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Symplectic Manifold**

by

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## A Spectral Sequence Associated with a Symplectic Manifold

A. M. VINOGRADOV AND C. DI PIETRO

ABSTRACT. With a symplectic manifold a spectral sequence converging to its de Rham cohomology is associated. A method of computation of its terms is presented together with some stabilization results. As an application a characterization of symplectic harmonic manifolds is given and a relationship with the  $\mathcal{C}$ -spectral sequence is indicated.

Let  $(M, \Omega)$  be a  $2n$ -dimensional symplectic manifold and  $\Lambda(M)$  be the algebra of differential forms on  $M$ . Consider the ideal  $\Lambda_{\mathcal{L}}(M)$  of  $\Lambda(M)$ , composed of all differential forms that vanish when restricted to any Lagrangian submanifold of  $M$ . This ideal is differentially closed and its powers constitute the *symplectic filtration* in the de Rham complex of  $M$ . The corresponding spectral sequence  $\{E_r^{p,q}, d_r^{p,q}\}$  is called the *symplectic spectral sequence* associated with  $(M, \Omega)$ .

A motivation for this construction comes from the theory of  $\mathcal{C}$ -spectral sequences (see [5]). Moreover, if  $M = T^*N$ , then the symplectic spectral sequence is nothing but the “classical part” of the  $\mathcal{C}$ -spectral sequence associated with the differential equation  $d\rho = 0$ ,  $\rho \in \Lambda^1(N)$ .

### 1. NOTATIONS AND PRELIMINARIES

In this section the notation is fixed and all necessary facts concerning symplectic manifolds (see [1, 2, 6] for further details) are collected.

Throughout the paper  $(M, \Omega)$  stands for a  $2n$ -dimensional symplectic manifold,  $\Lambda = \sum_k \Lambda^k$  for the algebra of differential forms on  $M$ ,  $H(M) = \sum_k H^k(M)$  for the de Rham cohomology of  $M$  and  $D = \sum_k D_k$  for the algebra of multivectors on  $M$ .

The isomorphism  $\Gamma_1: V \in D_1 \mapsto V \lrcorner \Omega \in \Lambda^1$  of  $C^\infty(M)$ -modules extends uniquely to a  $C^\infty(M)$ -algebra isomorphism  $\Gamma: D \rightarrow \Lambda$ .

$P = \Gamma^{-1}(\Omega)$  is called the *corresponding to  $\Omega$  Poisson bivector*.  $C^\infty(M)$ -linear operators

$$\top: \Lambda^k \rightarrow \Lambda^{k+2}, \quad \top\omega = \omega \wedge \Omega,$$

$$\perp: \Lambda^k \rightarrow \Lambda^{k-2}, \quad \perp\omega = P \lrcorner \omega,$$

acting on  $\Lambda$  are basic for our purposes. Put  $\top\Lambda = \text{im } \top$  and  $\Lambda_\epsilon = \ker \perp$ . Elements of  $\Lambda_\epsilon$  are called *effective forms*.

Another very useful fact is the *Hodge–Lepage* expansion (see, for instance, [2]):

**Proposition 1.** *Any  $\omega \in \Lambda^k$ ,  $k \geq 0$ , admits a unique expansion of the form*

$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 \cdots + \top^n\omega_n$$

with  $\omega_i \in \Lambda_\epsilon^{k-2i}$ . In particular,  $\Lambda$  splits into the direct sum  $\Lambda_\epsilon \oplus \top\Lambda$ .

There exists a symplectic analogue of the Hodge star-operator

$$*: \Lambda^k \rightarrow \Lambda^{2n-k}, \quad k \geq 0,$$

uniquely characterized by the following property:

$$\eta \wedge * \omega = (\perp^k(\eta \wedge \omega))\Omega^n, \quad \eta, \omega \in \Lambda^k.$$

The map  $\delta = \{\delta_k\}_k$ ,  $\delta_k = (-1)^{k+1} * d*: \Lambda^k \rightarrow \Lambda^{k-1}$ , is a  $(-1)$ -degree differential in  $\Lambda$  and  $[\perp, d] = \delta$  (see [1]).

The introduced operators are subject to the following graded commutation relations:

$$[d, \delta] = 0 \quad ; \quad [\top, d] = 0 \quad ; \quad [\top, \delta] = d \quad ; \quad [\perp, \delta] = 0, \quad (1)$$

the first of which shows that  $(\Lambda, d, \delta)$  is a bicomplex (see [1]).

**Proposition 2.** *The Hodge–Lepage expansion of  $d\omega$ ,  $\omega \in \Lambda_\epsilon$ , is of the form*

$$d\omega = (d\omega)_0 + \top(d\omega)_1, \quad (d\omega)_0, (d\omega)_1 \in \Lambda_\epsilon.$$

**Corollary 1.**  $\Lambda_\epsilon$  is  $\delta$ -closed.

## 2. THE TERM $E_0$

An explicit description of the symplectic filtration is based on the following fact of linear algebra.

**Proposition 3.**  $\Lambda_{\mathcal{L}} = \top\Lambda$ .

It shows that the operator  $\top^k$  respects the symplectic filtration by shifting it by  $2k$ . Moreover, it commutes with  $d$  and, so, induces an automorphism  $\tau = \{\tau_r^k\}$ ,  $\tau_r^k$  colon  $E_r^{p,q} \rightarrow E_r^{p+k, q+k}$ , of the symplectic spectral sequence.

**Proposition 4.**

- (1) *The term  $E_0^{p,q}$  is trivial if  $(p, q) \in \mathbb{Z}^2$  lies outside the triangle with vertexes at  $(0, 0)$ ,  $(0, n)$  and  $(n, n)$ .*
- (2) *The term  $E_0^{0,q}$  is naturally isomorphic to  $\Lambda_\epsilon^q$ .*
- (3)  *$\tau_0^p: E_0^{0, q-p} \rightarrow E_0^{p,q}$  is an isomorphism, if  $0 \leq p \leq q \leq n$ .*

The last assertion of Proposition 4 is inductively generalized to terms  $E_r$ ,  $r > 0$ .

**Proposition 5.** *Let  $r > 0$ . Then  $\tau_r^k: E_r^{p,q} \rightarrow E_r^{p+k, q+k}$  is an isomorphism, if  $\sum_{i=1}^r (i-1) \leq p \leq q \leq q+k \leq n - (2 + \sum_{i=1}^r (i-2))$ . In particular,  $\tau_1^p: E_1^{0, q-p} \rightarrow E_1^{p,q}$  is an isomorphism, if  $0 \leq p \leq q < n$ .*

3. THE TERM  $E_1$ 

The exact sequence constructed in this section gives a useful description of the first term of the symplectic spectral sequence. It is composed of the following two families of  $\mathbb{R}$ -homomorphisms;

$$\begin{aligned} \phi_{p,q}: H^{q-p}(M) &\rightarrow E_1^{p,q}, & 0 \leq p \leq q \leq n \\ [\omega]_{\text{im } d} &\mapsto [\top^p \omega]_{\text{im } d_0} \\ \psi_{p,q}: E_1^{p,q} &\rightarrow H^{q-p-1}(M), & 0 \leq p \leq q < n \\ [\top^p \rho]_{\text{im } d_0} &\mapsto [\eta]_{\text{im } d} \end{aligned}$$

with  $\top \eta = d\rho$ .

Put  $A^k = \{\omega \mid \omega \in \Lambda^k; d\omega \in \Lambda_{\epsilon}^{k+1}\}$ ,  $C^k = A^k / \text{im } d \cap \Lambda^k$  and consider  $\mathbb{R}$ -homomorphisms

$$\begin{aligned} \phi_{p,n}: C^{n-p} &\rightarrow E_1^{p,n}, & [\omega]_{\text{im } d} \mapsto [\top^p \omega]_{\text{im } d_0} \\ \psi_{p,n}: E_1^{p,n} &\rightarrow C^{n-(p+1)}, & [\top^p \rho]_{\text{im } d_0} \mapsto [\eta]_{\text{im } d}. \end{aligned}$$

The above defined homomorphisms together with the multiplication by the cohomology class of  $\Omega$  homomorphism  $\tau: H^*(M) \rightarrow H^{*+2}(M)$  form the following sequences

$$\begin{aligned} \dots \rightarrow H^{q-p}(M) &\xrightarrow{\phi_{p,q}} E_1^{p,q} \xrightarrow{\psi_{p,q}} H^{q-(p+1)}(M) \\ &\xrightarrow{\tau} H^{q+1-p}(M) \xrightarrow{\phi_{p,q+1}} E_1^{p,q+1} \rightarrow \dots \end{aligned} \quad (2)$$

whose left and right ends are

$$0 \rightarrow H^0(M) \xrightarrow{\phi_{p,p}} E_1^{p,p} \xrightarrow{\psi_{p,p}} 0 \xrightarrow{\tau} H^1(M) \xrightarrow{\phi_{p,p+1}} E_1^{p,p+1} \rightarrow \dots$$

and

$$\begin{aligned} \dots \rightarrow H^{n-(p+2)}(M) &\xrightarrow{\tau} C^{n-p} \xrightarrow{\phi_{p,n}} E_1^{p,n} \\ &\xrightarrow{\psi_{p,n}} C^{n-(p+1)} \xrightarrow{\tau^{p+1}} H^{n+p+1}(M) \rightarrow 0, \end{aligned}$$

respectively. Notice that sequence (2) involves the terms of the  $p$ -th column of  $E_1$ .

**Theorem 1.** *The sequence (2) is exact.*

## 4. STABILIZATION THEOREMS

Theorem 1 is key in studying stability of the symplectic spectral sequence. For instance, if  $\Omega$  is exact, then the homomorphism  $\tau: H^k(M) \rightarrow H^{k+2}(M)$  in (2) is trivial. This fact and Theorem 1 lead to the following result.

**Theorem 2.** *If  $\Omega$  is exact, then the symplectic spectral sequence stabilizes at the term  $E_2$ . Moreover, if a term  $E_2^{p,q}$  is different from zero, then either  $p = 0$ , or  $q = n$  and  $E_2^{0,q} \cong H^q(M)$ , if  $q \leq n$ , and  $E_2^{p,n} \cong H^{n+p}(M)$ , if  $p \geq 0$ .*

**Corollary 2.** *If  $\Omega$  is the standard symplectic form on  $M = T^*N$ ,  $\dim N = n$ , then the corresponding symplectic spectral sequence stabilizes at the second term and  $E_2^{0,q} \cong H^q(N)$ ,  $E_2^{p,q} = 0$ , for  $p > 0$ .*

Theorem 2 is generalized as follows.

**Theorem 3.** *Let  $t > 1$  be the minimal integer such that  $\Omega^t = dp$ . Then the symplectic spectral sequence stabilizes at the term  $E_{t-r+1}$  where  $r$  is an integer such that  $0 \leq r \leq \max\{0, 2t - (n + 1)\}$ .*

The estimate in the previous theorem can not be, generally, improved as the following example shows .

**Example 1.** *The symplectic spectral sequence associated with  $(C, \Omega) \times (\mathbb{R}^{2h}, \Omega_{\mathbb{R}^{2h}})$ , where  $(C, \Omega)$  is a closed symplectic  $2m$ -fold and  $(\mathbb{R}^{2h}, \Omega_{\mathbb{R}^{2h}})$  is the standard symplectic manifold, is not stable in terms  $E_h$ , if  $h \leq m + 1 = t$ , and becomes stable by starting from the term  $E_{m+2}$ .*

If cohomology classes  $[\Omega^t]$ ,  $t \leq n$ , are all nontrivial, then  $M$  is closed. Closed symplectic manifolds are characterized by the fact that  $E_\infty^{p,p} = E_2^{p,p} \cong \mathbb{R}$ , if  $0 \leq p \leq n$ . Moreover, it holds

**Theorem 4.** *The symplectic spectral sequence for a closed symplectic manifold stabilizes at the term  $E_2$ .*

## 5. THE BRYLINSKI CONJECTURE

**Definition 1.** *A form  $\eta$  is called symplectically harmonic iff  $\eta \in \ker d \cap \ker \delta$ .*

In [1] Brylinski conjectured that each cohomology class of a closed symplectic manifold contains at least one symplectically harmonic form. It was, however, disproved by counterexamples found by Mathieu (see [3]). So, the problem of characterization of symplectic manifolds for which the Brylinski conjecture holds arises. Call such symplectic manifolds *harmonic*. In view of (1) and Proposition 2 we have the following useful technical characterization of harmonic symplectic manifolds.

**Proposition 6.** *A symplectic manifold is harmonic iff any cohomology class is represented by a form  $\eta$  whose Hodge–Lepage expansion terms are all closed.*

This Proposition helps to prove

**Theorem 5.** *A closed symplectic manifold is harmonic iff  $\tau_2^{n-q}: E_2^{0,q} \rightarrow E_2^{n-q,n}$  is an isomorphism for all  $0 \leq q \leq n$ .*

**Corollary 3.** *For a closed harmonic manifold mappings  $\tau_2^{n-p-q}: E_2^{p,q} \rightarrow E_2^{n-q,n-p}$  are isomorphisms for  $p + q \leq n$  and  $0 \leq p \leq q$ . In particular,  $E_2$  is symmetric with respect to the line  $p + q = n$ .*

## 6. ASSOCIATED DIFFIETY

Let  $J^k(M, n)$  be the manifold of  $k$ -jets of  $n$ -dimensional submanifolds of a manifold  $M$ . The  $k$ -th prolongation  $M_{(k)}$  of a symplectic manifold  $(M, \Omega)$  is a submanifold in  $J^k(M, n)$  composed of  $k$ -th jets of Lagrangian submanifolds of  $M$ . By restricting the Cartan distribution on  $J^\infty(M, n)$  to  $M_{(\infty)}$  we obtain a diffiety which locally coincides with the infinite prolongation of the equation  $d\rho = 0, \rho \in \Lambda^1(N), \dim N = n$ . A natural projection  $M_{(\infty)} \rightarrow M$  induces a morphism of the symplectic spectral sequence of  $M$  to the  $\mathcal{C}$ -spectral sequence of the diffiety  $M_{(\infty)}$ . This allows to find out some useful interpretations for various terms and differentials of the symplectic spectral sequence. For instance, consider the following functional defined on compact Lagrangian submanifolds associated with an element  $\theta \in E_1^{0,n}$ :

$$L \mapsto \int_L \omega|_L, \quad \theta = [\omega], \quad \omega \in \Lambda^n \pmod{\mathcal{T}\Lambda}.$$

Then  $d_1^{0,n}(\theta) = 0$  is the Euler-Lagrange equation for extremals of this functional.

## 7. GENERALISATIONS

There are numerous analogues of the symplectic spectral sequence due to the fact that the underlying construction is of a rather general nature. Below we list some of its "neighbors".

- Let  $N$  be a submanifold of a symplectic manifold  $M$ . By restricting the symplectic filtration in  $\Lambda(M)$  to  $\Lambda(N)$  we get a spectral sequence converging to the de Rham cohomology of  $N$ . In particular, this way a spectral sequence is associated with a Hamilton–Jacobi equation  $\mathcal{E}$ . This spectral sequence is the "classical part" of the  $\mathcal{C}$ -spectral sequence associated with  $\mathcal{E}$ .
- Let  $M$  be a contact manifold and  $\mathcal{I} \subset \Lambda(M)$  the ideal composed of differential forms that vanish on all Legendre submanifolds of  $M$ . Powers of  $\mathcal{I}$  form the *contact filtration* in  $\Lambda(M)$ . This way one gets the *contact spectral sequence* associated with a contact manifold.
- By restricting the contact filtration to a submanifold  $N$  of a contact manifold  $M$  one gets a spectral sequence which is the classical part of the  $\mathcal{C}$ -spectral sequence associated with  $N$  interpreted as an (overdetermined) system of first order scalar differential equations (see [2, 5]).
- Let  $(M, P)$  be a Poisson manifold,  $P$  being the Poisson bivector. The *Poisson filtration* in the algebra  $D(M)$  of multivectors fields on  $M$  is that formed by powers of the principal ideal generated by  $P$ . The Poisson differential  $d_P, d_P(Q) = [[P, Q]], Q \in D(M)$ , respects this filtration and the corresponding to it spectral sequence is called *Poisson*. If  $P$  is nondegenerate the Poisson spectral sequence is, in a sense, dual to the corresponding symplectic one.

Constructions and results of this note naturally generalize to all these spectral sequences. Details will be given in a separate publication.

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