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Geometry Revisited**

by

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Iterated Differential Forms II: Riemannian Geometry Revisited

A. M. VINOGRADOV AND L. VITAGLIANO

ABSTRACT. A natural extension of Riemannian geometry to a much wider context is presented on the basis of the iterated form formalism developed in [1] and an application to general relativity is given.

In this note the formalism of iterated differential forms developed in the preceding note [1] (see also [2]) is used to decipher the conceptual meaning of Riemannian geometry in the context of differential calculus over (graded) commutative algebras. This is indispensable in order to extend Riemannian geometry to algebraic geometry and Secondary Calculus and to clarify some aspects in general relativity and, more generally, in field theory. In particular, we show that a Levi-Civita-like connection is naturally associated with a generic covariant 2-tensor whose symmetric part is nondegenerate and on this basis present new Einstein-like equations in general relativity. The super-case is briefly sketched. In the sequel the notation and definitions of [1] (see also [3, 4, 5, 6]) are used.

1. DIFFERENTIAL FORM CALCULUS AND CONNECTIONS

Let $\mathcal{G} = (G, \mu)$ be a grading group, \mathbb{k} be a field of zero-characteristic. In what follows a graded \mathbb{k} -algebra will be always assumed associative, unitary and graded commutative (see [1]). Accordingly, a graded morphism of algebras will be always assumed unitary. The necessary facts concerning differential forms over \mathcal{G} -graded \mathbb{k} -algebras we shall need are summarized below along the lines of [7].

Let B be a \mathcal{G} -graded \mathbb{k} -algebra, $\Lambda(B) = \bigoplus_k \Lambda^k(B)$ be the $\mathcal{G} \oplus \mathbb{Z}$ -graded \mathbb{k} -algebra of differential forms over B and $d_B : \Lambda(B) \rightarrow \Lambda(B)$ the exterior differential. If P is a graded B -module, denote by $D(B, P)$ the totality of all P -valued graded derivations of B . In a natural way $D(B, P)$ is a graded B -module.

Let $\varphi : B \rightarrow C$ be a morphism of \mathcal{G} -graded \mathbb{k} -algebras. It allows to supply any C -module Q with a structure of graded B -module which will be denoted by Q^φ . φ

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induces a morphism of $\mathcal{G} \oplus \mathbb{Z}$ -graded \mathbb{k} -algebras $\Lambda(\varphi) : \Lambda(B) \longrightarrow \Lambda(C)$. Denote by $\Lambda^1(\varphi) : \Lambda^1(B) \longrightarrow \Lambda^1(C)^\varphi$ its first homogeneous component (with respect to the \mathbb{Z} -grading). $\Lambda^1(\varphi)$ is a morphism of B -modules. Let $X \in D(B, \Lambda^l(C)^\varphi)$, $l \geq 0$. The insertion of X into a differential form $\sigma \in \Lambda(B)$, denoted by $i_X \sigma \in \Lambda(C)$, is defined inductively on the degree of σ by the formula

$$i_X(da \wedge \sigma) = X(a) \wedge (\Lambda(\varphi)(\sigma)) - (-1)^{X \cdot a + l} ((d_C \circ \varphi)(a)) \wedge i_X \sigma, \quad a \in B.$$

The insertion operator $i_X : \Lambda(B) \longrightarrow \Lambda(C)^\varphi$ is a derivation of the algebra $\Lambda(B)$ of bi-degree $(|X|, l-1)$. The Lie derivative along X is defined as $\mathcal{L}_X \stackrel{\text{def}}{=} [i_X, d] = i_X \circ d_B + (-1)^l d_C \circ i_X : \Lambda(B) \longrightarrow \Lambda(C)^\varphi$. \mathcal{L}_X is a derivation of the algebra $\Lambda(B)$ of bi-degree $(|X|, l)$.

Let $\Lambda^1(B)$ be a projective and finitely generated B -module. Then, for any $X, Y \in D(B, \Lambda(B))$ there exists a unique $Z \in D(B, \Lambda(B))$ such that $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_Z$. Put $Z \stackrel{\text{def}}{=} [X, Y]^{\text{fn}}$. The bracket $[\cdot, \cdot]^{\text{fn}} : D(B, \Lambda(B)) \times D(B, \Lambda(B)) \ni (X, Y) \longmapsto [X, Y]^{\text{fn}} \in D(B, \Lambda(B))$ is called Frölicher–Nijenhuis (F–N) bracket and supplies $D(B, \Lambda(B))$ with a structure of $\mathcal{G} \oplus \mathbb{Z}$ -graded \mathbb{k} -Lie-algebra.

Now let $\varphi : B_1 \longrightarrow B_2$ be a morphism of \mathcal{G} -graded \mathbb{k} -algebras. Consider the functor

$$D(B_1, \bullet^\varphi) : P \longmapsto D(B_1, P^\varphi),$$

P being a \mathcal{G} -graded B_2 -module. It is represented by the graded B_2 -module $\Lambda^1(B_1) \otimes_{B_1} B_2^\varphi$, i.e., for any graded B_2 -module P and $X \in D(B_1, P^\varphi)$ there exists a unique graded B_2 -module homomorphism $\psi_X : \Lambda^1(B_1) \otimes_{B_1} B_2^\varphi \longrightarrow P$ such that $X = \psi_X \circ d_\varphi$, where $d_\varphi : B_1 \longrightarrow \Lambda^1(B_1) \otimes_{B_1} B_2^\varphi$ is a natural derivation defined by

$$d_\varphi b \stackrel{\text{def}}{=} d_{B_1} b \otimes 1 \in \Lambda^1(B_1) \otimes_{B_1} B_2^\varphi, \quad b \in B_1.$$

Note that $\Lambda^1(\varphi) \circ d_{B_1} : B_1 \longrightarrow \Lambda^1(B_2)$ is a graded derivation, i.e., $\Lambda^1(\varphi) \circ d_{B_1} \in D(B_1, \Lambda^1(B_2)^\varphi)$. Denote $\psi = \psi_{\Lambda^1(\varphi) \circ d_{B_1}} : \Lambda^1(B_1) \otimes_{B_1} B_2^\varphi \longrightarrow \Lambda^1(B_2)$ and put $\Lambda_\varphi^1(B_1, B_2) = \text{im } \psi \subset \Lambda^1(B_2)$ (or, simply, $\Lambda^1(B_1, B_2)$ if the context does not allow a confusion).

Definition 1. A connection in φ is a graded derivation $\nabla : B_2 \longrightarrow \Lambda(B_1) \otimes_{B_1} B_2^\varphi$ such that, $\nabla(B_2) \subset \Lambda^1(B_1) \otimes_{B_1} B_2^\varphi$ and $\nabla \circ \varphi = d_\varphi$.

Let ∇ be a connection in φ and $i_\nabla : \Lambda^1(B_2) \longrightarrow \Lambda^1(B_1) \otimes_{B_1} B_2^\varphi$ the unique graded B_2 -module homomorphism such that $\nabla = i_\nabla \circ d_{B_2}$. By definition i_∇ is a left inverse of ψ . This proves that ψ is injective and $\Lambda^1(B_1) \otimes_{B_1} B_2^\varphi \simeq \Lambda_\varphi^1(B_1, B_2)$.

A connection ∇ in φ defines a natural transformation of functors $\nabla^\bullet : D(B_1, \bullet^\varphi) \longrightarrow D(B_2, \bullet)$ (see [7]). Namely,

$$\nabla^P : D(B_1, P^\varphi) \ni X \longmapsto \nabla_X^P \stackrel{\text{def}}{=} \psi_X \circ \nabla \in D(B_2, P),$$

P being a \mathcal{G} -graded B_2 -module. By definition $\nabla_X^P \circ \varphi = X$.

There are many situations when a new connection can be constructed on the basis of other ones. Some of them we need are described below.

Let $\nabla : B_2 \longrightarrow \Lambda(B_1) \otimes_{B_1} B_2^\varphi$ be a connection in $\varphi : B_1 \longrightarrow B_2$ and $\kappa : B_2 \longrightarrow B_2$ an automorphism of the algebra B_2 . Obviously, $\kappa : B_2^\varphi \longrightarrow B_2^{\kappa \circ \varphi}$ is B_1 -linear and, so, the map

$$\text{id}_{\Lambda(B_1)} \otimes \kappa : \Lambda(B_1) \otimes_{B_1} B_2^\varphi \longrightarrow \Lambda(B_1) \otimes_{B_1} B_2^{\kappa \circ \varphi}. \quad (1)$$

is well-defined. The composition

$$\nabla^\kappa \stackrel{\text{def}}{=} (\text{id}_{\Lambda(B_1)} \otimes \kappa) \circ \nabla \circ \kappa^{-1} : B_2 \longrightarrow \Lambda(B_1) \otimes_{B_1}^{\kappa \circ \varphi} B_2$$

is a connection in $\kappa \circ \varphi : B_1 \longrightarrow B_2$.

Let φ and ∇ be as above. ∇ extends naturally to a connection in $\Lambda(\varphi) : \Lambda(B_1) \longrightarrow \Lambda(B_2)$ as follows. Consider the bi-differential algebra $(\Lambda_2(B_1), d_1, d_2)$ of doubly iterated differential forms over B_1 (see [1]).

Note that the map

$$\nu : B_2 \ni b \longmapsto 1 \otimes b \in \Lambda(B_1) \otimes_{B_1} B_2^\varphi$$

is a morphism of \mathcal{G} -graded algebras and $\nabla \in \text{D}(B_2, (\Lambda(B_1) \otimes_{B_1} B_2^\varphi)^\nu)$. According to the above construction ∇ extends to a derivation

$$\mathcal{L}_\nabla : \Lambda(B_2) \longrightarrow \Lambda(\Lambda(B_1) \otimes_{B_1} B_2^\varphi).$$

There exists a canonical isomorphism of graded algebras,

$$\phi : \Lambda(\Lambda(B_1) \otimes_{B_1} B_2^\varphi) \xrightarrow{\cong} \Lambda_2(B_1) \otimes_{\Lambda(B_1)} \Lambda(B_2)^{\Lambda(\varphi)},$$

given by the formula

$$(\phi \circ d_{\Lambda(B_1) \otimes_{B_1} B_2^\varphi})(\omega \otimes b) = d_2 \omega \otimes b + \omega \otimes d_{B_2} b, \quad \omega \in \Lambda(B_1), b \in B_2.$$

Put

$$\Lambda \nabla \stackrel{\text{def}}{=} \phi \circ \mathcal{L}_\nabla : \Lambda(B_2) \longrightarrow \Lambda_2(B_1) \otimes_{\Lambda(B_1)} \Lambda(B_2)^{\Lambda(\varphi)}.$$

Then $\Lambda \nabla$ is a connection in $\Lambda(\varphi) : \Lambda(B_1) \longrightarrow \Lambda(B_2)$.

Finally, let B_1, B_2, B_3 be \mathcal{G} -graded algebras, $\varphi : B_2 \longrightarrow B_3$ and $\varphi' : B_1 \longrightarrow B_2$ be homomorphisms of graded algebras and $\nabla : B_3 \longrightarrow \Lambda(B_2) \otimes_{B_2} B_3^\varphi$, $\nabla' : B_2 \longrightarrow \Lambda(B_1) \otimes_{B_1} B_2^{\varphi'}$ be connections in φ, φ' , respectively. Similarly as above (see (1)) consider the homomorphism

$$\text{id}_{\Lambda(B_1)} \otimes \varphi : \Lambda(B_1) \otimes_{B_1} B_2^{\varphi'} \longrightarrow \Lambda(B_1) \otimes_{B_1} B_3^{\varphi \circ \varphi'}.$$

By the universal property of $\Lambda^1(B_2) \otimes_{B_2} B_3^\varphi$ there exists a unique homomorphism of graded B_3 -modules $\bar{\psi} : \Lambda^1(B_2) \otimes_{B_2} B_3^\varphi \longrightarrow \Lambda(B_1) \otimes_{B_1} B_3^{\varphi \circ \varphi'}$ such that $\bar{\psi} \circ d_\varphi = (\text{id}_{\Lambda(B_1)} \otimes \varphi) \circ \nabla'$. Put

$$\square \equiv \bar{\psi} \circ \nabla : B_3 \longrightarrow \Lambda(B_1) \otimes_{B_1} B_3^{\varphi \circ \varphi'}.$$

Then \square is a connection in $\varphi \circ \varphi' : B_1 \longrightarrow B_3$.

$$\begin{array}{ccccc}
 B_2 & \xrightarrow{\nabla'} & \Lambda(B_1) \otimes B_2^{\varphi'} & \xrightarrow{\text{id} \otimes \varphi} & \Lambda(B_1) \otimes B_3^{\varphi \circ \varphi'} \\
 \downarrow d_\varphi & & \nearrow \bar{\psi} & & \nearrow \\
 \Lambda^1(B_2) \otimes B_3^\varphi & & \square & & \\
 \uparrow \nabla & & & & \\
 B_3 & & & &
 \end{array}$$

Definition 2. \square is called the composition of connections ∇ and ∇' .

Abusing the notation we also denote this connection by $\nabla \circ \nabla'$.

2. LEVI-CIVITA-LIKE CONNECTIONS OVER SMOOTH ALGEBRAS

To simplify the exposition and for pedagogical reasons we shall limit to the case when the basic smooth algebra A is the smooth function algebra on a smooth manifold.

Consider an n -dimensional smooth manifold M , a local chart (x^1, \dots, x^n) on it and put $A \equiv C^\infty(M)$. Let $\Lambda = \Lambda_1$ be the \mathbb{Z} -graded algebra of (geometric) differential forms over M and $\Lambda_k = \Lambda(\Lambda_{k-1})$ be the \mathbb{Z}^k -graded algebra of k -times iterated (geometric) differential forms over M (see [8], where iterated differential forms were first introduced, and [9] for an alternative approach to iterated differential forms on super-manifolds). Denote by $\kappa : \Lambda_2 \longrightarrow \Lambda_2$ the canonical involution (see [1, 2]). Recall that $\kappa \circ d_1 \circ \kappa = d_2$, $\kappa \circ d_2 \circ \kappa = d_1$ and d_1 is a natural extension of the de Rham differential on M to Λ_2 . In the following we shall denote simply by $\omega\omega'$ (rather than $\omega \wedge \omega'$) the product of the iterated forms ω and ω' . Since $\Lambda_2^1 \equiv \Lambda^1(\Lambda)$ is a projective and finitely generated Λ -module the F-N bracket in $D(\Lambda, \Lambda_2)$ is well-defined.

The injective A -homomorphism $\iota_2 : T_2^0(M) \hookrightarrow \Lambda_2$ (see [1]) allows one to interpret covariant 2-tensors, in particular, (pseudo-)metric tensors, as iterated forms. This way pseudo-Riemannian geometry becomes a subject of the differential calculus over the algebra of iterated forms Λ_∞ . This interpretation allows to associate a linear connection in the tangent bundle of M with a (possibly non-symmetric) 2-tensor whose symmetric part is non-degenerate. According to the standard approach, the Levi-Civita connection associated with a pseudo-metric is defined implicitly as the unique one satisfying certain properties. On the contrary, we define it directly by applying to the considered 2-tensor a natural operator in the algebra Λ_∞ . This is one of numerous examples showing potentialities of the calculus of iterated forms.

Proposition 3. *Let $g \in \Lambda_2$ be a second iterated form of bi-degree $(1, 1)$. Then the following two assertions are equivalent:*

- (1) The map $D(M) \times D(M) \ni (X, Y) \mapsto (i_Y^{(2)} \circ i_X^{(1)})(g) \in A \subset \Lambda$ is A -bilinear, $\kappa(g) = g$ and the A -homomorphism $\lrcorner_g : D(M) \ni X \mapsto i_X^{(2)}g \in \Lambda^1 \subset \Lambda_2$ is bijective.
- (2) $g = \iota_2(g')$ for a Riemannian or pseudo-Riemannian metric g' over M .

Proposition 3 is a corollary of proposition 4 from [1] and motivates the following new point of view on what, in reality, the Riemannian geometry is.

Definition 4. A metric over a smooth algebra A , for instance, $A = C^\infty(M)$, is an iterated 1-form $g \in \Lambda_2$ satisfying hypothesis 1 of proposition 3.

Locally a metric $g \in \Lambda_2$ looks as $g = g_{\mu\nu}d_1x^\mu d_2x^\nu$ with $g_{\mu\nu} = g_{\nu\mu} \in A$, $1 \leq \mu, \nu \leq n$.

This way the Riemannian geometry is embedded in the context of iterated differential forms and, after that, it is generalized rather directly to more general 2-tensors. Namely, let $\tau \in \Lambda_2$ be an iterated form of bi-degree (1, 1) such that $\tau = \iota_2(\tau')$ for a covariant 2-tensor τ' over M . Locally τ looks as $\tau = \tau_{\mu\nu}d_1x^\mu d_2x^\nu$ with $\tau_{\mu\nu} \in A$, $1 \leq \mu, \nu \leq n$. τ can be uniquely written as $\tau = g + \omega$ where $\kappa(g) = g$ and $\kappa(\omega) = -\omega$. Clearly, $g = \iota_2(g')$ and $\omega = \iota_2(\omega')$, where g' and ω' are symmetric and skew-symmetric parts of τ' , respectively. From now on, suppose that g is nondegenerate, i.e., a pseudo-metric.

Consider the iterated 2-form $\gamma = -d_2d_1\tau$. It is of bi-degree (2, 2) and its local expression is

$$\gamma = \partial_\nu \partial_\beta \tau_{\alpha\mu} d_1x^\beta d_1x^\alpha d_2x^\mu d_2x^\nu + \gamma_{\beta\alpha\mu} d_1x^\alpha d_2x^\mu d_2d_1x^\beta + \tau_{\alpha\beta} d_2d_1x^\alpha d_2d_1x^\beta.$$

with $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\gamma_{\beta\alpha\mu} = \partial_\mu \tau_{\alpha\beta} + \partial_\alpha \tau_{\beta\mu} - \partial_\beta \tau_{\alpha\mu}$. Note that if $\omega = 0$, then the $\gamma_{\beta\alpha\mu}$'s are doubled first kind Christoffel symbols of g' . By this reason γ is naturally baptized to be the (*generalized*) *Christoffel form*.

Next, the homomorphism $\lrcorner^g = \frac{1}{2}\lrcorner_g^{-1} : \Lambda^1 \longrightarrow D(M)$ extends to Λ as an A -linear, $D(A, \Lambda)$ -valued derivation of Λ of degree -1 denoted by $\lrcorner_1^g : \Lambda \longrightarrow D(A, \Lambda)$. It should be noted that $D(A, \Lambda)$ is a Λ -module and $\lrcorner_1^g \in D(\Lambda, D(A, \Lambda))$. Recall that with any $X \in D(A, \Lambda)$ the insertion operator $i_X \in D(\Lambda, \Lambda)$ is associated and consider the map

$$\Delta : \Lambda \ni \sigma \mapsto \Delta(\sigma) \stackrel{\text{def}}{=} i_{\lrcorner_1^g(\sigma)} \in D(\Lambda, \Lambda).$$

Δ is an A -linear, $D(\Lambda, \Lambda)$ -valued derivation of Λ of degree -2 . Define the map

$$\lrcorner_2^g : \Lambda_2 \longrightarrow D(\Lambda, \Lambda_2),$$

by putting for any $\Omega \in \Lambda_2$ and $\omega \in \Lambda$

$$\lrcorner_2^g(\Omega)(\omega) \equiv (-1)^{\Omega \cdot \omega} i_{\Delta(\omega)}^{(2)} \Omega.$$

Note that $D(\Lambda, \Lambda_2)$ possesses a natural Λ_2 -module structure and \lrcorner_2^g is a $D(\Lambda, \Lambda_2)$ -valued derivation of Λ_2 of bi-degree $(-2, -1)$. Locally it looks as

$$\lrcorner_2^g(\Omega)(\omega) = (-1)^{|\Omega| \cdot (0, -1)} \frac{1}{2} g^{\mu\nu} (i_{i_{\partial_\mu}^{(2)}}^{(2)} \Omega)(i_{\partial_\nu} \omega) \in \Lambda_2,$$

where $\|g^{\mu\nu}\| = \|g_{\mu\nu}\|^{-1}$.

Definition 5. *The Λ_2 -valued derivation $\Gamma =]_2^g(\gamma) = -]_2^g(d_1 d_2 \tau)$ of the algebra Λ is called the Levi-Civita connection form of τ .*

This terminology is motivated by the fact that the local expression of Γ is

$$\Gamma = (d_1 d_2 x^\alpha + \Gamma_{\mu\beta}^\alpha d_1 x^\beta d_2 x^\mu) i_{\partial_\alpha},$$

with $\Gamma_{\mu\beta}^\alpha \equiv \frac{1}{2} g^{\alpha\delta} \gamma_{\delta\beta\mu}$ and in the case when $\omega = 0$ $\Gamma_{\rho\gamma}^\alpha$'s are nothing but Christoffel symbols of g' . So, with any covariant 2-tensor whose symmetric part is nondegenerate a linear connection in the tangent bundle is associated (see below).

Now consider the insertion into 2-iterated forms operator $i_\Gamma^{(2)} \in D(\Lambda_2, \Lambda_2)$ corresponding to the graded form-valued derivation Γ . The composition $\kappa \circ i_\Gamma^{(2)} \circ \kappa$ is again a derivation of Λ_2 . Put

$$\kappa(\Gamma) \stackrel{\text{def}}{=} [\kappa \circ i_\Gamma^{(2)} \circ \kappa, d_2] |_\Lambda \in D(\Lambda, \Lambda_2).$$

The graded form-valued derivation $T = \Gamma - \kappa(\Gamma)$ has the following local expression

$$T = T_{\mu\beta}^\alpha d_1 x^\beta d_2 x^\mu i_{\partial_\alpha}, \quad T_{\mu\beta}^\alpha = \Gamma_{\mu\beta}^\alpha - \Gamma_{\beta\mu}^\alpha = 3g^{\alpha\delta} \partial_{[\mu} \omega_{\beta\delta]} d_1 x^\beta d_2 x^\mu i_{\partial_\alpha},$$

where the $\omega_{\beta\delta}$'s are local components of ω , i.e., locally $\omega = \omega_{\beta\delta} d_1 x^\beta d_2 x^\delta$, and indexes in square bracket are skew-symmetrized.

Definition 6. *The Λ_2 -valued derivation T of the algebra Λ is called the torsion of τ .*

The graded F-N square of Γ

$$R = [\Gamma, \Gamma]^{\text{fn}} \in D(\Lambda, \Lambda_2)$$

is well defined and has the following local expression

$$R = [\Gamma, \Gamma]^{\text{fn}} = R_{\sigma\mu\delta}^\alpha d_1 x^\delta d_2 x^\sigma d_2 x^\mu i_{\partial_\alpha},$$

where $R_{\sigma\mu\delta}^\alpha = \partial_\sigma \Gamma_{\mu\delta}^\alpha - \partial_\mu \Gamma_{\sigma\delta}^\alpha + \Gamma_{\sigma\beta}^\alpha \Gamma_{\mu\delta}^\beta - \Gamma_{\mu\beta}^\alpha \Gamma_{\sigma\delta}^\beta$.

Definition 7. *The Λ_2 -valued derivation R of the algebra Λ is called the Riemann tensor of τ .*

The introduced above concepts of generalized Levi-Civita connection, torsion and Riemann tensor associated with the 2-tensor τ have standard counterparts. Namely, recall that $\tau = \iota_2(\tau')$, $\tau' \in T_2^0(M)$ being a ‘‘standard’’ covariant 2-tensor on M . The above computations show that there is a standard connection Γ' in the tangent bundle of M associated with τ' . Indeed, the covariant derivative along $X \in D(M)$ with respect to Γ' ∇'_X is well defined by

$$(\nabla'_X Y)(f) = (i_X^{(1)} \circ i_Y^{(2)} \circ \Gamma)(df) \in A \subset \Lambda_2.$$

Now, denote by T' and R' the ‘‘standard’’ torsion and Riemann curvature tensor of Γ' , respectively.

Proposition 8. For any $X, Y, Z \in D(M)$ and $f \in A$

$$T'(X, Y)(f) = (i_X^{(1)} \circ i_Y^{(2)} \circ T)(df) \in A \subset \Lambda_2.$$

$$R'(X, Y)(Z)(f) = (i_Z^{(1)} \circ i_X^{(2)} \circ i_Y^{(2)} \circ R)(df) \in A \subset \Lambda_2.$$

Proof. Obvious from local expression. \square

In new terms geodesics of Γ' are described as follows. Let $\frac{d}{dt} : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ be the “time current” field. Consider a smooth curve $\chi : I \rightarrow M$ on M , $I \subset \mathbb{R}$ being an interval, the algebra $\Lambda_2(I)$ of second iterated forms on I and the pull-back homomorphism $\chi^* : \Lambda_2 \rightarrow \Lambda_2(I)$. The *geodesic curvature* vector field (along χ) is the composition

$$K_\chi \stackrel{\text{def}}{=} i_{d/dt} \circ i_{d/dt}^{(2)} \circ \chi^* \circ \Gamma : \Lambda \rightarrow \Lambda(I).$$

Proposition 9. χ is an affinely parameterized geodesic of the metric g' if and only if $K_\chi = 0$.

This is obvious from the local expression

$$K_\chi(dx^\mu) = \Gamma_{\rho\alpha}^\mu \frac{dx^\rho}{dt} \frac{dx^\alpha}{dt} + \frac{d^2x^\mu}{dt^2}.$$

We now describe covariant derivatives of covariant tensors in new terms. First of all note that for any $l < k$, $\Lambda(\Lambda_l) \otimes_{\Lambda_l} \Lambda_k$ is naturally isomorphic to $\Lambda(\Lambda_l, \Lambda_k)$. Therefore, a connection in the inclusion $\Lambda_l \subset \Lambda_k$ (in the sense of section 1) may be understood as a derivation $\Lambda_k \rightarrow \Lambda(\Lambda_l, \Lambda_k)$. In the following we will adopt this point of view.

Define a new graded form-valued derivation $\nabla = d_2 - \Gamma : \Lambda \rightarrow \Lambda_2$. ∇ has local expression

$$\nabla = d_2 x^\mu (\partial_\mu - \Gamma_{\mu\beta}^\alpha d_1 x^\beta i_{\partial_\alpha}).$$

Note that $\text{im } \nabla \subset \Lambda(A, \Lambda)$.

Proposition 10. $\nabla : \Lambda \rightarrow \Lambda(A, \Lambda)$ is a connection in the natural inclusion $\varphi : A \hookrightarrow \Lambda$ in the sense of section 1.

As shown in section 1, in its turn ∇ induces a connection $\Lambda\nabla$ in $\Lambda(\varphi) : \Lambda \hookrightarrow \Lambda_2$. Note that $\Lambda(\varphi) = \kappa|_\Lambda$. Therefore,

$$\nabla_2 \equiv \Lambda(\kappa) \circ \nabla' \circ \kappa : \Lambda_2 \rightarrow \Lambda(\Lambda, \Lambda_2)$$

is a connection in the natural inclusion $\Lambda \subset \Lambda_2$. Iterating the procedure we get, at the k -th step, a connection ∇_k in the inclusion $\Lambda_{k-1} \subset \Lambda_k$. Note that the existence of the connections ∇_k enriches noteworthy the standard calculus in Riemannian geometry (consequences of this remark will be explored elsewhere). Denote by $\overset{k}{\nabla}$ the composition

$$\overset{k}{\nabla} = \nabla_k \circ \nabla_{k-1} \circ \cdots \circ \nabla_2 \circ \nabla_1 : \Lambda_k \rightarrow \Lambda(A, \Lambda_k).$$

$\overset{k}{\nabla}$ is a connection in the inclusion $A \subset \Lambda_k$. Accordingly, we can lift A -linearly a derivation $X \in D(A, \Lambda_k)$ to a derivation $\overset{k}{\nabla}_X = i_X^{(k+1)} \circ \overset{k}{\nabla} \in D(\Lambda_k, \Lambda_k)$.

Definition 11. $\overset{k}{\nabla}$ is called the k -th covariant derivative associated to Γ .

As an example, we report local expression of $\overset{2}{\nabla}$:

$$\overset{2}{\nabla} = d_3 x^\mu (\partial_\mu - \Gamma_{\mu\beta}^\alpha d_1 x^\beta i_{\partial_\alpha} - \Gamma_{\mu\beta}^\alpha d_2 x^\beta i_{\partial_\alpha}^{(2)} - d_1(\Gamma_{\mu\beta}^\alpha d_2 x^\beta) i_{i_{\partial_\alpha}}^{(2)}).$$

The terminology of definition 11 is motivated by the fact that the standard covariant derivatives with respect to Γ' of a covariant k -tensor $s' \in T_k^0(M)$ are naturally recovered from $\overset{k}{\nabla}$. Namely, we have

Proposition 12. Let $s = \iota_k(s') \in \Lambda_k$ and $X, Y_1, \dots, Y_k \in D(M)$. Then

$$(\overset{k}{\nabla}'_X s')(Y_1, \dots, Y_k) = (i_{Y_1}^{(1)} \circ \dots \circ i_{Y_k}^{(k)} \circ i_X^{(k+1)} \circ \overset{k}{\nabla})(s) \in A \subset \Lambda_k.$$

It is worth mentioning that, on the contrary to the the classical Levi-Civita connections, i.e., associated with metrics, the covariant derivative of a 2-tensor τ with respect to the associated connection is, generally, different from zero, i.e., $\overset{2}{\nabla}\tau \neq 0$. Namely, let $\Gamma^g : \Lambda_1 \rightarrow \Lambda_2$ be the (generalized) Levi-Civita connection of g and $\overset{k}{\nabla}^g$ the associated k -th covariant derivative. Then,

$$\overset{2}{\nabla}\tau = \overset{2}{\nabla}\omega = \overset{2}{\nabla}^g\omega + T(\omega),$$

where $T(\omega)$ is the iterated form locally given by

$$T(\omega) = T_{\mu[\nu}^\beta \omega_{\alpha]\beta} d_1 x^\alpha d_2 x^\nu d_3 x^\mu.$$

3. NATURAL EQUATIONS IN GENERAL RELATIVITY

The possibility to associate in a canonical way a Levi-Civita-like connection with a 2-tensor τ with non-degenerate symmetric part suggests an immediate application to general relativity. Indeed, it is natural to interpret the skew-symmetric part ω of τ as a matter field. Then, it is natural as well to assume that the space, i.e., the symmetric part g of τ , is shaped by the matter in such a way that the Ricci tensor $\text{Ric}(\tau)$ of the associated with τ connection vanishes. This way one gets the *natural* equations in general relativity. We use here the word “natural” in order to stress that in the proposed approach the credit is definitively given to the naturalness of the background mathematical language, but not to the “physical intuition” which could be easily misleading in this context.

In coordinates these natural equations look as follows. Let $R = R_{\mu\sigma}^\alpha dx^\delta \otimes dx^\sigma \otimes dx^\mu \otimes \frac{\partial}{\partial x^\alpha}$ be the associated with τ standard Riemann tensor. The Ricci tensor $\text{Ric}(\tau)$ of τ is defined as the 2-tensor locally given by $\text{Ric}(\tau) = R_{\mu\delta} dx^\mu \otimes dx^\delta$, where $R_{\mu\delta} = R_{\mu\alpha\delta}^\alpha$, $\mu, \delta = 1, \dots, n$ (the interpretation of the Ricci tensor in terms of iterated differential forms will be discussed separately). Then the natural equations for τ read $\text{Ric}(\tau) = 0$.

Now denote by $\overset{g}{\nabla}$ and $\text{Ric}(g)$ the covariant differential and the Ricci tensor of the pseudo-metric g , respectively, and put $\text{Ric}(g) = \overset{g}{R}_{\mu\delta} dx^\mu \otimes dx^\delta$. Then in terms of g and ω the natural equations for τ read

$$\begin{aligned} \overset{g}{R}_{\mu\nu} + \frac{9}{16} g^{\gamma\rho} g^{\delta\sigma} \partial_{[\delta\omega_{\mu\rho]} \partial_{[\gamma\omega_{\nu\sigma]}} = 0 \\ \overset{g}{\nabla}^\gamma (\partial_{[\mu\omega_{\nu\gamma]}) = 0 \end{aligned} \quad (2)$$

System (2) may be understood as Einstein-like equations for g in presence of a matter field (represented by ω) satisfying a suitable constitutive equation. Indeed, it can be shown that (2) are Einstein equations in presence of a perfect irrotational fluid with a definite equation of state (details will be discussed in a separate publication).

It is worth stressing that the matter in the above natural equations is treated in a simplified manner. More rich and exact versions of natural equations in the frames of the developed approach will be discussed separately.

4. LEVI-CIVITA CONNECTIONS OVER SMOOTH SUPER-ALGEBRAS

The previous results are straightforwardly generalized to supermanifolds. In this section, we shall sketch how Riemannian supergeometry (see, for instance, [10] and references therein) can be developed in the framework of iterated differential forms along the lines of section 2.

Consider a super-manifold S , a local chart $(\theta^1, \dots, \theta^n)$ on it and put $A = C^\infty(S)$. A is a \mathbb{Z}_2 -graded commutative algebra. Put $\bar{\alpha} = |\theta^\alpha| \in \mathbb{Z}_2$. Let $\Lambda = \Lambda_1$ be the $\mathbb{Z}_2 \oplus \mathbb{Z}$ -graded algebra of (geometric) differential forms over S and $\Lambda_2 = \Lambda(\Lambda)$ be the $\mathbb{Z}_2 \oplus \mathbb{Z}^2$ -graded algebra of doubly iterated geometric differential forms over S (see also [8]).

Let g be an even doubly iterated form of bi-degree $(1, 1)$ over A such that the map $D(S) \times D(S) \ni (X, Y) \mapsto (i_Y^{(2)} \circ i_X^{(1)})(g) \in A \subset \Lambda$ is A -bilinear, $\kappa(g) = g$ and the A -homomorphism $\int_g : D(M) \ni X \mapsto i_X^{(2)} g \in \Lambda^1 \subset \Lambda_2$ is bijective.

Definition 13. g is called a supermetric over S .

Locally, $g = g_{\mu\nu} d_1 \theta^\mu d_2 \theta^\nu$, $g_{\mu\nu} = (-1)^{\bar{\mu}\bar{\nu}} g_{\nu\mu} \in A$. Moreover, the matrix $\|g_{\nu\mu}\|$ is invertible in A .

Given a metric g over S the derivation $\int_2^g : \Lambda_2 \rightarrow D_\Lambda(\Lambda_2)$ is defined exactly as above. Its local expression is

$$\int_2^g(\Omega)(\omega) = (-1)^{|\Omega| \cdot (\bar{\nu}, 0, -1)} \frac{1}{2} g^{\mu\nu} (i_{\partial_\mu}^{(2)} \Omega)(i_{\partial_\nu} \omega) \in \Lambda_2,$$

where $\Omega \in \Lambda_2$, $\omega \in \Lambda$, $(\bar{\nu}, 0, -1) \in \mathbb{Z}_2 \oplus \mathbb{Z}^2$ is the multi-degree of i_{∂_ν} , $\partial_\nu = \frac{\partial}{\partial \theta^\nu}$ and $g^{\mu\nu} \in A$, $(-1)^{\bar{\gamma}\bar{\alpha}} g_{\nu\alpha} g^{\nu\gamma} = \delta_\alpha^\gamma$.

Definition 14. The Λ_2 -valued derivation $\Gamma = \int_2^g(\gamma) = -\int_2^g(d_1 d_2 g)$ of the algebra Λ is called the Levi-Civita connection form of g .

The local expression of Γ is

$$\Gamma = (d_1 d_2 \theta^\alpha + \Gamma_{\mu\beta}^\alpha d_1 \theta^\beta d_2 \theta^\mu) i_{\partial_\alpha},$$

with $\Gamma_{\mu\beta}^\alpha = \frac{1}{2} g^{\gamma\alpha} [(-1)^{\bar{\gamma}\cdot\bar{\gamma}+\bar{\mu}(\bar{\mu}+\bar{\gamma})} \partial_\mu g_{\beta\gamma} + (-1)^{\bar{\gamma}\cdot\bar{\gamma}+\bar{\beta}(\bar{\mu}+\bar{\beta}+\bar{\gamma})} \partial_\beta g_{\mu\gamma} - \partial_\gamma g_{\beta\mu}]$. Similarly, the Riemann tensor and the covariant derivative are introduced. For instance,

$$R = [\Gamma, \Gamma]^{\text{fn}} = R_{\gamma\beta\alpha}^\nu d_1 \theta^\alpha d_2 \theta^\gamma d_2 \theta^\beta i_{\partial_\nu}$$

where

$$\begin{aligned} R_{\gamma\beta\alpha}^\nu &= (-1)^{\bar{\alpha}(\bar{\beta}+\bar{\gamma})+\bar{\beta}(\bar{\beta}+\bar{\nu})} \partial_\beta \Gamma_{\alpha\gamma}^\nu - (-1)^{\bar{\beta}(\bar{\beta}+\bar{\gamma}+\bar{\nu})} \partial_\gamma \Gamma_{\alpha\beta}^\nu \\ &+ (-1)^{\bar{\mu}\bar{\beta}+\bar{\alpha}\bar{\gamma}} \Gamma_{\mu\beta}^\nu \Gamma_{\alpha\gamma}^\mu - (-1)^{\bar{\beta}\cdot(\bar{\alpha}+\bar{\gamma})+\bar{\mu}\bar{\gamma}} \Gamma_{\mu\gamma}^\nu \Gamma_{\alpha\beta}^\mu. \end{aligned}$$

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