text{Diff}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}), \mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$ becomes a unitary ring with respect to the composition operation. So, the ring $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ may be viewed as a subring of $\mathcal{C}\text{Diff}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}), \mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$ containing the identity operator. Moreover, if $X \in \mathcal{C}\text{D}(\Lambda_{k-1}(\mathcal{E}))$, then $X(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})) \subset \mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$. So, the restriction $X|_{\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})} \in \mathcal{C}\text{Diff}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}), \mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$ is well-defined.

Put

$$\mathcal{C}\text{D}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})) \stackrel{\text{def}}{=} \{X|_{\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})} \mid X \in \mathcal{C}\text{D}(\Lambda_{k-1}(\mathcal{E}))\} \subset \mathcal{C}\text{Diff}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}), \mathcal{C}_*\Lambda_{k-1}(\mathcal{E})).$$

As a ring $\mathcal{C}\text{Diff}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}), \mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$ is generated by its submodules $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ and $\mathcal{C}\text{D}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$.

Definition 3. A couple composed of a left graded $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -module Q and a homomorphism

$$\mathcal{C}\text{Diff}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}), \mathcal{C}_*\Lambda_{k-1}(\mathcal{E})) \ni \Delta \mapsto \Delta^Q \in \mathcal{C}\text{Diff}(Q, Q)$$

of unitary rings is called a $\Lambda_{k-1}\mathcal{C}$ -module (see, e.g., [5, 6]).

Example 4. $\mathcal{C}_*\Lambda_k^p(\mathcal{E}) = \mathcal{C}_*\Lambda_k(\mathcal{E}) \cap \Lambda_k^p(\mathcal{E}) \subset \mathcal{C}^p\Lambda_k^p(\mathcal{E})$ is a $\Lambda_{k-1}\mathcal{C}$ -module for any p . Indeed, for $Y = X|_{\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})} \in \mathcal{C}\text{D}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$, $X \in \mathcal{C}\text{D}(\Lambda_{k-1}(\mathcal{E}))$, define the operator Y^Q by putting

$$Y^Q(\Omega) = \mathcal{L}_X^{\{k\}}(\Omega) \in \mathcal{C}_*\Lambda_k^p(\mathcal{E}), \quad \Omega \in \mathcal{C}^p\Lambda_k^p(\mathcal{E}).$$

Then, in particular, $(\omega Y)^Q = \omega Y^Q$, $\omega \in \mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$, $Y \in \mathcal{C}\text{D}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$.

Note that the tensor product of $\Lambda_{k-1}\mathcal{C}$ -modules is a $\Lambda_{k-1}\mathcal{C}$ -module in a natural way.

Let Q be a $\Lambda_{k-1}\mathcal{C}$ -module and P_1, P_2 be $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -modules. For $\Delta \in \mathcal{C}\text{Diff}(P_1, P_2)$, $p \in P_1$ and $\phi \in P_2^* = \text{Hom}_{\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})}(P_2, \mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$ define $\Delta(p, \phi) \in \mathcal{C}\text{Diff}(\mathcal{C}_*\Lambda_{k-1}(\mathcal{E}), \mathcal{C}_*\Lambda_{k-1}(\mathcal{E}))$ by putting

$$\Delta(p, \phi)(\omega) \stackrel{\text{def}}{=} (-1)^{|\phi|(|p|+|\Delta|)+|\omega||p|}(\phi \circ \Delta)(\omega p), \quad \omega \in \mathcal{C}_*\Lambda_{k-1}(\mathcal{E}).$$

Tensor products we need now on are over the algebra $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ and we shall simplify the notation by using \otimes instead of $\otimes_{\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})}$.

Proposition 5. *There exists a unique $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -module homomorphism*

$$\mathcal{C}\text{Diff}(P_1, P_2) \ni \Delta \longrightarrow \Delta^Q \in \mathcal{C}\text{Diff}(P_1 \otimes Q, P_2 \otimes Q)$$

such that

$$\Delta^Q(p \otimes q, \phi \otimes \psi)(\omega) = (-1)^{|\psi|(|q|+|p|+|\Delta|+|\phi|)+|q|(|\phi|+|\omega|)}\psi(\Delta(p, \phi)^Q(\omega q)),$$

$p \in P_1$, $q \in Q$, $\phi \in P_2^*$, $\psi \in Q^*$, $\omega \in \mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$.

Now, let P_3 be another $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -module. Then

$$(\Delta_2 \circ \Delta_1)^Q = \Delta_2^Q \circ \Delta_1^Q$$

for any $\Delta_1 \in \mathcal{C}\text{Diff}(P_1, P_2)$, $\Delta_2 \in \mathcal{C}\text{Diff}(P_2, P_3)$. This simple fact allows to associate with a complex of $\mathcal{C}_*\Lambda_{k-1}(\mathcal{E})$ -modules

$$\cdots \longrightarrow P_{i-1} \xrightarrow{\Delta_{i-1}} P_i \xrightarrow{\Delta_i} P_{i+1} \xrightarrow{\Delta_{i+1}} \cdots$$

connected by \mathcal{C} -differential operators Δ_i 's a new one, namely,

$$\cdots \longrightarrow P_{i-1} \otimes Q \xrightarrow{\Delta_{i-1}^Q} P_i \otimes Q \xrightarrow{\Delta_i^Q} P_{i+1} \otimes Q \xrightarrow{\Delta_{i+1}^Q} \cdots$$

3. $(k-1)$ -IDF-SYMMETRIES AND $\Lambda_{k-1}\mathcal{C}$ -SPECTRAL SEQUENCE OF A SYSTEM OF PDES

Let \mathcal{E} be as above. The universal linearization $\ell_{\Lambda_{k-1}F}^{\{k\}} : \Lambda_{k-1}\mathcal{Z} \longrightarrow \Lambda_{k-1}P$ of $\Lambda_{k-1}F$ is a \mathcal{C} -differential operator and as such (proposition 2) restricts to \mathcal{E} .

Theorem 6. *There is a Lie-algebra isomorphism $\Lambda_{k-1}\text{Sym}(\mathcal{E}) \simeq \ker(\ell_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}) \subset \Lambda_{k-1}\mathcal{Z}|_{\mathcal{E}}$.*

An exact description of this isomorphism is as follows. Let $\bar{\chi} \in \ker(\ell_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}) \subset \Lambda_{k-1}\mathcal{Z}|_{\mathcal{E}}$. Then $\bar{\chi} = R_{\mathcal{E}}^{\Lambda_{k-1}\mathcal{Z}}(\chi)$ for some $\chi \in \Lambda_{k-1}\mathcal{Z}$. The evolutionary derivation $\mathfrak{D}_{\chi} : \Lambda_{k-1} \longrightarrow \Lambda_{k-1}$ can be restricted to \mathcal{E} , i.e., there exists a unique derivation $\mathfrak{D}_{\chi}|_{\mathcal{E}} \in \text{D}(\Lambda_{k-1}(\mathcal{E}), \Lambda_{k-1}(\mathcal{E}))$ such that $i_{\mathcal{E}}^* \circ \mathfrak{D}_{\chi} = \mathfrak{D}_{\chi}|_{\mathcal{E}} \circ i_{\mathcal{E}}^*$. Moreover, $\mathfrak{D}_{\chi}|_{\mathcal{E}} \in \text{D}_{\mathcal{C}}(\Lambda_{k-1}(\mathcal{E}))$. Then the derivation that corresponds to $\bar{\chi}$ via the isomorphism of theorem 6 is $[\mathfrak{D}_{\chi}|_{\mathcal{E}}]_{\mathcal{C}\text{D}(\Lambda_{k-1}(\mathcal{E}))} \in \Lambda_{k-1}\text{Sym}(\mathcal{E})$.

Put

$$\Lambda_{k-1}Q \stackrel{\text{def}}{=} \{\nabla|_{\mathcal{E}} \circ \ell_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}} \mid \nabla \in \mathcal{C}\text{Diff}(\Lambda_{k-1}P, \Lambda_{k-1})\} \subset \mathcal{C}\text{Diff}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}, \Lambda_{k-1}(\mathcal{E}))$$

and note that $\mathcal{C}\text{Diff}_{(p-1)}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}Q)$ is naturally embedded into $\mathcal{C}\text{Diff}_{(p)}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E}))$. In view of this embedding the following definition

$$\Lambda_{k-1}Q_p \stackrel{\text{def}}{=} \text{alt}_p(\mathcal{C}\text{Diff}_{(p-1)}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}Q)) \subset \mathcal{C}\text{Diff}_{(p)}^{\text{alt}}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E}))$$

makes sense. In particular, $\Lambda_{k-1}Q_1 = \Lambda_{k-1}Q$.

Proposition 7. *There is a $\Lambda_{k-1}(\mathcal{E})$ -module isomorphism*

$$\mathcal{C}^p\Lambda_k^p(\mathcal{E}) \simeq \mathcal{C}\text{Diff}_{(p)}^{\text{alt}}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E})) / \Lambda_{k-1}Q_p.$$

According to this proposition the homomorphism

$$\lambda_{\Lambda_{k-1}F} : \mathcal{C}\text{Diff}(\Lambda_{k-1}P|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E})) \longrightarrow \mathcal{C}\text{Diff}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E})),$$

sending $\nabla \in \mathcal{C}\text{Diff}(\Lambda_{k-1}P|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E}))$ to $\nabla \circ \ell_{\Lambda_{k-1}F}^{\{k\}}$, makes the following sequence of $\Lambda_{k-1}(\mathcal{E})$ -homomorphisms

$$\mathcal{C}\text{Diff}(\Lambda_{k-1}P|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E})) \xrightarrow{\lambda_{\Lambda_{k-1}F}} \mathcal{C}\text{Diff}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E})) \longrightarrow \mathcal{C}\Lambda_k^1(\mathcal{E}) \longrightarrow 0,$$

exact.

Recall (see [2, 5]) that the equation \mathcal{E}_F is said ℓ -normal if the sequence of $C^\infty(\mathcal{E})$ -homomorphisms

$$0 \longrightarrow \mathcal{C}\text{Diff}(P|_{\mathcal{E}}; C^\infty(\mathcal{E})) \xrightarrow{\lambda_F} \mathcal{C}\text{Diff}(\mathcal{A}|_{\mathcal{E}}; C^\infty(\mathcal{E})) \longrightarrow \mathcal{C}\Lambda^1(\mathcal{E}) \longrightarrow 0$$

with $\lambda_F(\nabla) \stackrel{\text{def}}{=} \nabla \circ \ell_F$, $\nabla \in \mathcal{C}\text{Diff}(P|_{\mathcal{E}}; C^\infty(\mathcal{E}))$, is exact. The ‘‘iterated’’ analogue of this notion is as follows.

Definition 8. *An equation \mathcal{E}_F is said $\ell^{\{k\}}$ -normal if the sequence of $\Lambda_{k-1}(\mathcal{E})$ -homomorphisms*

$$0 \longrightarrow \mathcal{C}\text{Diff}(\Lambda_{k-1}P|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E})) \xrightarrow{\lambda_{\Lambda_{k-1}F}} \mathcal{C}\text{Diff}(\Lambda_{k-1}\mathcal{A}|_{\mathcal{E}}; \Lambda_{k-1}(\mathcal{E})) \longrightarrow \mathcal{C}\Lambda_k^1(\mathcal{E}) \longrightarrow 0$$

is exact, i.e., $\ker \lambda_{\Lambda_{k-1}F} = 0$.

Proposition 9. *If \mathcal{E} is ℓ -normal, then it is $\ell^{\{k\}}$ -normal as well for any $k > 1$.*

The zeroth column differential $d_0^{0,\bullet} : \mathcal{H}\Lambda_k(\mathcal{E}) \longrightarrow \mathcal{H}\Lambda_k(\mathcal{E})$ in the zeroth term of the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence of \mathcal{E} is a \mathcal{C} -differential operator. Consider the operator $[d_0^{0,\bullet}]_p \stackrel{\text{def}}{=} (d_0^{0,\bullet})_{\mathcal{C}_*\Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_*\Lambda_k^{p-1}(\mathcal{E})}$,

$$[d_0^{0,\bullet}]_p : \mathcal{C}_*\Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_*\Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}) \longrightarrow \mathcal{C}_*\Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_*\Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E})$$

and denote by

$$\text{alt}_p : \mathcal{C}_*\Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_*\Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}) \longrightarrow \mathcal{C}_*\Lambda_k^p(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}) \quad (1)$$

the alternation map. Then a natural isomorphism $\mathcal{C}_\star \Lambda_k^p(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}) \simeq \Lambda_{k-1} \mathcal{C}E_0^{p,\bullet}$ takes place. Moreover, the inclusion $\mathcal{C}_\star \Lambda_k^p(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}) \subset \mathcal{C}_\star \Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_\star \Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E})$ is a right inverse of alt_p so that $\Lambda_{k-1} \mathcal{C}E_0^{p,\bullet}$ is a direct summand in $\mathcal{C}_\star \Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_\star \Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E})$. More precisely, the complex $(\Lambda_{k-1} \mathcal{C}E_0^{p,\bullet}, d_0^{p,\bullet})$ is a direct summand in the complex $(\mathcal{C}_\star \Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_\star \Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}), [d_0^{p,\bullet}]_p)$.

Now consider the restriction to \mathcal{E} of the adjoint operator of $\ell_{\Lambda_{k-1}F}^{\{k\}}$:

$$\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}} : \widehat{\Lambda_{k-1}P}|_{\mathcal{E}} \longrightarrow \widehat{\Lambda_{k-1}\mathcal{A}}|_{\mathcal{E}}$$

and its multiple extension

$$[\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}]_p : \widehat{\Lambda_{k-1}P}|_{\mathcal{E}} \otimes \mathcal{C}_\star \Lambda_k^p(\mathcal{E}) \longrightarrow \widehat{\Lambda_{k-1}\mathcal{A}}|_{\mathcal{E}} \otimes \mathcal{C}_\star \Lambda_k^p(\mathcal{E})$$

defined as $[\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}]_p \stackrel{\text{def}}{=} (\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}})^{\mathcal{C}_\star \Lambda_k^p(\mathcal{E})}$.

Proposition 10.

- $H^q(\mathcal{C}_\star \Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_\star \Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}), [d_0^{0,\bullet}]_p) \simeq 0$ for $q < n - 1$.
- $H^{n-1}(\mathcal{C}_\star \Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_\star \Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}), [d_0^{0,\bullet}]_p) \simeq \ker [\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}]_{p-1}$.
- $H^n(\mathcal{C}_\star \Lambda_k^1(\mathcal{E}) \otimes \mathcal{C}_\star \Lambda_k^{p-1}(\mathcal{E}) \otimes \mathcal{H}\Lambda_k(\mathcal{E}), [d_0^{0,\bullet}]_p) \simeq \text{coker} [\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}]_{p-1}$.

In the following theorem that collects the above results the map alt_p , abusing the notation, stands for the induced by projection (1) map in cohomology.

Theorem 11 (Two Lines Theorem). *Let $\mathcal{E} = \mathcal{E}_F^\infty$ be the infinite prolongation of an l -normal equation $\mathcal{E}_F = \{F = 0\}$. Then \mathcal{E} is $\ell^{\{k\}}$ -normal and*

- $\Lambda_{k-1} \mathcal{C}E_1^{0,\bullet}(\mathcal{E}) \simeq \Lambda_{k-2} \mathcal{C}E_1^{\bullet,\bullet}(\mathcal{E})$.
- $\Lambda_{k-1} \mathcal{C}E_1^{p,q}(\mathcal{E}) = 0$ for any $p > 0$ and $q < n - 1$.
- $\Lambda_{k-1} \mathcal{C}E_1^{p,n-1}(\mathcal{E}) \simeq \text{alt}_p(\ker [\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}]_{p-1})$ for any $p > 0$.
- $\Lambda_{k-1} \mathcal{C}E_1^{p,n}(\mathcal{E}) \simeq \text{alt}_p(\text{coker} [\widehat{\ell}_{\Lambda_{k-1}F}^{\{k\}}|_{\mathcal{E}}]_{p-1})$ for any $p > 0$.

4. SHORT CONCLUSION

Limits of a short note do not allow us neither to be completely conceptual in the exposition, nor to present all obtained results in full generality. Both require introducing numerous new notions and constructions that can be done only in frames of the forthcoming detailed and systematical exposition. The same concerns various applications, even quite direct ones as, for instance, secondary versions of notes [3, 7, 8]. By concluding we state that the “iterated” analogues of the l -lines theorem and all results contained in the last chapter of book [5] take place.

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ALEXANDRE VINOGRADOV, DMI, FACOLTÀ DI SCIENZE MM. FF. NN. UNIVERSITÀ DEGLI STUDI DI SALERNO, VIA PONTE DON MELILLO, 84084 FISCIANO (SA) AND INFN, SEZ. NAPOLI-SALERNO, ITALY

E-mail address: vinograd@unisa.it

LUCA VITAGLIANO, DMI, FACOLTÀ DI SCIENZE MM. FF. NN. UNIVERSITÀ DEGLI STUDI DI SALERNO, VIA PONTE DON MELILLO, 84084 FISCIANO (SA) AND INFN, SEZ. NAPOLI-SALERNO, ITALY

E-mail address: luca_vitagliano@fastwebnet.it