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Iterated Differential Forms V: $\mathcal{C}$–Spectral Sequence on $J^\infty(\pi)$

A. M. VINOGRADOV AND L. VITAGLIANO

ABSTRACT. In the preceding note [1] the $\Lambda_{k-1}\mathcal{C}$–spectral sequence, whose first term is composed of *secondary iterated differential forms*, was constructed for a generic diffiety. In this note the zero and first terms of this spectral sequence are explicitly computed for infinite jet spaces. In particular, this gives an explicit description of secondary covariant tensors on these spaces and some basic operations with them. On the basis of these results a description of the $\Lambda_{k-1}\mathcal{C}$–spectral sequence for infinitely prolonged PDE's will be given in the subsequent note.

Introduced in [1] secondary iterated differential forms on a generic diffiety $(\mathcal{O}, \mathcal{C})$ are elements of the first term of the $\Lambda_{k-1}\mathcal{C}$– spectral sequence associated with it. In the present note we give an explicit description of the zeroth and first terms of this spectral sequence for the infinite jet space of a vector bundle $\mathcal{O} = J^\infty(\pi)$, $\pi : E \longrightarrow M$. Moreover, analogues of basic operations of tensor analysis in secondary calculus on these spaces are constructed. This goal is got by a due generalization of the approach developed in [2] taking into account some improvements proposed in [3].

1. Notations and conventions

In what concerns geometry of infinite jet manifolds and general theory of iterated differential forms (shortly, IDFs), we follow [4] and [5], respectively. The notation is slightly simplified against [4, 5]. Namely, when the context allows we omit the reference to arguments "π" and "$J^\infty(\pi)$". For instance, we use $\mathcal{F}$ and Λ for algebras of smooth functions and differential forms on $J^\infty(\pi)$, respectively, instead of standard $\mathcal{F}(\pi)$ and $\Lambda(J^\infty(\pi))$, etc.

Also the algebra of horizontal differential forms on $J^\infty(\pi)$ is denoted by $\mathcal{H}\Lambda$ (rather than Λ_0) and the algebra of vertical differential forms by $\mathcal{C}^\bullet\Lambda$. Accordingly, d^{h} and d^{v} stand for horizontal and vertical differentials, respectively.

As usually, $\{\ldots, x^\mu, \ldots, u^j, \ldots\}$, $\mu = 1, \ldots, n$, $j = 1, \ldots, m$, $n = \dim M$, $m = \dim E - n$, denote an adapted to π local chart on E, assuming that $\{\ldots, x^\mu, \ldots\}$ is a local chart on M. We follow the notation of [1] in what concerns the theory of secondary IDF's.

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The wedge product of IDFs is denoted, according to [1], by the "dot". Finally, the fact that an isomorphism is *canonical* is stressed using the symbol $\simeq$.

2. IDFs on $J^\infty(\pi)$

Recall that, due to the fibered structure of $J^\infty(\pi)$, the de Rham differential d splits into vertical and horizontal parts: $d = d^{\mathsf{v}} + d^{\mathsf{h}}$. $(\Lambda, d^{\mathsf{v}}, d^{\mathsf{h}})$ is a bi–complex, for the first time introduced in [6] (see also [7, 8, 2]) and later called the *variational bicomplex* (see [3, 4], see also [9]). A variational bi–complex exists locally for any diffiety and may be understood as a local description of the $\mathcal{C}$–spectral sequence. Moreover, Λ^1 splits naturally as $\Lambda^1 = \mathcal{C}\Lambda^1 \oplus \mathcal{H}\Lambda^1$. Accordingly, Λ is factorized as $\Lambda \simeq \mathcal{C}^\bullet\Lambda \otimes_{\mathcal{F}} \mathcal{H}\Lambda$ ($\mathcal{C}^\bullet\Lambda \equiv \bigwedge^\bullet \mathcal{C}\Lambda^1$, $\mathcal{H}\Lambda \equiv \bigwedge^\bullet \mathcal{H}\Lambda^1$). In a dual manner, D splits as $\mathrm{D} = \mathcal{C}\mathrm{D} \oplus \mathrm{D}^{\mathsf{v}}$ (see [4]).

Note that d^{v} and d^{h} extends as derivations to Λ_k. Abusing the notation we continue to denote these extensions by d^{v} and d^{h}. Put $d^{\mathsf{v}}_m = \kappa_{1m} \circ d^{\mathsf{v}} \circ \kappa_{1m} : \Lambda_k \longrightarrow \Lambda_k$, $d^{\mathsf{h}}_m = \kappa_{1m} \circ d^{\mathsf{h}} \circ \kappa_{1m} : \Lambda_k \longrightarrow \Lambda_k$, $\kappa_{1m} : \Lambda_k \longrightarrow \Lambda_k$ being the involution that interchanges d_1 and d_m, $m \leq k$ and also $d^{\mathsf{v}}_K = d^{\mathsf{v}}_{k_1} \circ \cdots \circ d^{\mathsf{v}}_{k_s}$, $K = \{k_1, \ldots, k_s\} \subset \{1, \ldots, k\}$. $(\Lambda_k, (d^{\mathsf{v}}_1, d^{\mathsf{h}}_1), \ldots, (d^{\mathsf{v}}_k, d^{\mathsf{h}}_k))$, where $d^{\mathsf{v}}_1 = d^{\mathsf{v}}$ and $d^{\mathsf{h}}_1 = d^{\mathsf{h}}$, is a multiple bi–complex. Define $\mathcal{H}\Lambda^1_k \subset \Lambda^1_k$ as the Λ_{k-1}–submodule generated by $\pi^*_\infty(\Lambda^1_k(M))$, $\pi_\infty : J^\infty(\pi) \longrightarrow M$ being a natural projection. $\mathcal{C}\Lambda^1_k \subset \Lambda^1_k$ (see [1]) is a locally free Λ_{k-1}–submodule generated by $d^{\mathsf{v}}_k(\Lambda_{k-1})$. Elements

$$\mathcal{C}\Lambda^1_k \ni d^{\mathsf{v}}_k d^{\mathsf{v}}_K u^j_\sigma, \quad K \subset \{1, \ldots, k-1\},\ j = 1, \ldots, m,\ \sigma \text{ a lenght } n \text{ multi–index} \tag{1}$$

form a local basis of it and must be understood as iterated Cartan forms. Similarly, $\mathcal{H}\Lambda^1_k \subset \Lambda^1_k$ is a locally free Λ_{k-1}–submodule generated by $d^{\mathsf{h}}_k(\Lambda_{k-1})$ and elements

$$\mathcal{H}\Lambda^1_k \ni d_k d_K x^\mu, \quad K \subset \{1, \ldots, k-1\},\ \mu = 1, \ldots, n \tag{2}$$

form an its local basis.

Denote by $\mathcal{C}^\bullet\Lambda_k \subset \Lambda_k$ (resp. $\mathcal{H}\Lambda_k \subset \Lambda_k$) the Λ_{k-1}–subalgebra generated by 1 and $\mathcal{C}\Lambda^1_k$ (resp. $\mathcal{H}\Lambda^1_k$). Clearly, $d^{\mathsf{v}}_k(\mathcal{C}^\bullet\Lambda_k) \subset \mathcal{C}^\bullet\Lambda_k$ and $d^{\mathsf{h}}_k(\mathcal{H}\Lambda_k) \subset \mathcal{H}\Lambda_k$. Moreover, $\Lambda^1_k = \mathcal{C}\Lambda^1_k \oplus \mathcal{H}\Lambda^1_k$ and $\Lambda_k \simeq \mathcal{C}^\bullet\Lambda_k \otimes_{\Lambda_{k-1}} \mathcal{H}\Lambda_k$.

Put $\mathrm{D}^{\mathsf{v}}(\Lambda_{k-1}) \overset{\text{def}}{=} \{X \in \mathrm{D}(\Lambda_{k-1}, \Lambda_{k-1}) \mid X \circ \pi^*_\infty = 0\}$. Elements of $\mathrm{D}^{\mathsf{v}}(\Lambda_{k-1})$ are called vertical derivations of Λ_{k-1}. $\mathrm{D}(\Lambda_{k-1}, \Lambda_{k-1})$ splits as $\mathrm{D}(\Lambda_{k-1}, \Lambda_{k-1}) \simeq \mathcal{C}\mathrm{D}(\Lambda_{k-1}) \oplus \mathrm{D}^{\mathsf{v}}(\Lambda_{k-1})$ (see [1]). Moreover, $\mathcal{C}\mathrm{D}(\Lambda_{k-1}) \simeq \mathrm{Hom}_{\Lambda_{k-1}}(\mathcal{H}\Lambda^1_k, \Lambda_{k-1})$ and $\mathrm{D}^{\mathsf{v}}(\Lambda_{k-1}) \simeq \mathrm{Hom}_{\Lambda_{k-1}}(\mathcal{C}\Lambda^1_k, \Lambda_{k-1})$. Denote by $V^{\sigma,K}_j \in \mathrm{D}^{\mathsf{v}}(\Lambda_{k-1})$ and $D^K_\mu \in \mathcal{C}\mathrm{D}(\Lambda_{k-1})$ with $K \subset \{1, \ldots, k-1\}$, $j = 1, \ldots, m$, $\mu = 1, \ldots, n$ and σ being a lenght n multi–index, elements of the dual local bases of (1) and (2), respectively.

The following two sub–algebras of Λ_{k-1} will be of a special interest in the sequel.

- $\mathcal{C}_\star\Lambda_{k-1}$: the $\mathcal{F}$–subalgebra generated by elements of the form $d^{\mathsf{v}}_K f$, $f \in \mathcal{F}$, $K \subset \{1, \ldots, k-1\}$ (note that $\mathcal{C}_\star\Lambda_0 \equiv \mathcal{C}_\star\Lambda = \mathcal{C}^\bullet\Lambda$).
- $\mathcal{C}_\circ\Lambda_{k-1}$: the $C^\infty(E)$–subalgebra generated by elements of the form $d^{\mathsf{v}}_K g$, $g \in C^\infty(E)$, $K \subset \{1, \ldots, k-1\}$.

Clearly, $\mathcal{C}_\circ\Lambda_{k-1}\subset\mathcal{C}_\star\Lambda_{k-1}$. Moreover, $V_j^{\sigma,K}(\mathcal{C}_\circ\Lambda_{k-1})\subset\mathcal{C}_\circ\Lambda_{k-1}$ with $K\subset\{1,\dots,k-1\}$, $j=1,\dots,m$ and σ being a lenght n multi–index. Denote by $\Lambda_{k-1}\varkappa$ the $\mathcal{C}_\star\Lambda_{k-1}$–module of $\mathcal{C}_\star\Lambda_{k-1}$–valued derivations of $\mathcal{C}_\circ\Lambda_{k-1}$. $\Lambda_{k-1}\varkappa$ is locally generated by derivations $V_j^K\equiv V_j^{(0,\dots,0),K}$, $K\subset\{1,\dots,k-1\}$, $j=1,\dots,m$.

k–times IDFs–valued symmetries (see [1]) of $J^\infty(\pi)$ are easily described in terms of vertical derivations of Λ_{k-1}. Indeed, let $\chi=[X]\in\Lambda_{k-1}\mathrm{Sym}$, $X\in\mathrm{D}_{\mathcal{C}}(\Lambda_{k-1})$. There is a unique vertical representative of χ denoted by $\mathcal{Э}_\chi$. $\mathcal{Э}_\chi$ is called an *evolutionary derivation* of Λ_{k-1}. $\mathcal{Э}_\chi(\mathcal{C}_\star\Lambda_{k-1})\subset\mathcal{C}_\star\Lambda_{k-1}$, since $\mathcal{Э}_\chi\in\mathrm{D}_{\mathcal{C}}(\Lambda_{k-1})\cap\mathrm{D}^{\mathsf{v}}(\Lambda_{k-1})$.

Proposition 1. *The correspondence* $\Lambda_{k-1}\mathrm{Sym}\ni\chi\longmapsto\mathcal{Э}_\chi|_{\mathcal{C}_\circ\Lambda_{k-1}}\in\Lambda_{k-1}\varkappa$ *is an isomorphism of vector spaces, whose inverse looks locally as* $\Lambda_{k-1}\varkappa\ni\chi_K^jV_j^K\longmapsto(D_\sigma\chi_K^j)V_j^{\sigma,K}\in\Lambda_{k-1}\mathrm{Sym}$ *with* $\chi_K^j\in\mathcal{C}_\star\Lambda_{k-1}$, $D_\sigma=(D_1^{(0,\dots,0)})^{\sigma_1}\circ\cdots\circ(D_n^{(0,\dots,0)})^{\sigma_n}$, $K\subset\{1,\dots,k-1\}$, $j=1,\dots,m$ *and* $\sigma=(\sigma_1,\dots,\sigma_n)$ *being a lenght n multi–index.*

As a consequence $\Lambda_{k-1}\mathrm{Sym}$ inherits the structure of a $\mathcal{C}_\star\Lambda_{k-1}$–module. On the other hand $\Lambda_{k-1}\varkappa$ inherits the structure of a graded Lie–algebra denoted by $(\Lambda_{k-1}\varkappa,\{\cdot,\cdot\})$. In the following we identify $\Lambda_{k-1}\mathrm{Sym}$ and $\Lambda_{k-1}\varkappa$. Finally, note that d_m^{v} is an evolutionary derivation of Λ_{k-1} for any $m<k$. Denote by U_m the corresponding element in $\Lambda_{k-1}\varkappa$, i.e., $U_m=d_m^{\mathsf{v}}|_{\mathcal{C}_\circ\Lambda_{k-1}}$.

3. Adjoint graded $\mathcal{C}$–differential operators and horizontal modules

In what follows the algebra $\mathcal{C}_\star\Lambda_{k-1}$ is considered to be the ground algebra. In particular, all differential operators are operators over this algebra. Obviously, $D_\sigma(\mathcal{C}_\star\Lambda_{k-1})\subset\mathcal{C}_\star\Lambda_{k-1}$ for any lenght n multi–index σ. Let P,Q be locally free graded $\mathcal{C}_\star\Lambda_{k-1}$–modules of finite rank.

Definition 2. *A linear differential operator* $\square:P\longrightarrow Q$ *is called $\mathcal{C}$–differential if for any local basis* $\{e_1,\dots,e_r\}$ *of* P, $\square$ *is locally of the form* $\square(p)=(-1)^{|p|\cdot|\alpha|}\square_\alpha^\sigma D_\sigma p^\alpha$, *where* $\square_\alpha^\sigma\in Q$ *and* $p=p^\alpha e_\alpha\in P$, $|\alpha|\equiv|e_\alpha|$, $p^\alpha\in\mathcal{C}_\star\Lambda_{k-1}$, $\alpha=1,\dots,r$.

The totality of all $\mathcal{C}$–differential operators $\square:P\longrightarrow Q$ has a natural $\mathcal{C}_\star\Lambda_{k-1}$–module structure. This module is denoted by $\mathcal{C}\mathrm{Diff}(P,Q)$. Similarly, denote by $\mathcal{C}\mathrm{Diff}_{(p)}^{\mathrm{alt}}(P,Q)$ the $\mathcal{C}_\star\Lambda_{k-1}$–module of Q–valued, skew–symmetric, multi–$\mathcal{C}$–differential operators over P with $p\geq0$ entries ($\mathcal{C}\mathrm{Diff}_{(0)}^{\mathrm{alt}}(P,Q)\equiv Q$). $\mathcal{H}\Lambda_k$ is a $\mathcal{C}_\star\Lambda_{k-1}$–module and $d_k^{\mathsf{h}}:\mathcal{H}\Lambda_k\longrightarrow\mathcal{H}\Lambda_k$ is a $\mathcal{C}$–differential operator.

The theory of adjoint $\mathcal{C}$–differential operators in the category of $\mathcal{C}_\star\Lambda_{k-1}$–modules may be developed almost literally as in the standard non–graded case (see [4]). Here we limit ourselves to those elements of this theory that are necessary for the proposed below description of the $\Lambda_{k-1}\mathcal{C}$–spectral sequence terms .

Let P be a graded $\mathcal{C}_\star\Lambda_{k-1}$–module. Consider the map $w_k^P:\mathcal{C}\mathrm{Diff}(P,\mathcal{H}\Lambda_k)\longrightarrow\mathcal{C}\mathrm{Diff}(P,\mathcal{H}\Lambda_k)$, given by $w_k^P(\square)=d_k^{\mathsf{h}}\circ\square$, $\square\in\mathcal{C}\mathrm{Diff}(P,\mathcal{H}\Lambda_k)$. w_k^P is a differential, i.e., $w_k^P\circ w_k^P=0$. The cohomology space of w_k^P carries a natural $\mathcal{C}_\star\Lambda_{k-1}$–module structure defined by $\omega\cdot[\square]\overset{\mathrm{def}}{=}(-1)^{|\omega|\cdot|\square|}[\square\circ\omega]$, $\omega\in\mathcal{C}_\star\Lambda_{k-1}$, $\square\in\mathcal{C}\mathrm{Diff}(P,\mathcal{H}\Lambda_k)$, $d_k^{\mathsf{h}}\circ\square=0$.

The corresponding $\mathcal{C}_\star\Lambda_{k-1}$–module is denoted by $\widehat{P}$ and called the *adjoint to P module*. Put $\Lambda_{k-1}\mathcal{B} \equiv \widehat{\mathcal{C}_\star\Lambda_{k-1}}$.

Proposition 3. *$\Lambda_{k-1}\mathcal{B} \simeq \mathcal{C}_\star\Lambda_{k-1} \otimes_{\mathcal{F}} \mathcal{H}\Lambda_1^n$. Moreover, if P is a locally free $\mathcal{C}_\star\Lambda_{k-1}$–module of finite rank, then $\widehat{P} \simeq \mathrm{Hom}_{\mathcal{C}_\star\Lambda_{k-1}}(P, \Lambda_{k-1}\mathcal{B})$.*

In particular, proposition 3 tells that $\Lambda_{k-1}\mathcal{B}$ is a locally free $\mathcal{C}_\star\Lambda_{k-1}$–module of rank one. Thus, if P is a locally free $\mathcal{C}_\star\Lambda_{k-1}$–module of finite rank, then $\widehat{\widehat{P}} \simeq P$. Further on our exposition is based on identifications of proposition 3. Note that $\Lambda_{k-1}\mathcal{B}$ is embedded in $\ker d_k^{\mathfrak{h}} \subset \mathcal{H}\Lambda_k$ in a natural way by means of the correspondence $\Lambda_{k-1}\mathcal{B} \simeq \mathcal{C}_\star\Lambda_{k-1} \otimes_{\mathcal{F}} \mathcal{H}\Lambda_1^n \ni \omega \otimes \sigma \mapsto \omega \cdot \kappa_{1k}(\sigma) \in \mathcal{H}\Lambda_k$, $\omega \in \mathcal{C}_\star\Lambda_{k-1}$, $\sigma \in \mathcal{H}\Lambda_1^n$.

If P, Q are $\mathcal{C}_\star\Lambda_{k-1}$–modules and $\square \in \mathcal{C}\mathrm{Diff}(P, Q)$, then a differential operator $\widehat{\square} \in \mathcal{C}\mathrm{Diff}(\widehat{Q}, \widehat{P})$ is well–defined by $\widehat{\square}[\Delta] = (-1)^{|\Delta|\cdot|\square|}[\Delta \circ \square]$, $\Delta \in \mathcal{C}\mathrm{Diff}(Q, \mathcal{H}\Lambda_k)$, $d_k^{\mathfrak{h}} \circ \Delta = 0$. $\widehat{\square}$ is called the *adjoint to $\square$ operator*.

Let P be as above, $\xi_1, \ldots, \xi_{p-2} \in P$ and $\square \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(P, \widehat{P})$. Define a $\mathcal{C}$–differential operator $\square_{\xi_1,\ldots,\xi_{p-2}} : P \longrightarrow \widehat{P}$ by putting $\square_{\xi_1,\ldots,\xi_{p-2}}(\xi) \overset{\mathrm{def}}{=} \square(\xi_1, \ldots, \xi_{p-2}, \xi) \in \widehat{P}$. Also put

$$L_p^{(k)}(P) \overset{\mathrm{def}}{=} \{\square \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(P, \widehat{P}) \mid \widehat{\square_{\xi_1,\ldots,\xi_{p-2}}} = -\square_{\xi_1,\ldots,\xi_{p-2}},\ \forall \xi_1, \ldots, \xi_{p-2} \in P\}.$$

A $\mathcal{C}_\star\Lambda_{k-1}$–module P is called *horizontal* if it is of the form $P \simeq \mathcal{C}_\star\Lambda_{k-1} \otimes_{C^\infty(M)} P_0$ for a (possibly graded) $C^\infty(M)$–module P_0. Evolutionary derivations of Λ_{k-1} act naturally on horizontal modules. Namely, let $\chi \in \Lambda_{k-1}\mathrm{Sym}$ and ϑ_χ be the corresponding evolutionary derivation of Λ_{k-1}. For $\xi = \omega \otimes \xi_0 \in P$, $\omega \in \mathcal{C}_\star\Lambda_{k-1}$ and $\xi_0 \in P_0$ we put

$$\vartheta_\chi \xi \overset{\mathrm{def}}{=} \vartheta_\chi \omega \otimes \xi_0 \in P. \tag{3}$$

Since ϑ_χ is a vertical derivation, definition (3) extends unambiguously to the whole P. Now, fix an element $\xi \in P$ and define an operator $\ell_\xi^{\{k\}} : \Lambda_{k-1}\varkappa \longrightarrow P$ by putting $\ell_\xi^{\{k\}}(\chi) \overset{\mathrm{def}}{=} (-1)^{|\xi|\cdot|\chi|}\vartheta_\chi \xi$, $\chi \in \Lambda_{k-1}\varkappa$. $\ell_\xi^{\{k\}}$ is a $\mathcal{C}$–differential operator called the *universal Λ_{k-1}–linearization* of ξ.

Example 4. *$\mathcal{H}\Lambda_k$ is a horizontal module. Indeed, $\mathcal{H}\Lambda_k \simeq \mathcal{C}_\star\Lambda_{k-1} \otimes_{C^\infty(M)} \Lambda_k(M)$.*

Example 5. *$\Lambda_{k-1}\varkappa$ and $\widehat{\Lambda_{k-1}\varkappa}$ are horizontal modules. Indeed, let $W_{k-1} \subset \Lambda_{k-1}\varkappa$ be the $C^\infty(M)$–submodule of derivations that are locally of the form $\chi_K^j V_j^K$ with $\chi_K^j \in C^\infty(M)$, $K \subset \{1, \ldots, k-1\}$, $j = 1, \ldots, m$. W_{k-1} is well defined. Then $\Lambda_{k-1}\varkappa \simeq \mathcal{C}_\star\Lambda_{k-1} \otimes_{C^\infty(M)} W_{k-1}$ and $\widehat{\Lambda_{k-1}\varkappa} \simeq \mathcal{C}_\star\Lambda_{k-1} \otimes_{C^\infty(M)} \mathrm{Hom}_{C^\infty(M)}(W_{k-1}, \Lambda^n(M))$.*

4. Secondary IDFs on $J^\infty(\pi)$

The complexes $(\Lambda_{k-1}\mathcal{C}E_0^{0,\bullet} = \Lambda_k/\mathcal{C}\Lambda_k, d_{k,0}^{0,\bullet})$ and $(\mathcal{H}\Lambda_k, d_k^{\mathfrak{h}})$ are isomorphic in a natural way and further on we shall identify $\Lambda_{k-1}\mathcal{C}E_0^{0,\bullet}$ with $\mathcal{H}\Lambda_k$.

Remark 6. *For any $p > 0$, there exists a natural isomorphism of complexes*

$$\begin{array}{ccc} \Lambda_{k-1}\mathcal{C}E_0^{p,\bullet} & \xrightarrow{d_{0,k}^{p,\bullet}} & \Lambda_{k-1}\mathcal{C}E_0^{p,\bullet} \\ \eta_{k-1}\downarrow\wr & & \eta_{k-1}\downarrow\wr \\ \mathcal{C}\mathrm{Diff}_{(p)}^{\mathrm{alt}}(\Lambda_{k-1}\varkappa, \mathcal{H}\Lambda_k) & \xrightarrow{w_k^p} & \mathcal{C}\mathrm{Diff}_{(p)}^{\mathrm{alt}}(\Lambda_{k-1}\varkappa, \mathcal{H}\Lambda_k) \end{array},$$

where $w_k^p(\square) \overset{\mathrm{def}}{=} (-1)^p d^{\mathsf{h}} \circ \square$, $\square \in \mathcal{C}\mathrm{Diff}_{(p)}^{\mathrm{alt}}(\Lambda_{k-1}\varkappa, \mathcal{H}\Lambda_k)$. The isomorphism η_{k-1} is defined by

$$\eta_{k-1}([\omega]_{\mathcal{C}^{p+1}\Lambda_k})(\chi_1, \dots, \chi_p) = (-1)^{|\omega|\cdot(|\chi_1|+\cdots+|\chi_p|)+p(p-1)/2}[(i^{\{k\}}_{\ni_{\chi_1}} \circ \cdots \circ i^{\{k\}}_{\ni_{\chi_p}})(\omega)]_{\mathcal{C}\Lambda_k} \in \mathcal{H}\Lambda_k.$$

The following two theorems generalize the results of [6, 2] concerning the first term of the $\mathcal{C}$–spectral sequence to that of the $\Lambda_{k-1}\mathcal{C}$–spectral one. In particular, they give an explicit description of secondary IDFs on $J^\infty(\pi)$ and basic operations over them.

Theorem 7 (One Line Theorem)**.**

- $\Lambda_k\mathcal{C}E_1^{p,q} = 0$ *if either $q > n$, or $p > 0$ and $q < n$;*
- $\Lambda_{k-1}\mathcal{C}E_1^{0,\bullet} \simeq H(\mathcal{H}\Lambda_k, d_k^{\mathsf{h}}) \simeq \Lambda_{k-2}\mathcal{C}E_1^{\bullet,\bullet}$*;*
- $\Lambda_{k-1}\mathcal{C}E_1^{p,n} \simeq L_p^{(k)}(\Lambda_{k-1}\varkappa)$*, if $p > 0$.*

Corollary 8. *$\Lambda_{k-1}\mathcal{C}E$ stabilizes at the second term and $\Lambda_{k-1}\mathcal{C}E_2^{0,q} \simeq H^q(\Lambda_k, d_k) \simeq H^q(E)$, $q \le n$, and $\Lambda_{k-1}\mathcal{C}E_2^{p,n} = H^{p+n}(E)$, $p \ge 0$.*

There is a distinguished element $\mathfrak{u}$ in $\widehat{\mathcal{H}\Lambda_k} \simeq \mathrm{Hom}_{\mathcal{C}_\star\Lambda_{k-1}}(\mathcal{H}\Lambda_k, \Lambda_{k-1}\mathcal{B})$ whose local expression is

$$\begin{aligned}\mathfrak{u}(\Omega^{J_1\cdots J_r}_{\mu_1\cdots\mu_r} d_{J_1}x^{\mu_1}\cdots d_{J_r}x^{\mu_r}) &\overset{\mathrm{def}}{=} \Omega^{J_1\cdots J_r}_{\mu_1\cdots\mu_r} \otimes ((\kappa_{1k}\circ\nu)(d_{J_1}x^{\mu_1}\cdots d_{J_r}x^{\mu_r})) \\ &\in \mathcal{C}_\star\Lambda_{k-1}\otimes_{\mathcal{F}}\mathcal{H}\Lambda_1^n \simeq \Lambda_{k-1}\mathcal{B},\end{aligned}$$

where $\Omega^{J_1\cdots J_r}_{\mu_1\cdots\mu_r} \in \mathcal{C}_\star\Lambda_{k-1}$ and $\nu : \mathcal{H}\Lambda_k \longrightarrow \mathcal{H}\Lambda_k^n$ is the projection onto the homogeneous component of multi–degree $(0, \dots, 0, n) \in \mathbb{Z}^k$.

Theorem 9.

- *The differential $d_{k,1}^{0,n} : H^n(\mathcal{H}\Lambda_k, d_k^{\mathsf{h}}) \longrightarrow \widehat{\Lambda_{k-1}\varkappa}$ is given by*

  $$d_{k,1}^{0,n}[\Omega]_{\mathrm{im}\, d_k^{\mathsf{h}}} = \widehat{\ell}^{\{k\}}_\Omega(\mathfrak{u}), \quad \Omega \in \mathcal{H}\Lambda_k^n;$$

  and its local description is

  $$\begin{aligned}d_{k,1}^{0,n}([\Omega]_{\mathrm{im}\, d_k^{\mathsf{h}}})(\chi) &= (-1)^{\sigma+|\chi|\cdot|\Omega|}\chi_L^j(D_\sigma\circ V_j^{\sigma L})(A)\otimes dx^1\cdots dx^n \\ &\in \mathcal{C}_\star\Lambda_{k-1}\otimes_{\mathcal{F}}\mathcal{H}\Lambda_1^n \simeq \Lambda_{k-1}\mathcal{B},\end{aligned}$$

  assuming that $\Omega = A\, d_kx^1\cdots d_kx^n \in \mathcal{H}\Lambda_k^n$, $A \in \mathcal{C}_\star\Lambda_{k-1}$ and $\chi = \chi_L^j V_j^L \in \Lambda_{k-1}\varkappa$;

- *the differential* $d_{k,1}^{p,n} : L_p^{(k)}(\Lambda_{k-1}\varkappa) \longrightarrow L_{p+1}^{(k)}(\Lambda_{k-1}\varkappa)$ *acting on secondary iterated* p*–forms,* $p > 0$*, is given by*

$$
\begin{aligned}
&d_{k,1}^{p,n}(\Theta)(\chi_1, \dots, \chi_p) = \sum_{i=1}^{p} (-1)^{a(i)+i+1} (\mathcal{Э}_{\chi_i})(\Theta(\chi_1, \dots, \widehat{\chi}_i, \dots, \chi_p)) \\
&+ \sum_{i<j} (-1)^{c(i,j)+i+j} \Theta(\{\chi_i, \chi_j\}, \chi_1, \dots, \widehat{\chi}_i, \dots, \widehat{\chi}_j, \dots, \chi_p) \\
&+ \tfrac{1}{p} \sum_{i=1}^{p} (-1)^{i+1} [(-1)^{a(i)} (p-1) \widehat{\ell}_{\chi_i}^{\{k+1\}} (\Theta(\chi_1, \dots, \widehat{\chi}_i, \dots, \chi_p)) \\
&- (-1)^{b(i)} \widehat{\ell}_{\Theta(\chi_1, \dots, \widehat{\chi}_i, \dots, \chi_p)}^{\{k+1\}} (\chi_i)],
\end{aligned}
$$

$\Theta \in L_p^{(k)}(\Lambda_{k-1}\varkappa)$*, where* $a(i) = |\chi_i| \cdot (|\Theta| + |\chi_1| + \cdots + |\chi_{i-1}|)$*,* $b(i) = |\chi_i| \cdot (|\chi_{i+1}| + \cdots + |\chi_p|)$*,* $c(i,j) = a(i) + a(j) - |\Theta| \cdot (|\chi_i| + |\chi_j|) - |\chi_i| \cdot |\chi_j|$*;*

- *the action* $\mathcal{L}_\chi^{\{k\}} : L_p^{(k)}(\Lambda_{k-1}\varkappa) \longrightarrow L_p^{(k)}(\Lambda_{k-1}\varkappa)$ *of* $\chi \in \Lambda_{k-1}\mathrm{Sym}$ *on secondary iterated* p*–forms,* $p > 0$*, is given by*

$$
\begin{aligned}
(\mathcal{L}_\chi^{\{k\}}\Theta)(\chi_1, \dots, \chi_{p-1})(\chi_p) &= (\mathcal{Э}_\chi \circ \Theta)(\chi_1, \dots, \chi_p) \\
&- \sum_{i=1}^{p} (-1)^{|\chi| \cdot (|\chi_1| + \cdots + |\chi_{i-1}| + |\Theta|)} \Theta(\chi_1, \dots, \{\chi, \chi_i\}, \dots, \chi_p),
\end{aligned}
$$

$\Theta \in L_p^{(k)}(\Lambda_{k-1}\varkappa)$.

- *the insertion* $i_\chi^{\{k\}} : \widehat{\Lambda_{k-1}\varkappa} \longrightarrow H(\mathcal{H}\Lambda_k, d_k^{\mathfrak{h}})$ *of* $\chi \in \Lambda_{k-1}\mathrm{Sym}$ *in secondary iterated* 1*–forms is given by*

$$
i_\chi^{\{k\}}\Theta = (-1)^{|\chi| \cdot |\Theta|} [\Theta(\chi)]_{\mathrm{im}\, d_k^{\mathfrak{h}}}, \quad \Theta \in \widehat{\Lambda_{k-1}\varkappa},\ \Theta(\chi) \in \Lambda_{k-1}\mathcal{B} \hookrightarrow \ker d_k^{\mathfrak{h}}.
$$

- *the insertion* $i_\chi^{\{k\}} : L_p^{(k)}(\Lambda_{k-1}\varkappa) \longrightarrow L_{p-1}^{(k)}(\Lambda_{k-1}\varkappa)$ *of* $\chi \in \Lambda_{k-1}\mathrm{Sym}$ *in secondary iterated* p*–forms,* $p > 1$*, is given by*

$$
(i_\chi^{\{k\}}\Theta)(\chi_1, \dots, \chi_{p-2}) = (-1)^{|\chi| \cdot |\Theta|} \Theta(\chi, \chi_1, \dots, \chi_{p-2}).
$$

5. Secondary covariant tensors

Recall that $\tau \in \Lambda_{k-1}\mathcal{C}E_1^{(1,\dots,1),\bullet}$ is a secondary covariant k–tensor iff $\mathcal{L}_{I_m^{K'}}^{\{k\}}\tau = i_{I_m^K}^{\{k\}}\tau = 0$ for any $m < k$ and $K = \{k_1, \dots, k_s\}, K' = \{k'_1, \dots, k'_{s'}\} \subset \{1, \dots, k-1\}$, $s \geq 2$, $s' \geq 1$ (see [1]). In what follows we characterize such τ's. First of all, there is a well defined $\mathcal{F}$–linear surjective map $\mathfrak{p}_{k-1} : \Lambda_{k-1}\varkappa \longrightarrow \Lambda_0\varkappa \equiv \varkappa$ locally given by

$$
\mathfrak{p}_{k-1}(\chi_K^j V_j^K) \overset{\text{def}}{=} \mathfrak{p}_0(\chi^j) \tfrac{\partial}{\partial u^j}, \quad \chi_K^j \in \mathcal{C}_\star\Lambda_{k-1},\ K \subset \{1, \dots, k-1\},\ j = 1, \dots, m,
$$

$\mathsf{p}_0 : \mathcal{C}_\star\Lambda_{k-1} \longrightarrow \mathcal{F}$ being the projection onto the homogeneous component of multi–degree $(0,\ldots,0) \in \mathbb{Z}^{k-1}$. Then there is an injective morphism of $\mathcal{F}$–modules $\mathfrak{i}_k$: $(\mathcal{C}\Lambda^1)^{\otimes k-1} \otimes_\mathcal{F} \widehat{\varkappa} \longrightarrow \Lambda_k\mathcal{C}E_1^{(1,\ldots,1),\bullet} \subset \widehat{\Lambda_{k-1}\varkappa}$ given by

$$\mathfrak{i}_k(\omega_1 \otimes \cdots \otimes \omega_{k-1} \otimes \psi)(\chi) \stackrel{\mathrm{def}}{=} (\omega_1 \cdot \kappa_{12}(\omega_2) \cdot \cdots \cdot \kappa_{1k-1}(\omega_{k-1})) \otimes ((\psi \circ \mathfrak{p}_{k-1})(\chi)) \in \mathcal{C}_\star\Lambda_{k-1} \otimes_\mathcal{F} \mathcal{H}\Lambda_1^n \simeq \Lambda_{k-1}\mathcal{B},$$

$\omega_1,\ldots,\omega_{k-1} \in \mathcal{C}\Lambda^1$, $\psi \in \widehat{\varkappa}$, $\chi \in \Lambda_{k-1}\varkappa$.

Now we are able to characterize secondary IDFs on $J^\infty(\pi)$ in the same manner as it was done for ordinary IDFs in [5]

Proposition 10. $\tau \in \Lambda_{k-1}\mathcal{C}E_1^{(1,\ldots,1),\bullet}$ *is a secondary covariant* k*–tensor iff* $\tau \in \mathrm{im}\,\mathfrak{i}_k$.

6. Conclusions

The above given description of secondary IDFs on $J^\infty(\pi)$ was not conceptual in all its aspects just because the necessary for this purpose constructions can not be put in frames of a short note. This will be done in the subsequent detailed exposition together with basic elements of secondary tensor calculus and some applications to mechanics and field theory.

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