

Iterated Differential Forms IV: \mathcal{C} –Spectral Sequence

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A. M. VINOGRADOV AND L. VITAGLIANO

ABSTRACT. For the multiple differential algebra of iterated differential forms [1, 2, 3, 4] on a diffiety $(\mathcal{O}, \mathcal{C})$ [5, 6] an analogue of \mathcal{C} -spectral sequence [7, 8, 6] is constructed. The first term of it is naturally interpreted as the algebra of *secondary* [5, 6, 9] iterated differential forms on $(\mathcal{O}, \mathcal{C})$. This allows to develop *secondary tensor analysis* on generic diffieties, some simplest elements of which are sketched here. The presented here general theory will be specified to infinite jet spaces and infinitely prolonged PDEs in subsequent notes.

1. INTRODUCTION

Let $(\mathcal{O}, \mathcal{C})$ be a diffiety. Recall that an elementary diffiety is a pair $(\mathcal{E}^\infty, \mathcal{C}(\mathcal{E}))$, where \mathcal{E}^∞ is the infinite prolongation of a system \mathcal{E} of partial differential equations and $\mathcal{C}(\mathcal{E})$ the Cartan distribution on it (see [5, 7, 8]). More generally, a diffiety is a smooth, usually pro-finite manifold \mathcal{O} endowed with a finite dimensional integrable distribution \mathcal{C} , locally identical to an elementary diffiety (see [6, 9]).

Recall that Secondary Calculus on the diffiety $(\mathcal{E}^\infty, \mathcal{C}(\mathcal{E}))$ is the proper formalization of the idea of differential calculus on the “space of all solutions of a system of partial differential equations \mathcal{E} ”. By this reason the search for analogues in secondary calculus of objects, concepts and constructions of “traditional” mathematics is an actual and important problem, called the *seconдарization problem* (see [6, 9]). In this note this problem is solved for iterated differential forms (in the sequel, IDFs) on the basis of the algebraic approach developed in preceding notes [1, 2] of this series. Below we use the notation, definitions and results of these notes. It is worth noticing that the alternative approach to iterated forms proposed in [3, 4] is not very seconдарization friendly.

Recall that the solution of the seconдарization problem for usual, i.e., one time iterated forms, is given in [1, 2] (see also [6, 9]). Namely, the algebra of secondary differential forms is realized as the first term of the \mathcal{C} -spectral sequence. In this note an analogue of this sequence is constructed and its first term is interpreted as the algebra of *secondary IDFs*.

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2. PRELIMINARIES ON IDF'S

Let \mathcal{O} be a smooth, generally, pro-finite manifold, i.e., the inverse limit of a sequence of finite-dimensional submersions

$$\cdots \longleftarrow \mathcal{O}_{s-1} \longleftarrow \mathcal{O}_s \longleftarrow \mathcal{O}_{s+1} \longleftarrow \cdots . \quad (1)$$

Denote by $(\Lambda_k(\mathcal{O}_s), d_1, \dots, d_k)$ the \mathbb{Z}^k -graded multiple differential algebra of geometric k -times IDF's on the (usual) manifold \mathcal{O}_s (see [1, 2]). Sequence (1) induces the sequence of embeddings of graded multiple differential algebras

$$\cdots \subset \Lambda_k(\mathcal{O}_{s-1}) \subset \Lambda_k(\mathcal{O}_s) \subset \Lambda_k(\mathcal{O}_{s+1}) \subset \cdots .$$

Then $\Lambda_k(\mathcal{O}) \stackrel{\text{def}}{=} \bigcup_s \Lambda_k(\mathcal{O}_s)$ is a \mathbb{Z}^k -graded multiple differential algebra.

Definition 1. *Elements of the algebra $\Lambda_k(\mathcal{O})$ are called k -times IDF's on the pro-finite manifold \mathcal{O} .*

The theory of IDF's on pro-finite manifolds is a straightforward generalization of the theory presented in [1, 2] and by this reason we shall not go into its details here.

Denote by $D(\Lambda_k(\mathcal{O}), \Lambda_m(\mathcal{O}))$ the $\Lambda_m(\mathcal{O})$ -module of graded $\Lambda_m(\mathcal{O})$ -valued derivations of $\Lambda_k(\mathcal{O})$, $k \leq m$.

Proposition 2. *Let $X \in D(C^\infty(\mathcal{O}), \Lambda_k(\mathcal{O}))$ and $K \subset \{1, \dots, k\}$. Then there exists a unique $i_X^K \in D(\Lambda_k(\mathcal{O}), \Lambda_k(\mathcal{O}))$ such that*

$$i_X^K(d_{K'}f) = \begin{cases} d_{K' \setminus K}(X(f)), & \text{if } K \subset K' \\ 0, & \text{otherwise} \end{cases}, \quad K' \subset \{1, \dots, k\}.$$

For instance, if $i_m^K \stackrel{\text{def}}{=} i_{d_m|_{C^\infty(\mathcal{O})}}^K \in D(\Lambda_k(\mathcal{O}), \Lambda_k(\mathcal{O}))$, $m \leq k$, $K \subset \{1, \dots, k\}$, then $i_m^\emptyset = d_m$.

Recall that, according to proposition 3 of [1] covariant k -tensors on \mathcal{O} are naturally interpreted as k -times IDF's. Namely, for any k there exists an injective morphism of $C^\infty(\mathcal{O})$ -modules $\iota_k : T_k^0(\mathcal{O}) \hookrightarrow \Lambda_k(\mathcal{O})$. Then proposition 4 of [1] characterizes those IDF's that represent covariant tensors. Below we give another useful characterization of such IDF's.

Proposition 3. *A homogeneous element $\omega \in \Lambda_k(\mathcal{O})$ of multi-degree $(1, \dots, 1, 1) \in \mathbb{Z}^k$ is a covariant tensor (i.e., $\omega \in \text{im } \iota_k$) iff $i_m^K(\omega) = 0$ for any $m < k$ and $K = \{k_1, \dots, k_s\} \subset \{1, \dots, k\}$, $s \geq 2$.*

3. SECONDARY IDF'S

Let $(\Lambda_k(\mathcal{O}), d_1, \dots, d_k)$ be the multiple complex of k -times IDF's on \mathcal{O} and $\kappa_{1k} : \Lambda_k(\mathcal{O}) \longrightarrow \Lambda_k(\mathcal{O})$ the involution that interchanges differentials d_1 and d_k , $k \geq 1$. The ideal $\mathcal{C}\Lambda(\mathcal{O}) \subset \Lambda(\mathcal{O})$ of (ordinary) differential forms on \mathcal{O} vanishing on $\mathcal{C}(\mathcal{O})$ is

embedded naturally into $\Lambda_k(\mathcal{O})$ due to the embedding $\Lambda(\mathcal{O}) \equiv \Lambda_1(\mathcal{O}) \subset \Lambda_k(\mathcal{O})$. Denote by $\mathcal{C}\Lambda_k(\mathcal{O}) \subset \Lambda_k(\mathcal{O})$ the ideal generated by elements in the form

$$(d_K \circ \kappa_{1k})(\omega), \quad \omega \in \mathcal{C}\Lambda(\mathcal{O}), \quad K \subset \{1, \dots, k-1\}.$$

It is multi-differential, i.e.,

$$d_m(\mathcal{C}\Lambda_k(\mathcal{O})) \subset \mathcal{C}\Lambda_k(\mathcal{O}), \quad m \leq k. \quad (2)$$

Definition 4. Elements of $\mathcal{C}\Lambda_k(\mathcal{O})$ are called k -Cartan k -times IDFs.

Let $\mathcal{C}^p\Lambda_k(\mathcal{O})$ be the p -th power of $\mathcal{C}\Lambda_k(\mathcal{O})$. Then the sequence

$$\Lambda_k(\mathcal{O}) \supset \mathcal{C}\Lambda_k(\mathcal{O}) \supset \mathcal{C}^2\Lambda_k(\mathcal{O}) \supset \dots \supset \mathcal{C}^p\Lambda_k(\mathcal{O}) \supset \dots, \quad (3)$$

is a filtration of the multi-graded differential algebra $(\Lambda_k(\mathcal{O}), d_k)$ by means of multi-graded differential ideals.

Definition 5. The spectral sequence $\Lambda_{k-1}\mathcal{C}E(\mathcal{O}) = \{(\Lambda_{k-1}\mathcal{C}E_r(\mathcal{O}), d_{k,r})\}_r$ associated with filtration (3) is called the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence of the diffiety $(\mathcal{O}, \mathcal{C})$.

In particular, $\Lambda_0\mathcal{C}E(\mathcal{O}) = \mathcal{C}E(\mathcal{O})$.

Filtration (3) is regular and, therefore, the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence converges to the cohomology $H(\Lambda_k(\mathcal{O}), d_k)$. Hence,

$$\Lambda_{k-1}\mathcal{C}E_\infty(\mathcal{O}) \simeq H(\Lambda_k(\mathcal{O}), d_k) \simeq H(\mathcal{O}).$$

It follows from (2) and commutation relations $[d_m, d_k] = 0$, $m < k$, that d_m induces a well defined differential in each term of the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence. Denote by $d_{m,r}$ the so-obtained differential in the term $\Lambda_{k-1}\mathcal{C}E_r(\mathcal{O})$. Obviously, $[d_{m,r}, d_{n,r}] = 0$, $m, n \leq k$, so that $(\Lambda_{k-1}\mathcal{C}E_r(\mathcal{O}), d_{1,r}, \dots, d_{k,r})$ is a multiple differential algebra.

The zeroth term of the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence is described as follows. Let $\mathcal{C}^p\Lambda_k^p(\mathcal{O})$ denote the $\Lambda_{k-1}(\mathcal{O})$ -module of k -times iterated p -forms in $\mathcal{C}^p\Lambda_k(\mathcal{O})$. Note that $\mathcal{C}^p\Lambda_k(\mathcal{O}) = \mathcal{C}^p\Lambda_k^p(\mathcal{O}) \wedge \Lambda_k(\mathcal{O})$ and put

$$\mathcal{H}\Lambda_k(\mathcal{O}) \stackrel{\text{def}}{=} \Lambda_k(\mathcal{O}) / \mathcal{C}\Lambda_k(\mathcal{O}) = \Lambda_{k-1}\mathcal{C}E_0(\mathcal{O}).$$

Definition 6. $(\mathcal{H}\Lambda_k(\mathcal{O}), d_{k,0})$ is called the differential algebra of horizontal k -times IDFs on the diffiety $(\mathcal{O}, \mathcal{C})$.

Proposition 7. There exists a natural isomorphism

$$\Lambda_{k-1}\mathcal{C}E_0^{p,\bullet}(\mathcal{O}) \simeq \mathcal{C}^p\Lambda_k^p(\mathcal{O}) \otimes_{\Lambda_{k-1}} \mathcal{H}\Lambda_k(\mathcal{O}).$$

The isomorphism of proposition 7 supplies $\mathcal{C}^p\Lambda_k^p(\mathcal{O}) \otimes_{\Lambda_{k-1}} \mathcal{H}\Lambda_k(\mathcal{O})$ with a structure of a differential algebra. The corresponding differential \bar{d}_k looks as follows. Let $\omega \in \mathcal{C}^p\Lambda_k^p(\mathcal{O})$ and $\bar{\rho} = [\rho]_{\mathcal{C}\Lambda_k(\mathcal{O})} \in \mathcal{H}\Lambda_k(\mathcal{O})$, $\rho \in \Lambda_k(\mathcal{O})$. In its turn $d_k\omega \in \mathcal{C}^p\Lambda_k(\mathcal{O})$. Therefore, $d_k\omega = \sum_\alpha \omega_\alpha \wedge \sigma_\alpha$ for some $\omega_\alpha \in \mathcal{C}^p\Lambda_k^p(\mathcal{O})$ and $\sigma_\alpha \in \Lambda_k(\mathcal{O})$. Then

$$\bar{d}_k(\omega \otimes \bar{\rho}) = \sum_\alpha \omega_\alpha \otimes \overline{\sigma_\alpha \wedge \rho} + (-1)^p \omega \otimes d_{k,0}\bar{\rho} \in \mathcal{C}^p\Lambda_k^p(\mathcal{O}) \otimes_{\Lambda_{k-1}} \mathcal{H}\Lambda_k(\mathcal{O}),$$

where $\overline{\sigma_\alpha \wedge \rho} = [\sigma_\alpha \wedge \rho]_{\mathcal{C}\Lambda_k(\mathcal{O})} \in \mathcal{H}\Lambda_k(\mathcal{O})$.

Definition 8. $(\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O}), d_{1,1}, \dots, d_{k,1})$ is called the multiple differential algebra of secondary k -times IDF's. Accordingly, elements of $\Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O})$ are called secondary k -times iterated differential p -forms.

It takes place the secondary analogue of the fact that $(k+1)$ -times iterated 0-forms coincide with k -times iterated ones. Indeed, it holds the

Proposition 9. *There exists a natural isomorphism of graded multiple differential algebras*

$$\varphi_{k-1} : (\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O}), d_{1,1}, \dots, d_{k,1}) \longrightarrow (\Lambda_k\mathcal{C}E_1^{0,\bullet}(\mathcal{O}), d_{1,1}, \dots, d_{k,1}).$$

Since $\Lambda_{k-1}\mathcal{C}E_1^{0,\bullet}(\mathcal{O}) = H(\mathcal{H}\Lambda_k(\mathcal{O}), d_{k,0})$, according to the above proposition secondary k -times IDF's may be understood either as elements in the first term of the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence or as cohomologies of the differential algebra of *horizontal* $(k+1)$ -times IDF's.

The standard action of the permutation group S_k on IDF's is secondarized as follows. Let $\sigma \in S_{k-1}$ be a permutation of $\{1, \dots, k-1\}$. The associated automorphism $\kappa_\sigma : \Lambda_k(\mathcal{O}) \longrightarrow \Lambda_k(\mathcal{O})$ respects both the differential d_k and the filtration (3), i.e., $\kappa_\sigma \circ d_k = d_k \circ \kappa_\sigma$ and $\kappa_\sigma(\mathcal{C}\Lambda_k(\mathcal{O})) \subset \mathcal{C}\Lambda_k(\mathcal{O})$. Therefore, κ_σ induces an automorphism of the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence. Abusing the notation we continue denoting by κ_σ the induced automorphism in the first term of $\Lambda_{k-1}\mathcal{C}E(\mathcal{O})$. In particular, $\kappa_\sigma : \Lambda_{k-1}\mathcal{C}E_1(\mathcal{O}) \longrightarrow \Lambda_{k-1}\mathcal{C}E_1(\mathcal{O})$ is an isomorphism of the multiple differential algebras $(\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O}), d_{1,1}, \dots, d_{k,1})$ and $(\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O}), d_{\sigma(1),1}, \dots, d_{\sigma(k-1),1}, d_{k,1})$. Proposition 9 allows to define the action of the transposition τ exchanging k and $m < k$ on $\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O})$. Namely, observe that the action of κ_τ on $\Lambda_k\mathcal{C}E_1^{0,\bullet}(\mathcal{O})$ is well defined as it was explained above and the isomorphism φ_{k-1} transfers it then to $\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O})$.

Since $\mathcal{C}\Lambda_k(\mathcal{O})$ is a multi-graded differential ideal, $\Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O})$ inherits a multi-graded algebra structure. Namely, let $\theta = [[\omega]_{\mathcal{C}^{p+1}\Lambda_k(\mathcal{O})}]_{\text{imd}_{k,0}} \in \Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O})$, $\omega \in \mathcal{C}^p\Lambda_k(\mathcal{O})$, $d_k\omega \in \mathcal{C}^{p+1}\Lambda_k(\mathcal{O})$. Then

$$\theta \in \Lambda_{k-1}\mathcal{C}E_1^{(p_1, \dots, p_{k-1}, p), q}(\mathcal{O}) \stackrel{\text{def}}{\iff} \omega \text{ has multi-degree } (p_1, \dots, p_{k-1}, p+q) \in \mathbb{Z}^k, \quad (4)$$

$p_1, \dots, p_{k-1}, q \geq 0$. Definition (4) is correct and

$$\Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O}) = \bigoplus_{p_1, \dots, p_{k-1}, q} \Lambda_{k-1}\mathcal{C}E_1^{(p_1, \dots, p_{k-1}, p), q}(\mathcal{O}).$$

Proposition 10. *Let $\sigma \in S_k$ be a permutation of $\{1, \dots, k\}$. Then*

$$\kappa_{\sigma^{-1}}(\Lambda_{k-1}\mathcal{C}E_1^{(p_1, \dots, p_{k-1}, p_k), q}) \subset \Lambda_{k-1}\mathcal{C}E_1^{(p_{\sigma(1)}, \dots, p_{\sigma(k-1)}, p_{\sigma(k)}), q}.$$

4. IDF-SYMMETRIES OF A DIFFIETY

Iterated analogues of contact and, respectively, trivial contact vector fields on a diffiety are defined as follows (see [5, 6]):

$$D_C(\Lambda_{k-1}(\mathcal{O})) \stackrel{\text{def}}{=} \{X \in D(\Lambda_{k-1}(\mathcal{O}), \Lambda_{k-1}(\mathcal{O})) \mid \mathcal{L}_X^{\{k\}}(\mathcal{C}\Lambda_k(\mathcal{O})) \subset \mathcal{C}\Lambda_k(\mathcal{O})\}$$

and

$$\mathcal{CD}(\Lambda_{k-1}(\mathcal{O})) \stackrel{\text{def}}{=} \{X \in \mathcal{D}(\Lambda_{k-1}(\mathcal{O}), \Lambda_{k-1}(\mathcal{O})) \mid i_X^{\{k\}}(\mathcal{C}\Lambda_k(\mathcal{O})) = 0\}.$$

$\mathcal{D}_{\mathcal{C}}(\Lambda_{k-1}(\mathcal{O}))$ is a graded sub-algebra of the Lie algebra $\mathcal{D}(\Lambda_{k-1}(\mathcal{O}))$ and $\mathcal{CD}(\Lambda_{k-1}(\mathcal{O}))$ is an its graded ideal. Denote by

$$\Lambda_{k-1}\text{Sym}(\mathcal{O}) \stackrel{\text{def}}{=} \mathcal{D}_{\mathcal{C}}(\Lambda_{k-1}(\mathcal{O})) / \mathcal{CD}(\Lambda_{k-1}(\mathcal{O})).$$

the quotient Lie algebra. In particular, $\Lambda_0\text{Sym}(\mathcal{O}) = \text{Sym}(\mathcal{O})$.

Definition 11. *Elements in $\Lambda_{k-1}\text{Sym}(\mathcal{O})$ are called $(k-1)$ -IDF-symmetries of the diffeity $(\mathcal{O}, \mathcal{C})$.*

Just as symmetries of $(\mathcal{O}, \mathcal{C})$ act naturally on secondary differential forms, IDF-symmetries of $(\mathcal{O}, \mathcal{C})$ act naturally on secondary IDFs. Namely, let $\chi = [X] \in \Lambda_{k-1}\text{Sym}(\mathcal{O})$, $X \in \mathcal{D}_{\mathcal{C}}(\Lambda_{k-1}(\mathcal{O}))$. The Lie derivative $\mathcal{L}_X^{\{k\}} : \Lambda_k(\mathcal{O}) \rightarrow \Lambda_k(\mathcal{O})$ respects both the differential d_k and the filtration (3), i.e., $\mathcal{L}_X^{\{k\}} \circ d_k = d_k \circ \mathcal{L}_X^{\{k\}}$ and $\mathcal{L}_X^{\{k\}}(\mathcal{C}\Lambda_k(\mathcal{O})) \subset \mathcal{C}\Lambda_k(\mathcal{O})$. Therefore, $\mathcal{L}_X^{\{k\}}$ induces a morphism of the $\Lambda_{k-1}\mathcal{C}$ -spectral sequence. Denote by

$$\mathcal{L}_{\chi}^{\{k\}} : \Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O}) \rightarrow \Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O})$$

its action on the first term since it doesn't depend on the choice of the representative X of χ . $\mathcal{L}_{\chi}^{\{k\}}$ is a graded derivation of $\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O})$. Moreover, if $\chi_1, \chi_2 \in \Lambda_{k-1}\text{Sym}(\mathcal{O})$, then

$$[\mathcal{L}_{\chi_1}^{\{k\}}, d_{k,1}] = 0, \quad [\mathcal{L}_{\chi_1}^{\{k\}}, \mathcal{L}_{\chi_2}^{\{k\}}] = \mathcal{L}_{[\chi_1, \chi_2]}^{\{k\}}.$$

The definition of $\mathcal{L}_{\chi}^{\{k\}}$ is consistent with the natural isomorphism $\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O}) \simeq \Lambda_k\mathcal{C}E_1^{0,\bullet}(\mathcal{O})$. Namely, let χ and X be as above, so that $\mathcal{L}_X^{\{k\}} \in \mathcal{D}_{\mathcal{C}}(\Lambda_k(\mathcal{O}))$. Put $\chi' \stackrel{\text{def}}{=} [\mathcal{L}_X^{\{k\}}] \in \Lambda_k\text{Sym}(\mathcal{O})$. Then

$$\mathcal{L}_{\chi}^{\{k\}} = \varphi_{k-1}^{-1} \circ \mathcal{L}_{\chi'}^{\{k+1\}}|_{\Lambda_k\mathcal{C}E_1^{0,\bullet}(\mathcal{O})} \circ \varphi_{k-1}.$$

Now observe that $i_m^K \in \mathcal{D}_{\mathcal{C}}(\Lambda_{k-1}(\mathcal{O}))$ for any $m < k$ and $K \subset \{1, \dots, k-1\}$ and put

$$I_m^K \stackrel{\text{def}}{=} [i_m^K] \in \Lambda_{k-1}\text{Sym}(\mathcal{O}).$$

Then $d_{m,1} = \mathcal{L}_{I_m^K}^{\{k\}}$.

IDF-symmetries of $(\mathcal{O}, \mathcal{C})$ can be also inserted into secondary IDFs as follows. Indeed, let χ and X be as above and $\theta = [[\omega]_{\mathcal{C}^{p+1}\Lambda_k(\mathcal{O})}]_{\text{im } d_{k,0}} \in \Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O})$, $\omega \in \mathcal{C}^p\Lambda_k(\mathcal{O})$, $d_k\omega \in \mathcal{C}^{p+1}\Lambda_k(\mathcal{O})$, be a secondary IDF. Then $i_X^{\{k\}}\omega \in \mathcal{C}^{p-1}\Lambda_k(\mathcal{O})$ and $d_k(i_X^{\{k\}}\omega) = \mathcal{L}_X^{\{k\}}\omega - i_X^{\{k\}}(d_k\omega) \in \mathcal{C}^p\Lambda_k(\mathcal{O})$ so that

$$i_{\chi}^{\{k\}}\theta \stackrel{\text{def}}{=} [[i_X^{\{k\}}\omega]_{\mathcal{C}^p\Lambda_k(\mathcal{O})}]_{\text{im } d_{k,0}} \in \Lambda_{k-1}\mathcal{C}E_1^{p-1,\bullet}(\mathcal{O})$$

is a well defined secondary IDF. Moreover, the map

$$i_{\chi}^{\{k\}} : \Lambda_{k-1}\mathcal{C}E_1^{p,\bullet}(\mathcal{O}) \rightarrow \Lambda_{k-1}\mathcal{C}E_1^{p-1,\bullet}(\mathcal{O})$$

is a graded derivation of $\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O})$ not depending on the choice of X in χ .

For $\chi_1, \chi_2 \in \Lambda_{k-1}\text{Sym}(\mathcal{O})$ the following secondary analogues of well-known (graded) commutation relations hold

$$[i_{\chi_1}^{\{k\}}, d_{k,1}] = \mathcal{L}_{\chi_1}^{\{k\}}, \quad [i_{\chi_1}^{\{k\}}, i_{\chi_2}^{\{k\}}] = 0, \quad [i_{\chi_1}^{\{k\}}, \mathcal{L}_{\chi_2}^{\{k\}}] = i_{[\chi_1, \chi_2]}^{\{k\}}.$$

The definition of $i_{\chi}^{\{k\}}$ is consistent with the natural isomorphism $\Lambda_{k-1}\mathcal{C}E_1(\mathcal{O}) \simeq \Lambda_k\mathcal{C}E_1^{0,\bullet}(\mathcal{O})$. Namely, if χ and X be as above, then $i_X^{\{k\}} \in \text{D}_C(\Lambda_k(\mathcal{O}))$ and

$$i_{\chi}^{\{k\}} = \varphi_{k-1}^{-1} \circ \mathcal{L}_{\chi''}^{\{k+1\}}|_{\Lambda_k\mathcal{C}E_1^{0,\bullet}(\mathcal{O})} \circ \varphi_{k-1}.$$

with $\chi'' \stackrel{\text{def}}{=} [i_X^{\{k\}}] \in \Lambda_k\text{Sym}(\mathcal{O})$.

Definition 12. Let $\tau \in \Lambda_{k-1}\mathcal{C}E_1^{(1,\dots,1,1),\bullet}(\mathcal{O})$ be a secondary IDF of multi-degree $(1, \dots, 1, 1)$. τ is called a secondary covariant k -tensor iff $\mathcal{L}_{I_m^K}^{\{k\}}\tau = i_{I_m^K}^{\{k\}}\tau = 0$ for any $m < k$ and $K = \{k_1, \dots, k_s\}, K' = \{k'_1, \dots, k'_{s'}\} \subset \{1, \dots, k-1\}, s \geq 2, s' \geq 1$.

Definition 12 mimics characterization of covariant tensors given in proposition 3.

5. OTHER RELATED \mathcal{C} -SPECTRAL SEQUENCES

Now on put $\mathcal{C}_k\Lambda_k(\mathcal{O}) = \mathcal{C}\Lambda_k(\mathcal{O})$ and observe that $\mathcal{C}_k\Lambda_k(\mathcal{O})$ is not unique canonical multi-differential ideal in $\Lambda_k(\mathcal{O})$. Indeed, such are ideals

$$\mathcal{C}_m\Lambda_k(\mathcal{O}) \stackrel{\text{def}}{=} \kappa_{mk}(\mathcal{C}_k\Lambda_k(\mathcal{O})),$$

for any $m \leq k$. Here $\kappa_{mk} : \Lambda_k(\mathcal{O}) \rightarrow \Lambda_k(\mathcal{O})$ is the involution that interchanges differentials d_m and d_k . More generally, let $K = \{m_1, \dots, m_r\} \subset \{1, \dots, k\}$. Put

$$\mathcal{C}_K\Lambda_k(\mathcal{O}) \stackrel{\text{def}}{=} \mathcal{C}_{m_1}\Lambda_k(\mathcal{O}) + \dots + \mathcal{C}_{m_r}\Lambda_k(\mathcal{O}).$$

Then $\mathcal{C}_K\Lambda_k(\mathcal{O}) \subset \Lambda_k(\mathcal{O})$ is a multi-differential for any $K \subset \{1, \dots, k\}$.

Definition 13. Elements in $\mathcal{C}_K\Lambda_k(\mathcal{O})$ are called K -Cartan k -times IDFs.

All these ideals transform $(\Lambda_k(\mathcal{O}), d_k)$ into a multi-filtered complex. Homological algebra of such kind complexes and appearing this way numerous \mathcal{C} -spectral-like sequences will be considered separately. They enrich noteworthy secondary calculus and, as a consequence, geometrical theory of nonlinear PDEs with new powerful instruments.

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