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MICHAEL BÄCHTOLD

Abstract. We study Ricci flat 4–metrics of any signature under the assumption that they allow a Lie algebra of Killing fields with 2–dimensional orbits along which the metric degenerates and whose orthogonal distribution is not integrable. It turns out that locally there is a unique (up to a sign) metric which satisfies the conditions. This metric is of signature $(++--)$ and, moreover, homogeneous possessing a 6–dimensional symmetry algebra.

Introduction

In this article we find all metrics g in 4 dimensions with vanishing Ricci tensor that satisfy the conditions

(i) g allows a Lie algebra of Killing vector fields with 2–dimensional orbits,
(ii) the tensor g degenerates when restricted to the orbits,
(iii) the distribution orthogonal to the orbits is not integrable.

This is a particular case of the problem of finding all Ricci flat metrics with bidimensional Killing orbits. A systematical study of this problem was started in [6, 3], where the vacuum Einstein equations where solved for metrics of arbitrary signature under the assumption of integrability of the orthogonal to orbits distribution. A class of Lorentzian solutions with two commuting symmetries and a transversal and completely integrable orthogonal to orbits distribution was already found in the 70's [4, 1, 2].

Here we prove that under conditions (i)-(iii) there exists only one, up to a sign, Ricci flat metric (theorem 2). This outcome differs drastically from the cases considered previously where one finds an abundance of solutions. In particular, our case does not admit any solutions with an abelian Killing algebra, nor of a Lorentzian signature. This fact suggests that condition (iii) is, in a sense, not very compatible with the Einstein equations. We stress that the results of this work and that of [3] give an exhausting exact descriptions of Ricci flat metrics that are subjected to conditions (i) and (ii).

To solve the Einstein equations we work in a non-holonomic $\mathfrak{g}$-invariant frame field. This choice is motivated by the observation that $\mathfrak{g}$-invariant Ricci flat metrics are in one to one correspondence with Ricci flat metrics on the Lie algebroid of $\mathfrak{g}$-invariant vector fields, at least, for free actions. This Lie algebroid is associated to any smooth

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action on a manifold and forms a vector bundle over the space of orbits. Furthermore, the concept of curvature tensor makes sense for metrics on Lie algebroids.

Overview. The article consist of two main sections. In the first of them metrics satisfying the conditions (i)-(iii) without the assumption of Ricci flatness are described. Here it is shown that the Killing algebras to be considered as generators of orbits are necessarily bidimensional. Further we describe the orthogonal to orbits distribution and give normal forms for the action of $\mathfrak{g}$ on $\mathcal{D}^{\perp}$. Section 2 is dedicated to solving the Einstein equations. First, it is shown that in the case of an abelian $\mathfrak{g}$ these have no solutions. Then we treat the nonabelian case and give an exact description of all solutions in terms of the chosen frame by proving further that these are locally isometric to each other up to a sign. Finally, we compute the full Killing algebra of these metrics.

Notations and Conventions.

- All objects are assumed to be smooth, i.e., C^∞, M stands for a generic 4–fold we are working with and $U \subset M$ for a generic open subset in M;
- By a *metric* g we understand a symmetric non-degenerate $(0,2)$ tensor of arbitrary (but constant) signature;
- The $C^\infty(M)$ -module of vector fields is denoted by $D(M)$. The Lie derivative along a vector field X is denoted by $X(f)$ on functions and $L_X g$ on higher order tensors and $i_X g$ is the insertion of X into g. Basic vector fields of a local chart $x^1, \ldots, x^n$ are denoted by $\partial_1, \ldots, \partial_n$.
- The module of k-forms is denoted with $\Lambda^k(M)$. Unless specified differently, a set of of forms $\omega_1, \ldots, \omega_j \in \Lambda^1(M)$ will be called *linearly independent* if they are independent over $C^\infty(M)$, i.e. if they are independent at every point. A set of functions $x^1, \ldots, x^r \in C^\infty(U)$ will be called independent if their differentials are so.
- Abelian and nonabelian bidimensional Lie algebras are denoted by $\mathcal{A}_2$ and $\mathcal{G}_2$, respectively.
- We use the term *symmetry* as a synonym for *infinitesimal symmetry*.
- The Lie algebra of *all* Killing symmetries of g is denoted with $\mathrm{Kill}(g)$. A subalgebra $\mathfrak{g} \subset \mathrm{Kill}(\mathrm{g})$ will be called a *Killing algebra* of the metric.
- $\mathfrak{g}^{(1)}$ denotes the first *derived subalgebra* of a Lie algebra: $\mathfrak{g}^{(1)} = \mathrm{Span}\{[\xi, \zeta] \mid \xi, \zeta \in \mathfrak{g}\}$;
- Distributions are denoted by calligraphic letters $\mathcal{P}$ that refer to the module of sections of the distribution. We say that forms $\omega_1, \ldots, \omega_k \in \Lambda^1(M)$ *define or describe* a distribution, if they generate the module $\mathrm{Ann}(\mathcal{P})$ of one-forms annihilating $\mathcal{P}$;
- $\mathcal{D}$ denotes the distribution spanned by the vector fields of the Killing algebra $\mathfrak{g}$, $\mathcal{D}^{\perp}$ stands for the g–orthogonal to $\mathcal{D}$ distribution and $\mathcal{C} := \mathcal{D} \cap \mathcal{D}^{\perp}$;
- The algebra of $\mathfrak{g}$-*invariant* functions is denoted by $C^\infty(M)^{\mathfrak{g}} = \{f \in C^\infty(M) \mid X(f) = 0 \quad \forall X \in \mathfrak{g}\}$ and similarly for other invariant objects.

- Since most considerations of this paper are of local nature, we often omit to explicitly state arguments of the form: "after restriction to a possibly smaller open set", etc.

1. Metrics subject to conditions (i)-(iii)

1.1. **Preliminaries.** Let M be a 4–dimensional manifold supplied with a metric g. We assume that g allows a Killing algebra $\mathfrak{g} \subset \mathrm{Kill}(g)$ which spans a 2–dimensional distribution $\mathcal{D}$. Obviously, $\mathcal{D}$ is Frobenius and its integral submanifolds are called the Killing leaves or, equivalently, orbits. We further assume that the bidimensional orthogonal distribution $\mathcal{D}^\perp := \{X \in D(M) \mid g(X,Y) = 0 \quad \forall Y \in \mathcal{D}\}$ is not integrable and that the intersection $\mathcal{C} := \mathcal{D} \cap \mathcal{D}^\perp$ is nontrivial. This last condition is equivalent to saying that the metric becomes degenerate when restricted to the Killing leaves. Of course, this can only happen if g is of Lorentzian $(- + ++)$ or Kleinian $(+ + --)$ signature.

Almost everywhere, i.e., in an open dense subset of M, the dimension of $\mathcal{C}$ is 1 since an open region where $\mathcal{D} = \mathcal{D}^\perp$ would imply that $\mathcal{D}^\perp$ is integrable. So, we shall throughout assume that $\dim \mathcal{C} = 1$. Finally, note that the distribution $\mathcal{C}^\perp$ is 3 dimensional and contains all the others: $\mathcal{C}^\perp = \mathcal{D} + \mathcal{D}^\perp$.

The following proposition shows that under our assumptions $\mathfrak{g}$ can only be 2–dimensional, i.e., $\mathfrak{g}$ is either $\mathcal{A}_2$ or $\mathcal{G}_2$.

Proposition 1. *Let g be a metric possessing a Killing algebra $\mathfrak{g}$ with bidimensional orbits. If $\dim \mathfrak{g} > 2$, then $\mathcal{D}^\perp$ is integrable.*

See [6] for a proof.

1.2. **The orthogonal distribution.** The aim of this section is to bring to normal forms the action of $\mathfrak{g}$ on $\mathcal{D}^\perp$. This will allow us to introduce frame fields adapted to the considered situation, which will be used in solving the Einstein equations. Some basic facts we need in this section concerning distributions, can be found in the book [5].

The first simple yet important observation is

Proposition 2. *The distribution $\mathcal{D}^\perp$ is invariant under the action of $\mathfrak{g}$. In other words, the Killing algebra is a symmetry algebra of $\mathcal{D}^\perp$ as well.*

Proof. Take $A \in \mathcal{D}^\perp$, $X, Y \in \mathfrak{g}$ and use the standard formula for the Lie derivative

$$X(\underbrace{g(A,Y)}_{=0}) = \underbrace{L_X(g)}_{=0}(A,Y) + g([X,A],Y) + \underbrace{g(A,[X,Y])}_{=0}$$

Hence $g([X,A],Y) = 0$. Since $\mathcal{D}$ is spanned by fields $Y \in \mathfrak{g}$ this implies that $[X,A] \in \mathcal{D}^\perp$. □

The next fact we need is that $\mathcal{D}^\perp$ is contained in a 3–dimensional integrable distribution. Namely,

Proposition 3. *The distribution $\mathcal{C}^\perp$ is integrable.*

Proof. We can span $\mathcal{C}^\perp$ by two Killing fields from $\mathcal{D}$ and one field from $\mathcal{D}^\perp$. Then by the previous proposition all their commutators are either in $\mathcal{D}$ or $\mathcal{D}^\perp$ and hence in $\mathcal{C}^\perp$. □

In this connection it is convenient to introduce the following terminology.

Definition 1. A bidimensional non-integrable distribution $\mathcal{P}$ on a 4–fold will be called *semi-integrable* if it is contained in a 3–dimensional integrable distribution.

Remark 1. Such a 3–dimensional integrable distribution containing $\mathcal{P}$ is unique. Indeed, it must coincide with the distribution generated by $\mathcal{P}$ and all its commutators:

$$\mathcal{P}^{(1)} := \mathrm{Span}\{X, [X, Y] \mid X, Y \in \mathcal{P}\}$$

Now we come to the main results of this section, which allow us to construct the adapted frames for metrics with (i)-(iii).

Proposition 4. *Let M be a 4–dimensional manifold supplied with a 2–dimensional semi-integrable distribution $\mathcal{P}$. Suppose $\mathcal{A}_2$ acts on M as a symmetry algebra of $\mathcal{P}$. If the orbits of the action are bidimensional and run inside the leaves of $\mathcal{P}^{(1)}$, then for a generic point there exists a local chart $x, u, p, z \in C^\infty(U)$ in terms of which $\mathcal{P}$ is described by*

$$\mathrm{d}z, \quad \mathrm{d}u - p\mathrm{d}x \in \mathrm{Ann}(\mathcal{P})$$

while the symmetries are

$$\partial_u, \quad \partial_x \in \mathcal{A}_2$$

In the proof we shall use the following simple

Lemma 1. *If $\mathcal{P}$ is a bidimensional, non-integrable distribution and $S \in D(M)$ a non-trivial symmetry of it, then the set of points where S lies outside of $\mathcal{P}$ is open and dense.*

Proof. Otherwise there would be a region where S is a characteristic symmetry, but a bidimensional distribution with a nontrivial characteristic symmetry is integrable. □

Now we pass to prove proposition 4

Proof. Fix a basis $S_1, S_2 \in \mathcal{A}_2$. Since $\mathcal{P}^{(1)}$ is integrable we can locally choose a function $z \in C^\infty(U)$ so that $\mathcal{P}^{(1)}$ is given by $\mathrm{d}z = 0$. This way one finds the required coordinate function z which is unique up to transformation $z \mapsto f(z)$.

Choose then any 1–form $\theta' \in \mathrm{Ann}(\mathcal{P})$ which together with $\mathrm{d}z$ generates $\mathrm{Ann}(\mathcal{P})$ and scale it so that

$$i_{S_1}\theta' = 1.$$

This scaling is generically possible by lemma 1 and since θ' is linearly independent of $\mathrm{d}z$. Note that this choice of θ' comes with the freedom of adding a term $f \cdot \mathrm{d}z$ with arbitrary $f \in C^\infty(U)$. Making use of it we change θ' to a form θ such that

$$L_{S_i}(\theta) = 0, \quad i = 1, 2$$

Indeed, since S_1, S_2 are symmetries of $\mathcal{P}$ we have $L_{S_i}\theta' \in \mathrm{Ann}(\mathcal{P})$, i.e.,

$$L_{S_i}\theta' = a_i\theta' + b_i\mathrm{d}z, \quad i = 1, 2$$

with $a_i, b_i \in C^\infty(U)$. The standard formula $i_{S_1} \circ L_{S_i} = L_{S_i} \circ i_{S_1} - i_{[S_i,S_1]}$ applied to θ' shows that $a_i = 0$ or in other words, $L_{S_i}\theta' = b_i\mathrm{d}z$. Since S_1, S_2 commute we also have $L_{S_2}L_{S_1}\theta' = L_{S_1}L_{S_2}\theta'$, implying the "compatibility condition" $S_1(b_2) = S_2(b_1)$ for the overdetermined system

$$S_1(f) = -b_1, \quad S_2(f) = -b_2.$$

Hence this system admits a solution f and we get the desired result with $\theta = \theta' + f\mathrm{d}z$.

Since $\mathcal{P}$ is non-integrable, $\mathrm{d}\theta \neq 0$. Moreover, it follows from $L_{S_1}(\theta) = 0$ and $i_{S_1}(\theta) = 1$ that $i_{S_1}(\mathrm{d}\theta) = 0$. In other words, $S_1 \in \ker(\mathrm{d}\theta)$ and hence $\ker(\mathrm{d}\theta) \neq 0$. Since the rank of the 2–form $\mathrm{d}\theta$ is an even integer ≤ 4 we conclude that it is exactly 2.

Put now $p := -i_{S_2}(\theta)$. Then

$$0 = L_{S_2}(\theta) = \mathrm{d}(i_{S_2}\theta) + i_{S_2}(\mathrm{d}\theta) \quad \Leftrightarrow \quad i_{S_2}(\mathrm{d}\theta) = \mathrm{d}p$$

and

$$S_2(p) = i_{S_2}(\mathrm{d}p) = i_{S_2}(i_{S_2}(\mathrm{d}\theta)) = 0.$$

Similarly,

$$S_1(p) = i_{S_1}(i_{S_2}(\mathrm{d}\theta)) = -i_{S_2}(i_{S_1}(\mathrm{d}\theta)) = 0.$$

Thus $S_i(p) = 0, \quad i = 1, 2$.

Now consider a (local) function $x \in C^\infty(U)$ such that $S_2(x) = 1$, and which is constant along the 2–dimensional leaves of $\ker(\mathrm{d}\theta)$. To find such a function note that S_2 is a symmetry of the integrable distribution $\ker(\mathrm{d}\theta)$ and hence, is related to a vector field $\tilde{S}_2$ on the quotient manifold $U/\ker(\mathrm{d}\theta)$. We may then take x as the pullback of a function $\tilde{x}$ on $U/\ker(\mathrm{d}\theta)$ with $\tilde{S}_2(\tilde{x}) = 1$.

For $\omega := -p\mathrm{d}x$ it follows that $\ker(\mathrm{d}\omega) = \ker(\mathrm{d}\theta)$ and therefore, $\mathrm{d}\omega = \lambda\mathrm{d}\theta, \quad \lambda \in C^\infty(U)$. Using previous relations it is further easily seen that

$$i_{S_2}(\mathrm{d}\omega) = \mathrm{d}p = i_{S_2}(\mathrm{d}\theta)$$

which implies $\lambda = 1$. Thus $\mathrm{d}\omega = \mathrm{d}\theta$ and, so, $\theta = \omega + \mathrm{d}u$ for a function $u \in C^\infty(U)$.

Functions x, u, p, z constructed above form, obviously, a local chart on M and it remains to show that $S_1 = \partial_u$ and $S_2 = \partial_x$. But $1 = i_{S_1}(\theta) = S_1(u) + i_{S_1}(\omega) = S_1(u)$. This says that the u–component of S_1 is equal to 1. Other components vanish because of already established relations $S_1(x) = S_1(p) = S_1(z) = 0$. This proves that $S_1 = \partial_u$ and, similarly, that $S_2 = \partial_x$. □

The analog of proposition 4 in the nonabelian case is

Proposition 5. *Let M be a 4 dimensional manifold supplied with a 2–dimensional semi-integrable distribution $\mathcal{P}$. Suppose $\mathcal{G}_2$ acts on M as a symmetry algebra of $\mathcal{P}$. If the orbits of the action are bidimensional and run inside the leaves of $\mathcal{P}^{(1)}$, then for* *a generic point there exists a local chart $x, u, p, z \in C^\infty(U)$ in terms of which $\mathcal{P}$ is described by*

$$\mathrm{d}z, \quad \mathrm{d}u - p\mathrm{d}x \in \mathrm{Ann}(\mathcal{P})$$

while the symmetries are

$$\partial_u, \quad u\partial_u + p\partial_p \in \mathcal{G}_2$$

Proof. Fix a basis S_1, S_2 of $\mathcal{G}_2$ with the property $[S_1, S_2] = S_1$. The coordinate z and an 1–form θ' such that $i_{S_1}(\theta') = 1$ and $L_{S_1}(\theta') = 0$ are constructed exactly as in the previous case. Since $\mathrm{d}z$ and θ' generate $\mathrm{Ann}(\mathcal{P})$

$$L_{S_2}(\theta') = a\theta' + b\mathrm{d}z, \quad a, b \in C^\infty(U),$$

The identity $i_{S_1} \circ L_{S_2} = L_{S_2} \circ i_{S_1} + i_{[S_1,S_2]}$ applied to θ' shows that $a = 1$ while $[S_1, S_2] = S_1$ gives $S_1(b) = 0$. Now construct a form $\theta = \theta' + f\mathrm{d}z$ such that

$$L_{S_1}(\theta) = 0, \quad L_{S_2}(\theta) = \theta.$$

This is, obviously, equivalent to

$$S_1(f) = 0, \quad S_2(f) = f - b.$$

As it is easy to see, the compatibility condition for this system is exactly $S_1(b) = 0$ and, so, it possesses solutions. Hence the required form θ exists. Also note that by construction $i_{S_1}\theta = 1$.

Put

$$u := i_{S_2}\theta$$

The identities $i_{S_2} \circ L_{S_1} = L_{S_1} \circ i_{S_2} + i_{[S_1,S_2]}$ and $i_{S_2} \circ L_{S_2} = L_{S_2} \circ i_{S_2}$ applied to θ give

$$S_1(u) = 1, \quad S_2(u) = u \quad \Leftrightarrow i_{S_1}\mathrm{d}u = 1, \quad i_{S_2}\mathrm{d}u = u.$$

This shows that $\mathrm{d}u$ and $\mathrm{d}z$ are independent.

Consider now the non-closed form

$$\alpha := \mathrm{d}u - \theta = di_{S_2}(\theta) - L_{S_2}(\theta) = -i_{S_2}(\mathrm{d}\theta).$$

By the same arguments as in the proof of the previous proposition one can see that $\mathrm{d}\theta$ has rank 2. Hence $\mathrm{d}\theta \wedge \mathrm{d}\theta = 0$ and

$$\mathrm{d}\alpha \wedge \alpha = \mathrm{d}\theta \wedge i_{S_2}\mathrm{d}\theta = \frac{1}{2} i_{S_2}(\mathrm{d}\theta \wedge \mathrm{d}\theta) = 0$$

which means that $\ker(\alpha)$ is integrable. Hence there exist (locally) two independent functions x, p such that $\alpha = p\mathrm{d}x$ and, so, $\alpha = \mathrm{d}u - p\mathrm{d}x$. Functions x, u, p restricted to a generic hypersurface $z = \mathit{const}$ are independent. Indeed, otherwise the distribution $\mathcal{P}$ would be integrable. This also shows that functions x, u, p, z are (locally) independent.

Finally, observe that

$$i_{S_i}(\alpha) = 0 \quad \Leftrightarrow \quad S_i(x) = 0, \quad i = 1, 2.$$

Now the relations $i_{S_1}\mathrm{d}\alpha = 0$ and $i_{S_2}\mathrm{d}\alpha = \alpha$ together with $\mathrm{d}\alpha = \mathrm{d}p\wedge\mathrm{d}x$ and the previous equation yield

$$S_1(p) = 0, \quad S_2(p) = p$$

Together with previous relations this shows that $S_1 = \partial_u$ and $S_2 = u\partial_u + p\partial_p$. □

2. Solving the Einstein equations

2.1. **The abelian case.** In this section we show that under assumptions (i)-(iii) there are no Ricci flat metrics in the case $\mathfrak{g} = \mathcal{A}_2$.

Consider the chart x, u, p, z constructed in proposition 4. In terms of this chart Killing fields in question are ∂_x and ∂_u, while the orthogonal distribution $\mathcal{D}^\perp$ is spanned by ∂_p and $\partial_x + p\partial_u$. In the following non-holonomic frame field:

$$\begin{array}{lll} e_1 & := & \partial_u \quad \in \mathcal{D} \\ e_2 & := & \partial_p \quad \in \mathcal{D}^\perp \\ e_3 & := & \partial_x + p\partial_u \quad \in \mathcal{C} \\ e_4 & := & \partial_z \end{array}$$

the matrix $g_{ij} := g(e_i, e_j)$ is of the form

$$\begin{pmatrix} a & 0 & 0 & s_1 \\ 0 & b & 0 & s_2 \\ 0 & 0 & 0 & s_3 \\ s_1 & s_2 & s_3 & s_4 \end{pmatrix} \tag{1}$$

with $a, b, s_1, s_2, s_3, s_4 \in C^\infty(U)$. Since the chosen frame consists of $\mathfrak{g}$–invariant fields all the components are $\mathfrak{g}$–invariant as well and so, depend only on (p, z). Since

$$\det(g_{ij}) = -ab(s_3)^2,$$

non-degeneracy of the metric requires that functions a, b and s_3 be everywhere nonzero.

Now we compute the component $ric(e_3, e_3)$ of the Ricci tensor. Recall that for a frame $e_1, \ldots, e_4 \in D(U)$ the corresponding components of the Ricci tensor are given by:

$$ric(e_i, e_j) = e_j(\gamma^h_{hi}) - e_h(\gamma^h_{ji}) + \gamma^h_{jk}\gamma^k_{hi} - \gamma^h_{hk}\gamma^k_{ji} - c^k_{jh}\gamma^h_{ki}$$

(summation over crossed indices is understood). Here γ^l_{ij}'s are the Christoffel symbols defined by $\nabla_{e_i} e_j = \gamma^l_{ij} e_l$ and as it is well-known

$$\begin{aligned} \gamma^l_{ij} &= \frac{1}{2} g^{lh}(-e_h(g_{ij}) + e_i(g_{hj}) + e_j(g_{hi})) \\ &\quad -\frac{1}{2}(c^l_{ji} + g^{kl} g_{hi} c^h_{jk} + g^{kl} g_{hj} c^h_{ik}) \end{aligned}$$

with the c^h_{ij}'s being structure "constants", i.e., $[e_i, e_j] = c^h_{ij} e_h$.

For the considered frame the only non-vanishing commutator is $[e_2, e_3] = e_1$ and a straightforward computation gives

$$ric(e_3, e_3) = \frac{a}{2b}$$

But since the function a does not vanish we have

Theorem 1. *There are no Ricci flat metrics satisfying conditions (i)-(iii) for an abelian Lie algebra $\mathfrak{g}$.*

2.2. **The nonabelian case.** In the local chart x, u, p, z of proposition 5 the symmetries are $S_1 = \partial_u, \quad S_2 = u\partial_u + p\partial_p$ and the orthogonal distribution is spanned by $\partial_x + p\partial_u$ and ∂_p. As before it is convenient to pass to a $\mathfrak{g}$–invariant frame:

$$\begin{aligned} e_1 &:= p\partial_u \quad \in \mathcal{D} \\ e_2 &:= \partial_x + p\partial_u \quad \in \mathcal{D}^\perp \\ e_3 &:= p\partial_p \quad \in \mathcal{C} \\ e_4 &:= \partial_z \end{aligned}$$

As before the metric matrix in this frame is of the form (1) with functions $a, b, s_1, s_2, s_3, s_4 \in C^\infty(M)^{\mathcal{G}_2}$, depending on (x, z) only.

In the nonabelian case there is, additionally, an 1–dimensional distribution, namely, that generated by $\mathfrak{g}^{(1)}$, i.e., by ∂_u in the considered chart. Denote it by $\overline{\mathcal{D}}$. A peculiarity of this frame is that $e_1 \in \overline{\mathcal{D}}$. The only nonzero commutators are:

$$[e_3, e_2] = e_1, \quad [e_3, e_1] = e_1 \tag{2}$$

Summing up we have

Corollary 1. *In the nonabelian case there exists a $\mathcal{G}_2$-invariant frame $e_1, e_2, e_3, e_4 \in D(M)$ with $e_1 \in \overline{\mathcal{D}}$, $e_2 \in \mathcal{D}^\perp$ and $e_3 \in \mathcal{C}$, that satisfies commutation relations (2)*

Such a frame is not unique as the following proposition shows.

Proposition 6. *Frames described in corollary (1) are related one to another by a "gauge transformation"*

$$\begin{aligned} e_1 &\mapsto f \cdot e_1 \\ e_2 &\mapsto f \cdot e_2 - e_2(f) \cdot e_3 \\ e_3 &\mapsto e_3 \\ e_4 &\mapsto \alpha \cdot e_4 + \beta \cdot (e_1 - e_2) + e_2(\beta) \cdot e_3 \end{aligned}$$

where f is a nowhere vanishing $\mathcal{G}_2$-invariant function, α is a nowhere vanishing function which is constant along the 3–dimensional leaves of $\mathcal{C}^\perp$ and $\beta \in C^\infty(M)^{\mathcal{G}_2}$ is a solution of the first order differential equation

$$e_2(\beta) = \frac{1}{f}(\beta e_2(f) - \alpha e_4(f)).$$

Proof. The proof is straightforward if one proceeds by passing from one frame to another in the order: e_1, e_3, e_2, e_4 and taking into account that any $\mathcal{G}_2$-invariant frame consists of linear combinations of $e_1, \ldots, e_4$ with coefficients in $C^\infty(M)^{\mathcal{G}_2}$. □

We shall now solve the Einstein equations in various steps.

Step 1. First it is convenient, using the gauge freedom, to pass to a new frame, say X_1, X_2, X_3, X_4, such that $g(X_1, X_1) = \pm 1$. This is possible by proposition 6, since a is $\mathcal{G}_2$-invariant and nowhere zero. In such a frame the metric components take the form

$$\begin{pmatrix} \epsilon & 0 & 0 & t_1 \\ 0 & c & 0 & t_2 \\ 0 & 0 & 0 & t_3 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix}$$

with $\epsilon = \pm 1$ and some new components $c, t_1, \ldots, t_4$ which are $\mathcal{G}_2$-invariant.

Step 2. Put $R_{(i)(j)} := ric(X_i, X_j)$. A direct computation then gives:

$$R_{(3)(3)} = \frac{2c + \epsilon}{2c}.$$

So, for a Ricci flat metric $c = -\frac{\epsilon}{2}$.

Step 3. Taking into account the previous result we obtain

$$R_{(3)(1)} = \frac{t_1 - 2t_2}{t_3}$$

and $t_1 = 2t_2$ for Ricci flat metrics.

Step 4. From

$$R_{(3)(2)} = \frac{X_2(t_3)}{2t_3}$$

we subsequently get $X_2(t_3) = 0$ which means that t_3 is constant along the 3–dimensional leaves of $\mathcal{C}^\perp$.

Step 5. Next

$$R_{(3)(4)} = -\epsilon X_2(t_2)$$

which means that t_2 is also constant along $\mathcal{C}^\perp$.

Step 6. Finally,

$$R_{(1)(2)} = \frac{2t_2^2 - \epsilon t_4}{t_3^2}$$

allows us to eliminate t_4. Namely, $t_4 = 2\epsilon t_2^2$.

Up to this point the metric is of the form

$$\begin{pmatrix} \epsilon & 0 & 0 & 2t_2 \\ 0 & -\frac{\epsilon}{2} & 0 & t_2 \\ 0 & 0 & 0 & t_3 \\ 2t_2 & t_2 & t_3 & \epsilon 2t_2^2 \end{pmatrix}$$

with two arbitrary functions t_2, t_3 which are constant along $\mathcal{C}^\perp$ and $t_3 \neq 0$. Any metric of this form is Ricci flat, but we can still use a remaining gauge freedom in X_4 to eliminate t_2, t_3. This is done by changing X_4 to

$$\tilde{X}_4 = \frac{\epsilon}{t_3}(X_4 - 2\epsilon t_2(X_1 - X_2))$$

and leaving X_1, X_2, X_3 intact. It is easily seen that this field commutes with the first three, and in this final frame the metric assumes the form

$$\epsilon \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \epsilon = \pm 1 \tag{3}$$

The frame is now completely fixed up to a sign of X_1, X_2. Thus we have proven

Theorem 2. *Let g be a Ricci flat metric that possesses a symmetry algebra $\mathfrak{g} \subset Kill(g)$ satisfying properties (i)-(iii). Then $\mathfrak{g}$ is the 2–dimensional nonabelian Lie algebra $\mathcal{G}_2$ and there exist (locally) a non-holonomic frame field X_1, X_2, X_3, X_4 satisfying conditions of corollary 1 in which the components of the metric are that of (3). Such a frame is unique up to the signs of X_1 and X_2.*

Corollary 2. *Two Ricci flat metrics g_1, g_2 satisfying conditions (i)-(iii), are either locally isomorphic or $g_1 \simeq -g_2$.*

This is an immediate consequence of theorem 2 and

Lemma 2. *Given two frame fields $e_1, \ldots, e_4$ and $\tilde{e}_1, \ldots, \tilde{e}_4$ satisfying commutation relations (2), then there is a local diffeomorphism of the underlying manifold which sends one of them to another.*

Proof. A straightforward application of Lie's third theorem. □

2.2.1. *The full Killing algebra of the solutions.* Henceforth we shall denote a frame in which g assumes the form (3) by $X_1, \ldots, X_4$. We may introduce coordinates $x^1, \ldots, x^4$ in which

$$X_1 = \exp(x^3)\partial_1 \qquad X_3 = \partial_3$$
$$X_2 = \partial_2 + \exp(x^3)\partial_1 \qquad X_4 = \partial_4$$

Note also that in this chart the vector fields:

$$V_1 := \partial_1 \qquad V_3 := \partial_3 + x^1\partial_1$$
$$V_2 := \partial_2 \qquad V_4 := \partial_4$$

are symmetries of the frame $X_1, \ldots, X_4$, and since the metric has constant coefficients in that frame the fields $V_1, \ldots, V_4$ are also symmetries of g. Moreover, $V_1, \ldots, V_4$ form an transitively acting Lie algebra which is isomorphic to $\mathcal{G}_2 \oplus \mathcal{A}_2$. So, we have proven

Corollary 3. *The solutions described in theorem 2 are homogeneous metrics.*

Furthermore, we have

Proposition 7. *The full Killing algebra of the metrics (3) is 6–dimensional. Its commutation relations are given below.*

Proof. To find further symmetries of g we take $V_1,\ldots,V_4$ as a frame and present vector fields on M in the form $Z = f^i V_i$ with $f^i \in C^\infty(M), \quad i = 1,\ldots,4$. Using the fact that V_i's are symmetries of g and the formula

$$L_{fX}(g) = fL_X(g) + i_X(g)df$$

where the last term is a symmetric product of one forms, the condition that Z is a symmetry becomes:

$$L_Z(g) = i_{V_i}(g) \cdot df^i = 0.$$

This equality is equivalent to

$$df^i = \Lambda^{ij} \cdot i_{V_j}(g) \tag{4}$$

with some functions $\Lambda^{ij}, \quad i,j = 1,\ldots,4$, that are skew symmetric in the indices, $\Lambda^{ij} = -\Lambda^{ji}$. It follows from equation (4) that

$$\mathrm{d}\Lambda^{ij} \wedge \omega_j + \Lambda^{ij}\mathrm{d}\omega_j = 0 \tag{5}$$

with $\omega_j := i_{V_j}(g)$. Equations (5) form a system of 24 first order linear differential equations for the 6 unknown functions $\Lambda^{ij}, \quad i < j$. The trivial solution $\Lambda^{ij} = 0$ corresponds to the already known symmetries V_j. In the chart $(x^1,\ldots,x^4)$ as above, these equations can be easily solved and the general solution is

$$\Lambda^{1,4} = c_1 e^{x^3} + c_2 e^{2x^3}, \quad \Lambda^{2,4} = c_1 + 2c_2 e^{x^3}$$

with vanishing remaining components. Here c_1, c_2 are arbitrary constants corresponding to two "new" symmetries:

$$V_5 := e^{x^3}\partial_1 + x^3\partial_2 + \frac{1}{2}x^2\partial_4, \quad c_1 = 1, c_2 = 0,$$

and

$$V_6 := \frac{1}{2}e^{2x^3}\partial_1 + 2e^{x^3}\partial_2 + x_1\partial_4, \quad c_1 = 0, c_2 = 1$$

Thus the Killing algebra of the metrics (3) is spanned by $V_1,\ldots V_6$ whose commutation relations are

$$[V_1,V_3] = V_1, \quad [V_2,V_5] = \frac{1}{2}V_4, \quad [V_3,V_5] = V_2, \quad [V_1,V_6] = V_4, \quad [V_3,V_6] = V_6$$

with all other commutators vanishing. The center of this algebra is spanned by V_4, the first derived subalgebra is spanned by V_1, V_2, V_4, V_6 and the second one coincides with the center. □

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