

**A geometric study of the dispersionless
Boussinesq type equation**

by

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A geometric study of the dispersionless Boussinesq type equation

PAUL KERSTEN, I. KRASIL'SHCHIK, A. VERBOVETSKY

ABSTRACT. We discuss the dispersionless Boussinesq type equation, which is equivalent to the Benney–Lax equation, being a system of equations of hydrodynamical type. This equation was discussed in [4]. The results include: a description of local and non-local Hamiltonian and symplectic structures, hierarchies of symmetries, hierarchies of conservation laws, recursion operators for symmetries and generating functions of conservation laws (cosymmetries). Highly interesting are the appearances of operators that send conservation laws and symmetries to each other but are neither Hamiltonian, nor symplectic. These operators give rise to a noncommutative infinite-dimensional algebra of recursion operators.

INTRODUCTION

Below we deal with the *dispersionless Boussinesq type equation* (the dB-equation), which is the system

$$\begin{aligned}w_t &= u_x, \\u_t &= ww_x + v_x, \\v_t &= -uw_x - 3wu_x,\end{aligned}\tag{1}$$

being equivalent to the Benney–Lax equation, and which is known to be integrable, [4]. In particular, it possesses a bi-Hamiltonian structure. System (1) is of hydrodynamical type and can be obtained as a reduction of the Khokhlov–Zabolotskaya equation.

Using the methods developed in [6], we rediscover the above mentioned bi-Hamiltonian structure and show that it is only a part of the infinite-dimensional space of operators that take conservation laws of (1) (their generating functions, more precisely) to symmetries. These operators, in a standard way, generate an infinite associative (but not commutative) algebra of recursion operators for symmetries.

Every recursion operator, applied to a known symmetry (e.g., a point one), gives rise to an infinite family of symmetries (both local and nonlocal ones). Contrary to the known examples, these families are not *hierarchies* in the usual sense, because their jet order does not grow infinitely but remains at level 1 for all symmetries we found.

Dually, there exists an infinite-dimensional space of operators that take symmetries of the dB-equation to generating functions (or *cosymmetries*) and only some of these

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operators determine symplectic (or inverse Noether) structures on the equation. In a similar way, we obtain an infinite algebra of recursion operators for cosymmetries and infinite families of conservation laws (also of 1st order).

Below we present a detailed analysis of all these structures. In Section 1 a very informal introduction to the theoretical background is given. Section 2 contains preparatory material needed to achieve the main results. These results are exposed in Section 3 and discussed in concluding remarks (Sections 4 and 5).

1. BACKGROUND

As it was mentioned in the Introduction, our computations are based on the results of paper [6] (see also [7]). For the general theoretical background we also refer the reader to books [1, 10, 12]. Here we shall give an informal description of the computational scheme we use in subsequent sections.

We consider a system \mathcal{E} of evolution equations

$$u_t = F(x, t, u, u_1, \dots, u_k), \quad (2)$$

where both $u = (u^1, \dots, u^m)$ and $F = (F^1, \dots, F^m)$ are vectors and $u_t = \partial u / \partial t$, $u_s = \partial^s u / \partial x^s$. For simplicity, we restrict ourselves to the case of one-dimensional space variable x , though everything works in the general situation as well. We are interested in *symmetries* and *conservation laws* of system (2) and in various operators that relate these objects to each other (recursion operators, Hamiltonian and symplectic structures).

1.1. Symmetries and conservation laws. A *symmetry* of equation (2) is a vector field (an *evolutionary* field)

$$\mathfrak{D}_\varphi = \sum_{i \geq 0} \sum_{j=1}^m D_x^i(\varphi^j) \frac{\partial}{\partial u_i^j}, \quad (3)$$

where $\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector function depending on x, t, u, u_1, \dots, u_s and satisfying the equations

$$D_t(\varphi^j) = \sum_{i,l} \frac{\partial F^j}{\partial u_i^l} D_x^i(\varphi^l), \quad j = 1, \dots, m. \quad (4)$$

Here and below

$$D_x = \frac{\partial}{\partial x} + \sum_{i,l} u_{i+1}^l \frac{\partial}{\partial u_i^l}, \quad D_t = \frac{\partial}{\partial t} + \sum_{i,l} D_x^i(F^l) \frac{\partial}{\partial u_i^l}$$

are *total derivatives* with respect to x and t . We identify fields \mathfrak{D}_φ with functions φ (the *generating functions*). Thus, a symmetry is a function that satisfies the equation

$$\ell_{\mathcal{E}}(\varphi) = 0, \quad (5)$$

where

$$\ell_{\mathcal{E}} = D_t - \ell_F \quad (6)$$

and

$$\ell_F = \begin{pmatrix} \sum_i \partial F^1 / \partial u_i^1 D_x^i & \dots & \sum_i \partial F^1 / \partial u_i^m D_x^i \\ \dots & \dots & \dots \\ \sum_i \partial F^m / \partial u_i^1 D_x^i & \dots & \sum_i \partial F^m / \partial u_i^m D_x^i \end{pmatrix}. \quad (7)$$

Operator (6) is called the *linearization operator* for the equation \mathcal{E} . The set of symmetries is a Lie algebra denoted by $\text{sym}(\mathcal{E})$.

A *conservation law* for equation (2) is a horizontal 1-form

$$\omega = X dx + T dt$$

closed with respect to the horizontal de Rham differential

$$d_h = dx \wedge D_x + dt \wedge D_t,$$

i.e., such that

$$D_x(T) = D_t(X),$$

where $T = T(x, t, u, u_1, \dots)$, $X = X(x, t, u, u_1, \dots)$. A conservation law is *trivial* if it is of the form $\omega = d_h f$, i.e.,

$$X = D_x(f), \quad T = D_t(f), \quad f = f(x, t, u, u_1, \dots).$$

The space of equivalence classes of conservation laws modulo trivial ones coincides with the 1st horizontal de Rham cohomology group for \mathcal{E} and is denoted by $H_h^1(\mathcal{E})$.

Remark 1. If the number of the space variables x equals n , then this space coincides with $H_h^n(\mathcal{E})$.

To any conservation law $\omega = X dx + T dt$ there corresponds its *generating function*

$$\psi_\omega = \delta X = \left(\frac{\delta X}{\delta u^1}, \dots, \frac{\delta X}{\delta u^m} \right), \quad (8)$$

where δ denotes the *Euler operator* and

$$\frac{\delta}{\delta u^j} = \sum_{i \geq 0} (-1)^i D_x^i \circ \frac{\partial}{\partial u_i^j} \quad (9)$$

is the *variational derivative* with respect to u^j . Any generating function (8) satisfies the equation

$$\ell_{\mathcal{E}}^*(\psi_\omega) = 0, \quad (10)$$

where

$$\begin{aligned} \ell_{\mathcal{E}}^* &= -D_t + \ell_F^* \\ &= -D_t + \begin{pmatrix} \sum_i (-1)^i D_x^i \circ \partial F^1 / \partial u_i^1 & \dots & \sum_i (-1)^i D_x^i \circ \partial F^m / \partial u_i^1 \\ \dots & \dots & \dots \\ \sum_i (-1)^i D_x^i \circ \partial F^1 / \partial u_i^m & \dots & \sum_i (-1)^i D_x^i \circ \partial F^m / \partial u_i^m \end{pmatrix} \end{aligned} \quad (11)$$

is the operator adjoint to $\ell_{\mathcal{E}}$.

Solutions of equation (10) are called *cosymmetries* and in general not all of them are generating functions of conservation laws. The space of cosymmetries will be denoted by $\text{sym}^*(\mathcal{E})$. The Euler operator determines the embedding

$$\delta: H_h^1(\mathcal{E}) \rightarrow \text{sym}^*(\mathcal{E}). \quad (12)$$

Symmetries and cosymmetries may be understood as vector fields and differential 1-forms, respectively, on the equation \mathcal{E} and there is a natural pairing between them: if $\varphi = (\varphi^1, \dots, \varphi^m) \in \text{sym}(\mathcal{E})$ and $\psi = (\psi^1, \dots, \psi^m) \in \text{sym}^*(\mathcal{E})$, we set

$$\langle \psi, \varphi \rangle = \psi^1 \varphi^1 + \dots + \psi^m \varphi^m. \quad (13)$$

At first glance, the right-hand side of (13) looks like a function, but the ‘‘physical meaning’’ of $\langle \psi, \varphi \rangle$ is quite different. Namely, applying D_t to $\langle \psi, \varphi \rangle$ we have

$$D_t \langle \psi, \varphi \rangle = \langle D_t(\psi), \varphi \rangle + \langle \psi, D_t(\varphi) \rangle = -\langle \ell_F^*(\psi), \varphi \rangle + \langle \psi, \ell_F(\varphi) \rangle$$

and, consequently,

$$D_t \langle \psi, \varphi \rangle = D_x(T_{\psi, \varphi})$$

for some $T_{\psi, \varphi}$. Though the conservation law

$$\langle \psi, \varphi \rangle dx + T_{\psi, \varphi} dt \quad (14)$$

is not defined uniquely, its cohomology class depends on φ and ψ only and we obtain

$$\langle \cdot, \cdot \rangle: \text{sym}^*(\mathcal{E}) \times \text{sym}(\mathcal{E}) \rightarrow H_h^1(\mathcal{E}).$$

Note that for any $\omega \in H_h^1(\mathcal{E})$ one has

$$\langle \delta\omega, \varphi \rangle = \mathfrak{D}_{\varphi}(\omega).$$

1.2. Recursion operators, Hamiltonian and symplectic structures. In this subsection we shall discuss a local theory and shall explain how nonlocal components are incorporated in all construction in Subsection 1.4. All operators considered below are matrix operators in total derivatives of the form

$$\Delta = \left(\sum_{i,s} a_{jl}^{is} D_x^i D_t^s \right). \quad (15)$$

We call such operators *C-differential operators*. In particular, we shall deal with operators

$$\Delta = \left(\sum_i a_{jl}^i D_x^i \right) \quad (16)$$

in D_x only.

In the case of equation (2) we may consider the operator $\ell_{\mathcal{E}}$ to act from the space \varkappa of vector functions $\varphi = (\varphi^1, \dots, \varphi^m)$ to the same space. Then $\ell_{\mathcal{E}}^*$ acts from \varkappa^* to \varkappa^* , where \varkappa^* is the dual space.

Remark 2. A coordinate-free description of actions of $\ell_{\mathcal{E}}$ and $\ell_{\mathcal{E}}^*$ may be found in [1].

From Subsection 1.1 we have

$$\text{sym}(\mathcal{E}) = \ker \ell_{\mathcal{E}} \subset \mathfrak{X}, \quad \text{sym}^*(\mathcal{E}) = \ker \ell_{\mathcal{E}}^* \subset \mathfrak{X}^*$$

and we are looking for \mathcal{C} -differential operators of the form (16) acting as follows

$$\mathcal{R}: \mathfrak{X} \rightarrow \mathfrak{X}, \tag{17}$$

$$\mathcal{H}: \mathfrak{X}^* \rightarrow \mathfrak{X}, \tag{18}$$

$$\mathcal{S}: \mathfrak{X} \rightarrow \mathfrak{X}^*, \tag{19}$$

$$\mathcal{R}^*: \mathfrak{X}^* \rightarrow \mathfrak{X}^*, \tag{20}$$

and such that

$$\mathcal{R}(\text{sym}(\mathcal{E})) \subset \text{sym}(\mathcal{E}), \tag{21}$$

$$\mathcal{H}(\text{sym}^*(\mathcal{E})) \subset \text{sym}(\mathcal{E}), \tag{22}$$

$$\mathcal{S}(\text{sym}(\mathcal{E})) \subset \text{sym}^*(\mathcal{E}), \tag{23}$$

$$\mathcal{R}^*(\text{sym}^*(\mathcal{E})) \subset \text{sym}^*(\mathcal{E}). \tag{24}$$

To find such operators makes the first step in constructing the structures we are interested in. Namely,

- operators \mathcal{R} are recursion operators for symmetries;
- operators \mathcal{H} satisfying conditions described in Remark 3 below are Hamiltonian structures on the equation \mathcal{E} ;
- operators \mathcal{S} satisfying conditions described in Remark 4 below are symplectic structures on the equation \mathcal{E} ;
- operators \mathcal{R}^* .

Remark 3 (Hamiltonianity conditions). Let \mathcal{H} be an operator of type (18). Then for any two conservation laws ω_1 and ω_2 of equation (2) (or their equivalence classes in $H_h^1(\mathcal{E})$) one can define the bracket

$$\{\omega_1, \omega_2\}_{\mathcal{H}} = \langle \delta\omega_1, \mathcal{H}(\delta\omega_2) \rangle = \mathfrak{D}_{\mathcal{H}(\delta\omega_2)}(\delta\omega_1). \tag{25}$$

\mathcal{H} is a *Hamiltonian structure* if this bracket is skew-symmetric and satisfies the Jacobi identity. The first condition means that \mathcal{H} is a skew-adjoint operator ($\mathcal{H} = -\mathcal{H}^*$) and thus may be understood as a *bivector* on \mathcal{E} , while the second one amounts to

$$\llbracket \mathcal{H}, \mathcal{H} \rrbracket = 0, \tag{26}$$

where $\llbracket \cdot, \cdot \rrbracket$ is the *variational Schouten bracket*, [5]. Recall also that two Hamiltonian structures are *compatible* (or constitute a *Hamiltonian pair*) if and only if

$$\llbracket \mathcal{H}_1, \mathcal{H}_2 \rrbracket = 0. \tag{27}$$

Efficient coordinate formulas to check conditions (26) and (27) will be given in Subsection 1.5.

Remark 4 (symplectic conditions). Let \mathcal{S} be an operator of type (19). Then it can be understood as a map

$$\mathcal{S}: \text{sym}(\mathcal{E}) \times \text{sym}(\mathcal{E}) \rightarrow H_h^1(\mathcal{E}) \quad (28)$$

by setting

$$\mathcal{S}(\varphi_1, \varphi_2) = \langle \mathcal{S}(\varphi_1), \varphi_2 \rangle. \quad (29)$$

We say that \mathcal{S} is a *two-form* on \mathcal{E} if

$$\mathcal{S}(\varphi_1, \varphi_2) = -\mathcal{S}(\varphi_2, \varphi_1) \quad (30)$$

for all $\varphi_1, \varphi_2 \in \text{sym}(\mathcal{E})$ (this is equivalent to $\mathcal{S} = -\mathcal{S}^*$). For such a form one can define its differential

$$\delta\mathcal{S}: \text{sym}(\mathcal{E}) \times \text{sym}(\mathcal{E}) \times \text{sym}(\mathcal{E}) \rightarrow H_h^1(\mathcal{E})$$

by

$$\begin{aligned} \delta\mathcal{S}(\varphi_1, \varphi_2, \varphi_3) &= \mathfrak{D}_{\varphi_1}\mathcal{S}(\varphi_2, \varphi_3) - \mathfrak{D}_{\varphi_2}\mathcal{S}(\varphi_1, \varphi_3) - \mathfrak{D}_{\varphi_3}\mathcal{S}(\varphi_1, \varphi_2) \\ &\quad - \mathcal{S}(\{\varphi_1, \varphi_2\}, \varphi_3) + \mathcal{S}(\{\varphi_1, \varphi_3\}, \varphi_2) - \mathcal{S}(\{\varphi_2, \varphi_3\}, \varphi_1), \end{aligned} \quad (31)$$

where the bracket $\{\varphi, \varphi'\}$ is uniquely defined by the formula

$$\mathfrak{D}_{\{\varphi, \varphi'\}} = [\mathfrak{D}_\varphi, \mathfrak{D}_{\varphi'}].$$

We say that \mathcal{S} is a *symplectic structure* if the corresponding two-form is closed with respect to δ , i.e.,

$$\delta\mathcal{S} = 0. \quad (32)$$

Again, computational formulas to check conditions (30) and (32) will be given in Subsection 1.5.

Remark 5 (Nijenhuis conditions). Any operator (17) satisfying (21) can be considered as a recursion on symmetries, i.e., it takes a symmetry of the equation \mathcal{E} to a symmetry. Similarly, any operator (24) is a recursion operator for cosymmetries. Thus, applying \mathcal{R} to a given symmetry φ one obtains the whole family

$$\varphi_0 = \varphi, \quad \varphi_1 = \mathcal{R}\varphi, \dots, \varphi_i = \mathcal{R}^i\varphi, \dots \quad (33)$$

Commutativity of such families closely relates to integrability of \mathcal{E} .

For any recursion operator $\mathcal{R}: \text{sym}(\mathcal{E}) \rightarrow \text{sym}(\mathcal{E})$ its *Nijenhuis torsion*

$$N_{\mathcal{R}}: \text{sym}(\mathcal{E}) \times \text{sym}(\mathcal{E}) \rightarrow \text{sym}(\mathcal{E}) \quad (34)$$

is defined by

$$N_{\mathcal{R}}(\varphi_1, \varphi_2) = \{\mathcal{R}\varphi_1, \mathcal{R}\varphi_1\} - (\mathcal{R}(\{\mathcal{R}\varphi_1, \varphi_2\}) - \{\varphi_1, \mathcal{R}\varphi_2\} + \mathcal{R}(\{\varphi_1, \varphi_2\})),$$

while for a symmetry φ its action on \mathcal{R} is defined by

$$(\mathfrak{D}_\varphi\mathcal{R})(\varphi') = \{\varphi, \mathcal{R}\varphi'\} - \mathcal{R}\{\varphi, \varphi'\}.$$

Both operations can be expressed in terms of the *Nijenhuis bracket* and as it was shown in [9] the conditions

$$N_{\mathcal{R}} = 0, \quad \mathfrak{D}_\varphi\mathcal{R} = 0 \quad (35)$$

imply commutativity of family (33).

1.3. **Δ -coverings.** To construct operators possessing properties (21)–(24), we solve the following operator equations

$$\ell_{\mathcal{E}} \circ \mathcal{R} = A_{\mathcal{R}} \circ \ell_{\mathcal{E}} \quad (36)$$

$$\ell_{\mathcal{E}} \circ \mathcal{H} = A_{\mathcal{H}} \circ \ell_{\mathcal{E}}^* \quad (37)$$

$$\ell_{\mathcal{E}}^* \circ \mathcal{S} = A_{\mathcal{S}} \circ \ell_{\mathcal{E}} \quad (38)$$

$$\ell_{\mathcal{E}}^* \circ \mathcal{R}^* = A_{\mathcal{R}^*} \circ \ell_{\mathcal{E}}^* \quad (39)$$

with respect to \mathcal{R} , \mathcal{H} , \mathcal{S} , and \mathcal{R}^* for some \mathcal{C} -differential operators $A_{\mathcal{R}}$, $A_{\mathcal{H}}$, $A_{\mathcal{S}}$, and $A_{\mathcal{R}^*}$.

Remark 6. It is easily shown that

$$A_{\mathcal{R}} = \mathcal{R}, \quad A_{\mathcal{H}} = -\mathcal{H}, \quad A_{\mathcal{S}} = -\mathcal{S}, \quad A_{\mathcal{R}^*} = \mathcal{R}^*$$

and for any solution \mathcal{R} of equation (36) its adjoint \mathcal{R}^* is a solution of (39) and vice versa, while for any solution \mathcal{H} of equation (37) its inverse \mathcal{H}^{-1} (if it makes sense) is a solution of (38) and vice versa. In addition, for any solution \mathcal{H} its adjoint \mathcal{H}^* is a solution as well and the same for \mathcal{S} .

All four problems (36)–(39) can be formulated in the following general form. Let P , Q , P' , Q' be some spaces of vector functions on the equation \mathcal{E} and $\Delta: P \rightarrow P'$, $\nabla: Q \rightarrow Q'$ be \mathcal{C} -differential operators. How to find all \mathcal{C} -differential operators $\mathcal{X}: P \rightarrow Q$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\mathcal{X}} & Q \\ \Delta \downarrow & & \downarrow \nabla \\ P' & \xrightarrow{A_{\mathcal{X}}} & Q' \end{array}$$

is commutative for some \mathcal{C} -differential operator $A_{\mathcal{X}}$? For simplicity, we shall assume that \mathcal{X} is an operator in D_x only, i.e., of the form (16).

Remark 7. This problem has an obvious set of trivial solutions. Namely, any \mathcal{X} of the form $\mathcal{X} = \mathcal{X}' \circ \Delta$. We shall look for equivalence classes of solutions modulo trivial ones.

To this end, we use the following construction. Assume that the operator Δ acts on functions $v = (v^1, \dots, v^r)$ and consider the system of equations

$$\begin{cases} u_t = F(x, t, u, u_1, \dots, u_k), \\ \Delta(v) = 0 \end{cases} \quad (40)$$

(recall that the coefficient of Δ may depend on the function u and its derivatives). We say that system (40) is the Δ -covering (or the Δ -extension) of the equation \mathcal{E} .

Example 1 (the $\ell_{\mathcal{E}}$ -covering). Let us take the operator $\ell_{\mathcal{E}} = D_t - \ell_F$ for Δ . Then the corresponding extension is called the $\ell_{\mathcal{E}}$ -covering of \mathcal{E} . For example, if \mathcal{E} is the Burgers equation

$$u_t = u_{xx} + uu_x,$$

then $\ell_{\mathcal{E}} = D_t - D_x^2 - uD_x - u_x$ and the $\ell_{\mathcal{E}}$ covering is of the form

$$\begin{cases} u_t = u_{xx} + uu_x, \\ v_t = v_{xx} + uv_x + vu_x. \end{cases}$$

Example 2 (the $\ell_{\mathcal{E}}^*$ -covering). If we take the operator $\ell_{\mathcal{E}}^* = -D_t - \ell_F^*$ for Δ , then this extension is called the $\ell_{\mathcal{E}}^*$ -covering. For the Burgers equation, $\ell_{\mathcal{E}}^* = -D_t - D_x^2 + D_x \circ u - u_x = -D_t - D_x^2 + uD_x$ and thus

$$\begin{cases} u_t = u_{xx} + uu_x, \\ v_t = -v_{xx} + uv_x \end{cases}$$

is the $\ell_{\mathcal{E}}^*$ -covering.

Denote by \tilde{D}_x and \tilde{D}_t the total derivatives in the space of the Δ -covering. Then, since ∇ is a \mathcal{C} -differential operator, we can consider the operator $\tilde{\nabla}$ which is obtained from ∇ by changing D_x and D_t to \tilde{D}_x and \tilde{D}_t , respectively. Consider now the linear differential equation

$$\tilde{\nabla}(a) = 0, \tag{41}$$

where $a = (a^1, \dots, a^s)$ is a vector function linear in the variables $v, v_1, \dots, v_i, \dots$, i.e.,

$$a^j = \sum_{i,l} a_{il}^j v_i^l, \tag{42}$$

a_{il}^j being functions on \mathcal{E} .

Theorem 1. *Equivalence classes of solutions of the equation*

$$\nabla \circ \mathcal{X} = A_{\mathcal{X}} \circ \Delta$$

modulo trivial solutions are in one-to-correspondence with the solutions of equation (41) of the form (42). The operator \mathcal{X} corresponding to solution (42) is of the form

$$\mathcal{X} = \left(\sum_i a_{il}^j D_x^i \right). \tag{43}$$

1.4. Nonlocal theory. Δ -coverings are a particular case of the general geometric construction that allows to introduce nonlocal objects related to the initial differential equation (see [11]). Here we shall present all needed constructions in the coordinate form.

Let \mathcal{E} be an equation of the form (2) and $w = (w^1, \dots, w^r)$ be a new unknown vector function in x and t (the case of infinite number of w 's is included). A system $\tilde{\mathcal{E}}$

$$\begin{cases} w_x^j = X^j(x, t, w, u, u_1, \dots, u_p), \\ w_t^j = T^j(x, t, w, u, u_1, \dots, u_p), \end{cases} \quad (44)$$

$j = 1, \dots, r$, is called a *covering* over \mathcal{E} if its compatibility is provided by \mathcal{E} . The number r of new variables is called the *dimension* of the covering while w 's are said to be *nonlocal variables* in this covering.

Example 3. The system

$$w_x = u, \quad w_t = u_{xx} + \frac{1}{2}u^2$$

is a covering over the KdV equation.

Total derivatives on $\tilde{\mathcal{E}}$ are denote \tilde{D}_x and \tilde{D}_t and they are of the form

$$\tilde{D}_x = D_x + \sum_{j=1}^m X^j \frac{\partial}{\partial w^j}, \quad \tilde{D}_t = D_t + \sum_{j=1}^m T^j \frac{\partial}{\partial w^j}.$$

Thus, if Δ is a \mathcal{C} -differential operator on \mathcal{E} , we can define the operator $\tilde{\Delta}$ on $\tilde{\mathcal{E}}$ substituting D_x by \tilde{D}_x and D_t by \tilde{D}_t .

Compatibility conditions for (44) read

$$\tilde{D}_t X^j = \tilde{D}_x T^j \text{ mod } \tilde{\mathcal{E}}.$$

If X^j and T^j do not depend on w , then these conditions mean that the forms

$$\omega^1 = X^1 dx + T^1 dt, \dots, \omega^j = X^j dx + T^j dt, \dots$$

are conservation laws for the equation \mathcal{E} . Conversely, to any system of conservation laws we can put into correspondence a covering.

The system $\tilde{\mathcal{E}}$ is a differential equation itself and thus we can consider its symmetries, cosymmetries, conservation laws, etc. These objects will be referred to as *nonlocal* with respect to the initial equation \mathcal{E} . In particular, nonlocal symmetries are determined by the equation

$$\ell_{\tilde{\mathcal{E}}}(\tilde{\varphi}) = 0,$$

while nonlocal cosymmetries are defined by

$$\ell_{\tilde{\mathcal{E}}}^*(\tilde{\psi}) = 0.$$

On the other hand, since $\ell_{\mathcal{E}}$ is a \mathcal{C} -differential operator, the operator $\tilde{\ell}_{\mathcal{E}}$ is well defined. In general, $\ell_{\tilde{\mathcal{E}}} \neq \tilde{\ell}_{\mathcal{E}}$ and the equations

$$\tilde{\ell}_{\mathcal{E}}(\tilde{\varphi}) = 0$$

and

$$\tilde{\ell}_{\mathcal{E}}^*(\tilde{\psi}) = 0$$

can be considered. Their solutions are called *shadows* of symmetries and cosymmetries, respectively.

Remark 8. Δ -coverings are (usually, infinite-dimensional) coverings in the sense of this subsection. Consider, for example, the $\ell_{\mathcal{E}}$ -covering which can be presented in the form

$$u_t = F(x, t, u, u_1, \dots, u_k), \quad v_t = \ell_F(v)$$

and set

$$\omega^0 = v, \quad \omega^1 = v_x, \quad \omega^2 = v_{xx}, \dots$$

Then the $\ell_{\mathcal{E}}$ -covering is equivalent to the system

$$\begin{cases} \omega_x^j = \omega^{j+1}, \\ \omega_t^j = \tilde{D}_x^j(\ell_F(\omega^0)), \end{cases}$$

$j = 0, 1, 2, \dots$. In a similar way, the $\ell_{\mathcal{E}}^*$ -covering is equivalent to

$$\begin{cases} \rho_x^j = \rho^{j+1}, \\ \rho_t^j = -\tilde{D}_x^j(\ell_F^*(\rho^0)). \end{cases}$$

From now on, the notation ω and ρ is used for nonlocal variables in the $\ell_{\mathcal{E}}$ - and $\ell_{\mathcal{E}}^*$ -coverings, respectively.

Comparing the results of this and the previous subsections, we can formulate the statement fundamental for all subsequent computations.

Theorem 2. *Let \mathcal{E} be an equation of the form (2). Then equivalence classes of solutions $\mathcal{R}, \mathcal{H}, \mathcal{S}, \mathcal{R}^*$ of equations (36), (37), (38), (39) are in one-to-one correspondence with*

- shadows of symmetries in the $\ell_{\mathcal{E}}$ -covering for (36),
- shadows of symmetries in the $\ell_{\mathcal{E}}^*$ -covering for (37),
- shadows of cosymmetries in the $\ell_{\mathcal{E}}$ -covering for (38),
- shadows of cosymmetries in the $\ell_{\mathcal{E}}^*$ -covering for (39).

All shadows are taken to be linear with respect to nonlocal variables in the corresponding covering.

Since $\tilde{\mathcal{E}}$ is an equation, we can consider its coverings. In this way, coverings over coverings, etc., arise. Our next remark is concerned with special coverings over $\ell_{\mathcal{E}}$ - and $\ell_{\mathcal{E}}^*$ -coverings and with interpretation of the corresponding nonlocal variables.

Remark 9 (nonlocal vectors and covectors). Consider an equation \mathcal{E} and its $\ell_{\mathcal{E}}^*$ -covering. Let φ be a symmetry of \mathcal{E} . Then, since φ satisfies the equation $D_t\varphi = \ell_F\varphi$ and ρ is defined by $D_t\rho = -\ell_F^*\rho$, we have

$$D_t(\rho\varphi) = D_x(T_\varphi)$$

for some function T_φ on the $\ell_{\mathcal{E}}^*$ -covering (see the end of Subsection 1.1). Consequently,

$$\rho\varphi dx + T_\varphi dt$$

is a conservation law on the $\ell_{\mathcal{E}}^*$ -covering. Denote by ρ^φ the nonlocal variable in the corresponding covering, i.e.,

$$\rho_x^\varphi = \rho\varphi, \quad \rho_t^\varphi = T_\varphi. \quad (45)$$

We call ρ^φ the *nonlocal vector* corresponding to φ .

In a similar way, any cosymmetry ψ of \mathcal{E} determines the conservation law

$$\psi\omega \, dx + T_\psi \, dt$$

on the $\ell_{\mathcal{E}}$ -covering. The nonlocal variable ω^ψ defined by

$$\omega_x^\psi = \psi\omega, \quad \omega_t^\psi = T_\psi \quad (46)$$

is called the *nonlocal covector* (or *nonlocal form*) corresponding to ψ .

Take a set of symmetries $\varphi^1, \dots, \varphi^r$ and the covering corresponding to this set by (45). Then we can consider shadows of symmetries and cosymmetries in this covering linear with respect to the variables ρ and ρ^φ , i.e., shadows of the form $a = (a^1, \dots, a^m)$, where

$$a^j = \sum_{l=1}^m \sum_{i \geq 0} a_i^{jl} \rho_i^l + \sum_{s=1}^r a_s^j \rho^{\varphi^s}. \quad (47)$$

Proposition 1. *The operators corresponding to functions (47) are of the form*

$$\Delta = \left(\sum_{i \geq 0} a_i^{jl} D_x^i + \sum_{s=1}^r a_s^j D_x^{-1} \circ \varphi_s^l \right), \quad j, l = 1, \dots, m, \quad (48)$$

and they enjoy properties (37) and (39) for shadows of symmetries and cosymmetries, respectively. In addition, if (47) is a shadow of symmetry, then the vector functions

$$a_s = (a_s^1, \dots, a_s^m), \quad s = 1, \dots, r,$$

are symmetries of equation \mathcal{E} , and if (47) is a shadow of cosymmetry, then these functions are cosymmetries.

In the same way, one can consider a set ψ_1, \dots, ψ_r of cosymmetries and the covering over the $\ell_{\mathcal{E}}$ -covering given by (46). Let us take a shadow $b = (b^1, \dots, b^m)$,

$$b^j = \sum_{l=1}^m \sum_{i \geq 0} b_i^{jl} \omega_i^l + \sum_{s=1}^r b_s^j \omega^{\psi_s} \quad (49)$$

in this covering.

Proposition 2. *The operators corresponding to functions (49) are of the form*

$$\Delta = \left(\sum_{i \geq 0} b_i^{jl} D_x^i + \sum_{s=1}^r b_s^j D_x^{-1} \circ \psi_s^l \right), \quad j, l = 1, \dots, m, \quad (50)$$

and they enjoy properties (36) and (38) for shadows of symmetries and cosymmetries, respectively. In addition, if (49) is a shadow of symmetry, then the vector functions

$$b_s = (b_s^1, \dots, b_s^m), \quad s = 1, \dots, r,$$

are symmetries of \mathcal{E} , and if (49) is a shadow of cosymmetry, then these functions are cosymmetries.

Note that all operators (48) and (50) are obtained in the *weakly nonlocal form* (cf. [2]).

Remark 10. In the sequel, we shall use a short notation

$$\langle a_1, \dots, a_r \mid D_x^{-1} \mid \varphi_1, \dots, \varphi_r \rangle = \sum_{s=1}^r \alpha_s^j D_x^{-1} \circ \varphi_s^l$$

and

$$\langle b_1, \dots, b_r \mid D_x^{-1} \mid \psi_1, \dots, \psi_r \rangle = \sum_{s=1}^r b_s^j D_x^{-1} \circ \psi_s^l$$

for the nonlocal part of operators (48) and (50), resp.

1.5. General outline of the computational scheme. The computations described in Sections 2 and 3 are divided into two parts. The first one plays an auxiliary role and consists of the following steps:

- Construction of a number of conservation laws and the corresponding nonlocal variables for the initial equation.
- Construction of local and nonlocal symmetries that will serve as seed symmetries for infinite families and to be used to construct necessary nonlocal vectors.
- Construction of local and nonlocal cosymmetries that will serve as seed cosymmetries for infinite families and to be used to construct necessary nonlocal covectors.
- Construction of nonlocal vectors and covectors.

In the second part we reveal the main structures associated to the dB-equation. Namely,

- We solve the equation

$$\tilde{\ell}_{\mathcal{E}}(b) = 0$$

in the $\ell_{\mathcal{E}}$ -covering to construct *recursion operators for symmetries* of the form (50).

- We solve the equation

$$\tilde{\ell}_{\mathcal{E}}^*(a) = 0$$

in the $\ell_{\mathcal{E}}^*$ -covering to construct *recursion operators for cosymmetries* of the form (48).

- We solve the equation

$$\tilde{\ell}_{\mathcal{E}}(a) = 0 \tag{51}$$

in the $\ell_{\mathcal{E}}^*$ -covering to find operators of the form (48) that take cosymmetries to symmetries. *Hamiltonian structures* on \mathcal{E} are distinguished by Theorem 3 below.

- We solve the equation

$$\tilde{\ell}_{\mathcal{E}}^*(b) = 0 \quad (52)$$

in the $\ell_{\mathcal{E}}$ -covering to find operators of the form (50) that take symmetries to cosymmetries. Symplectic structures on \mathcal{E} are distinguished by Theorem 4.

We shall now present two criteria that allow to distinguish Hamiltonian and symplectic structures between local solutions of (51) and (52).

Theorem 3. *Let $a = (a^1, \dots, a^m)$ be a solution of (51), where $a^j = \sum_{l,i} a_i^{jl} \rho_i^l$. Consider the function $A = \sum_j a^j \rho^j$. Then the corresponding \mathcal{C} -differential operator is a Hamiltonian structure on the equation \mathcal{E} if and only if*

$$\sum_j \frac{\delta A}{\delta \rho^j} \rho^j = -2A \quad (53)$$

and

$$\delta \sum_j \frac{\delta A}{\delta u^j} \cdot \frac{\delta A}{\delta \rho^j} = 0. \quad (54)$$

In addition, two Hamiltonian structures with the functions A and A' are compatible if and only if

$$\delta \sum_j \left(\frac{\delta A}{\delta u^j} \cdot \frac{\delta A'}{\delta \rho^j} + \frac{\delta A'}{\delta u^j} \cdot \frac{\delta A}{\delta \rho^j} \right) = 0. \quad (55)$$

Theorem 4. *Let $b = (b^1, \dots, b^m)$ be a solution of (52), where $b^j = \sum_{l,i} b_i^{jl} \omega_i^l$. Consider the function $B = \sum_j b^j \omega^j$. Then the corresponding \mathcal{C} -differential operator is a symplectic structure on the equation \mathcal{E} if and only if*

$$\sum_j \frac{\delta B}{\delta \omega^j} \omega^j = -2B \quad (56)$$

and

$$\delta \sum_j \frac{\delta B}{\delta u^j} \omega^j = 0. \quad (57)$$

Remark 11. To use these two results correctly, note that

- (1) The variables ρ_i^j and ω_i^j should be considered as *odd* ones.
- (2) The operator δ is the *total* Euler operator in the corresponding covering equation, i.e.,

$$\delta = \left(\frac{\delta}{\delta u^1}, \dots, \frac{\delta}{\delta u^m}, \frac{\delta}{\delta \rho^1}, \dots, \frac{\delta}{\delta \rho^m} \right)$$

for the $\ell_{\mathcal{E}}^*$ -covering and

$$\delta = \left(\frac{\delta}{\delta u^1}, \dots, \frac{\delta}{\delta u^m}, \frac{\delta}{\delta \omega^1}, \dots, \frac{\delta}{\delta \omega^m} \right)$$

for the $\ell_{\mathcal{E}}$ -covering.

Remark 12. Conditions (53) and (56) guarantee that the corresponding operators are skew-adjoint. Conditions (54), (55), and (57) are equivalent to (26), (27), and (32), respectively. An amazing property of evolution equations in “general position” was established in [3] and [8] and is the following one.

Theorem 5. *Let \mathcal{E} be an evolution equation of the form (2) and assume that it is not a first-order scalar equation. Then if the symbol of the right-hand side is nondegenerate, one has*

- *any skew-adjoint \mathcal{C} -differential operator satisfying (37) satisfies (54) as well (any bivector is Poissonian),*
- *any two skew-adjoint \mathcal{C} -differential operators satisfying (37) satisfy (55) as well (any two Hamiltonian structures are compatible),*
- *any skew-adjoint \mathcal{C} -differential operator satisfying (38) satisfies (57) as well (any two-form is closed).*

2. PREPARATORY COMPUTATIONS

We now come back to the dB-equation

$$w_t = u_x, \quad u_t = ww_x + v_x, \quad v_t = -uw_x - 3wu_x$$

and first of all note that it is homogeneous with respect to the following gradings (weights)

$$|x| = -1, \quad |t| = -2, \quad |w| = 2, \quad |u| = 3, \quad |v| = 4$$

and corresponding weights of all variables and their polynomial functions. All constructions used in subsequent computations become homogeneous and we can restrict ourselves to homogeneous components.

2.1. Nonlocal variables. We looked for the coverings associated with conservation laws of the dB-equation and found the variables $p_1, p_2, p_3, p_5, \dots$, defined by

$$\begin{aligned} (p_1)_x &= w, & (p_1)_t &= u, \\ (p_2)_x &= u, & (p_2)_t &= \frac{1}{2}(2v + w^2), \\ (p_3)_x &= v + w^2, & (p_3)_t &= -uw, \\ (p_5)_x &= (u^2 + 2vw + 2w^3)/2, & (p_5)_t &= uv. \end{aligned}$$

The subscript refers to the grading of the corresponding variable. Thus,

$$|p_1| = 1, \quad |p_2| = 2, \quad |p_3| = 3, \quad |p_5| = 5.$$

Further computations show that exactly one new conservation law arises at each grading level except for gradings 4, 8, 12, \dots . We use the corresponding nonlocal variables to construct nonlocal symmetries and cosymmetries in Subsections 2.2 and 2.3.

Remark 13. Void positions with gradings divisible by 4 are occupied by nonlocal variables of the next level of nonlocality.

$$\begin{aligned} (q_0)_x &= p_1, & (q_0)_t &= p_2, \\ (q_4)_x &= p_3w - 3p_5, & (q_4)_t &= p_3u - 2p_6, \end{aligned}$$

etc. Moreover, we found two independent nonlocal conservation laws for each of gradings 8 and 12.

2.2. Symmetries. The defining equations for symmetries (or their shadows in the nonlocal case) of the dB-equation are

$$\begin{aligned} \tilde{D}_t(\varphi^w) &= \tilde{D}_x(\varphi^u), \\ \tilde{D}_t(\varphi^u) &= w\tilde{D}_x(\varphi^w) + w_1\varphi^w + \tilde{D}_x(\varphi^v), \\ \tilde{D}_t(\varphi^v) &= -u\tilde{D}_x(\varphi^w) - 3u_1\varphi^w - 3w\tilde{D}_x(\varphi^u) - w_1\varphi^u. \end{aligned} \tag{58}$$

The grading of a symmetry is understood as the grading of the corresponding vector field and hence

$$|\varphi| = |\varphi^w| - |w| = |\varphi^u| - |u| = |\varphi^v| - |v|.$$

2.2.1. (x, t) -independent local symmetries. Solving equations (58) we found a number of local symmetries independent of x and t . Here are the results for grading ≤ 7 :

$$\begin{aligned} \varphi_{-4}^w &= 0, & \varphi_1^w &= w_1, & \varphi_2^w &= u_1, \\ \varphi_{-4}^u &= 0, & \varphi_1^u &= u_1, & \varphi_2^u &= w_1w + v_1, \\ \varphi_{-4}^v &= 1, & \varphi_1^v &= v_1, & \varphi_2^v &= -w_1u - 3u_1w, \end{aligned}$$

and

$$\begin{aligned} \varphi_3^w &= 2w_1w + v_1, \\ \varphi_3^u &= -w_1u - u_1w, \\ \varphi_3^v &= -3w_1w^2 - u_1u - 2v_1w, \\ \varphi_5^w &= w_1(v + 3w^2) + u_1u + v_1w, \\ \varphi_5^u &= u_1v + v_1u, \\ \varphi_5^v &= w_1(-u^2 - 3w^3) - 4u_1uw + v_1(v - w^2), \\ \varphi_6^w &= 2w_1uw + u_1(2v + w^2) + 2v_1u, \\ \varphi_6^u &= w_1(-2u^2 + 2vw + w^3) - 4u_1uw + v_1(2v + w^2), \\ \varphi_6^v &= w_1u(-2v - 7w^2) + u_1(-2u^2 - 6vw - 3w^3) - 6v_1uw, \\ \varphi_7^w &= w_1(-3u^2 + 12vw + 14w^3) - 6u_1uw + 6v_1(v + w^2), \end{aligned}$$

$$\begin{aligned}\varphi_7^u &= 6w_1u(-v - 2w^2) + u_1(-3u^2 - 6vw - 4w^3) - 6v_1uw, \\ \varphi_7^v &= 6w_1w(u^2 - 3vw - 3w^3) + 6u_1u(-v + 2w^2) + v_1(-3u^2 - 12vw - 10w^3).\end{aligned}$$

The subscript in the notation above equals the grading of the corresponding symmetry. We computed this kind of symmetries up to grading 15 and found that they exist at all levels except for 4, 8, 12.

Remark 14. Note that only the first three symmetries are classical ones. All other symmetries are higher despite the fact that their jet order equals 1.

2.2.2. Nonlocal symmetries. Nonlocal symmetries (or, to be more precise, their shadows) arise at gradings 4, 8, 12, etc. We computed these symmetries up to grading 16 and found that at level 4 there exists one symmetry, at level 8 two of them, at level 12 three symmetries arise, etc. For example, at level 4 we have the symmetry $\varphi_{4,1}$ with the components

$$\begin{aligned}\varphi_{4,1}^w &= -20p_3w_1 - 12p_2u_1 - 6p_1(2w_1w + v_1) + 9u^2 + 16vw + 16w^3 \\ \varphi_{4,1}^u &= -20p_3u_1 - 12p_2(w_1w + v_1) + 6p_1(w_1u + u_1w) + 8u(3v + w^2), \\ \varphi_{4,1}^v &= -20p_3v_1 + 12p_2(w_1u + 3u_1w) + 6p_1(3w_1w^2 + u_1u + 2v_1w) \\ &\quad - 39u^2w + 16v^2 - 12w^4,\end{aligned}$$

while at level 8 we have two symmetries $\varphi_{8,1}$ and $\varphi_{8,2}$ with the components

$$\begin{aligned}\varphi_{8,1}^w &= -1764p_7w_1 - 980p_6u_1 + 1176p_5(-2w_1w - v_1) \\ &\quad + 280p_3(w_1v + 3w_1w^2 + u_1u + v_1w) + 42p_2(2w_1u + 2u_1v + u_1w^2 + 2v_1u) \\ &\quad + 14(54u^2v + 9u^2w^2 + 48v^2w + 96vw^3 + 52w^5), \\ \varphi_{8,1}^u &= -1764p_7u_1 - 980p_6(w_1w + v_1) + 1176p_5(w_1u + u_1w) + 280p_3(u_1v + v_1u) \\ &\quad + 42p_2(-2w_1u^2 + 2w_1vw + w_1w^3 - 4u_1uw + 2v_1v + v_1w^2) \\ &\quad + 28u(-27u^2w + 36v^2 + 24vw^2 - w^4), \\ \varphi_{8,1}^v &= -1764p_7v_1 + 980p_6(w_1u + 3u_1w) + 1176p_5(3w_1w^2 + u_1u + 2v_1w) \\ &\quad + 280p_3(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2) \\ &\quad + 42p_2(-2w_1uv - 7w_1uw^2 - 2u_1u^2 - 6u_1vw - 3u_1w^3 - 6v_1uw) \\ &\quad + 7(-27u^4 - 468u^2vw - 190u^2w^3 + 64v^3 - 144vw^4 - 96w^6)\end{aligned}$$

and

$$\begin{aligned}\varphi^w(8, 2) &= -2184p_7w_1 - 1456p_6u_1 + 2184p_5(-2w_1w - v_1) \\ &\quad + 1040p_3(w_1v + 3w_1w^2 + u_1u + v_1w) \\ &\quad + 312p_2(2w_1uw + 2u_1v + u_1w^2 + 2v_1u) \\ &\quad + 52p_1(-3w_1u^2 + 12w_1vw + 14w_1w^3 - 6u_1uw + 6v_1v + 6v_1w^2),\end{aligned}$$

$$\begin{aligned}
\varphi^u(8, 2) &= -2184p_7u_1 - 1456p_6(w_1w + v_1) + 2184p_5(w_1u + u_1w) \\
&\quad + 1040p_3(u_1v + v_1u) \\
&\quad + 312p_2(-2w_1u^2 + 2w_1vw + w_1w^3 - 4u_1uw + 2v_1v + v_1w^2) \\
&\quad + 52p_1(-6w_1uv - 12w_1uw^2 - 3u_1u^2 - 6u_1vw - 4u_1w^3 - 6v_1uw), \\
\varphi^v(8, 2) &= -2184p_7v_1 + 1456p_6(w_1u + 3u_1w) + 2184p_5(3w_1w^2 + u_1u + 2v_1w) \\
&\quad + 1040p_3(-w_1u^2 - 3w_1w^3 - 4u_1uw + v_1v - v_1w^2) \\
&\quad + 312p_2(-2w_1uv - 7w_1uw^2 - 2u_1u^2 - 6u_1vw - 3u_1w^3 - 6v_1uw) \\
&\quad + 52p_1(6w_1u^2w - 18w_1vw^2 - 18w_1w^4 - 6u_1uv \\
&\quad + 12u_1uw^2 - 3v_1u^2 - 12v_1vw - 10v_1w^3).
\end{aligned}$$

2.2.3. *(x, t)-dependent symmetries.* These symmetries arise at levels 0, 4, 8, etc., and at each level we have two new symmetries $\bar{\varphi}_{0,1}^w, \bar{\varphi}_{0,2}^w, \bar{\varphi}_{4,1}^w, \bar{\varphi}_{4,2}^w, \dots$. For example, we have

$$\begin{aligned}
\bar{\varphi}_{0,1}^w &= xw_1 - 2w, & \bar{\varphi}_{0,2}^w &= tu_1 + 2w, \\
\bar{\varphi}_{0,1}^u &= xu_1 - 3u, & \bar{\varphi}_{0,2}^u &= t(w_1w + v_1) + 3u, \\
\bar{\varphi}_{0,1}^v &= xv_1 - 4v, & \bar{\varphi}_{0,2}^v &= -t(w_1u + 3u_1w) + 4v
\end{aligned}$$

and

$$\begin{aligned}
\bar{\varphi}_{4,1}^w &= x(w_1v + 3w_1w^2 + u_1u + v_1w) - 3p_1(2w_1w + v_1) - 4p_2u_1 - 5p_3w_1, \\
\bar{\varphi}_{4,1}^u &= x(u_1v + v_1u) + 3p_1(w_1u + u_1w) - 4p_2(w_1w + v_1) - 5p_3u_1, \\
\bar{\varphi}_{4,1}^v &= -x(w_1u^2 + 3w_1w^3 + 4u_1uw - v_1v + v_1w^2) + 3p_1(3w_1w^2 + u_1u + 2v_1w) \\
&\quad + 4p_2(w_1u + 3u_1w) - 5p_3v_1, \\
\bar{\varphi}_{4,2}^w &= t(2w_1uw + 2u_1v + u_1w^2 + 2v_1u) + 4p_1(2w_1w + v_1) + 6p_2u_1 + 8p_3w_1, \\
\bar{\varphi}_{4,2}^u &= -t(2w_1u^2 - 2w_1vw - w_1w^3 + 4u_1uw - 2v_1v - v_1w^2) \\
&\quad - 4p_1(w_1u + u_1w) + 6p_2(w_1w + v_1) + 8p_3u_1, \\
\bar{\varphi}_{4,2}^v &= -t(2w_1uv + 7w_1uw^2 + 2u_1u^2 + 6u_1vw + 3u_1w^3 + 6v_1uw) \\
&\quad - 4p_1(3w_1w^2 + u_1u + 2v_1w) - 6p_2(w_1u + 3u_1w) + 8p_3v_1.
\end{aligned}$$

Note that only the two symmetries are local.

2.2.4. *Distribution of symmetries.* We present the distribution of symmetries described above in Table 1.

2.3. **Cosymmetries.** The defining equations for cosymmetries (or their shadows) of the dB-equation are

$$\tilde{D}_t(\psi^w) = w\tilde{D}_x(\psi^u) - u\tilde{D}_x(\psi^v) + 2u_1\psi^v,$$

-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
φ_{-4}					φ_1	φ_2	φ_3		φ_5	φ_6	φ_7		φ_9	φ_{10}	φ_{11}	
								$\varphi_{4,1}$				$\varphi_{8,1}$				$\varphi_{12,1}$
												$\varphi_{8,2}$				$\varphi_{12,2}$
																$\varphi_{12,3}$
				$\bar{\varphi}_{0,1}$				$\bar{\varphi}_{4,1}$				$\bar{\varphi}_{8,1}$				$\bar{\varphi}_{12,1}$
				$\bar{\varphi}_{0,2}$				$\bar{\varphi}_{4,2}$				$\bar{\varphi}_{8,2}$				$\bar{\varphi}_{12,2}$

TABLE 1. Distribution of symmetries along gradings

$$\begin{aligned}\tilde{D}_t(\psi^u) &= \tilde{D}_x(\psi^w) - 3w\tilde{D}_x(\psi^v) - 2w_1\psi^v, \\ \tilde{D}_t(\psi^v) &= \tilde{D}_x(\psi^u).\end{aligned}\tag{59}$$

The grading of a cosymmetry is understood as the grading of the corresponding differential form and thus

$$|\psi| = |\psi^w| + |w| = |\psi^u| + |u| = |\psi^v| + |v|.$$

2.3.1. *(x, t)-independent local cosymmetries.* We computed these cosymmetries up to grading 16 and found that they exist at all levels except for 1, 5, 9, etc. Here we present the results for gradings ≤ 8 :

$$\begin{aligned}\psi_2^w &= 1, & \psi_3^w &= 0, & \psi_4^w &= w, \\ \psi_2^u &= 0, & \psi_3^u &= 1, & \psi_4^u &= 0, \\ \psi_2^v &= 0, & \psi_3^v &= 0, & \psi_4^v &= \frac{1}{2}, \\ \psi_6^w &= v + 3w^2, & \psi_7^w &= uw, & \psi_8^w &= \frac{1}{14}(-3u^2 + 12vw + 14w^3) \\ \psi_6^u &= u, & \psi_7^u &= \frac{1}{2}(2v + w^2), & \psi_8^u &= -\frac{3}{7}uw, \\ \psi_6^v &= w, & \psi_7^v &= u, & \psi_8^v &= \frac{3}{7}(v + w^2).\end{aligned}$$

2.3.2. *Nonlocal cosymmetries.* Nonlocal cosymmetries independent of x and t were found at levels 9 (one cosymmetry), 13 (two cosymmetries), and 17 (three cosymmetries). For example, we have

$$\begin{aligned}\psi_{9,1}^w &= \frac{1}{14}(-42p_7 - 84p_5w + 20p_3(v + 3w^2) + 12p_2uw \\ &\quad + p_1(-3u^2 + 12vw + 14w^3)), \\ \psi_{9,1}^u &= \frac{1}{7}(-14p_6 + 10p_3u + 3p_2(2v + w^2) - 3p_1uw), \\ \psi_{9,1}^v &= \frac{1}{7}(-21p_5 + 10p_3w + 6p_2u + 3p_1(v + w^2)), \\ \psi_{13,1}^w &= \frac{1}{3}(195p_{11} + 484p_9w - 126p_7(v - 3w^2) - 70p_6uw\end{aligned}$$

$$\begin{aligned}
& + 14p_5(3u^2 - 12vw - 14w^3) + 5p_3(2v^2 + 12vw^2 + 11w^4) \\
& + p_2u(-2u^2 + 6vw + 3w^3)), \\
\psi_{13,1}^u &= \frac{1}{12}(297p_{10} - 504p_7u - 140p_6(2v + w^2) + 336p_5uw + 80p_3uv \\
& + 3p_2(-8u^2w + 4v^2 + 4vw^2 + w^4)), \\
\psi_{13,1}^v &= \frac{1}{3}(242p_9 - 126p_7w - 70p_6u - 84p_5(v + w^2) + 10p_3(u^2 + 2vw + 2w^3) \\
& + 3p_2u(2v + w^2)),
\end{aligned}$$

and

$$\begin{aligned}
\psi_{13,2}^w &= \frac{1}{9}(-351p_{11} - 847p_9w + 210p_7(v + 3w^2) + 112p_6uw \\
& + 21p_5(-3u^2 + 12vw + 14w^3) - 5p_3(2v^2 + 12vw^2 + 11w^4) \\
& + p_1(-3u^2v - 6u^2w^2 + 6v^2w + 14vw^3 + 9w^5)), \\
\psi_{13,2}^u &= \frac{1}{9}(-132p_{10} + 210p_7u + 56p_6(2v + w^2) - 126p_5uw - 20p_3uv \\
& - p_1u(u^2 + 6vw + 4w^3)), \\
\psi_{13,2}^v &= \frac{1}{18}(-847p_9 + 420p_7w + 224p_6u + 252p_5(v + w^2) \\
& - 20p_3(u^2 + 2vw + 2w^3) + p_1(-6u^2w + 6v^2 + 12vw^2 + 7w^4)).
\end{aligned}$$

2.3.3. *(x, t)-dependent cosymmetries.* These cosymmetries exist at levels 1, 5, 9, etc. The first one,

$$\bar{\psi}_{1,1}^w = x, \quad \bar{\psi}_{1,1}^u = t, \quad \bar{\psi}_{1,1}^v = 0,$$

is local, all others are nonlocal: two at level 5,

$$\begin{aligned}
\bar{\psi}_{5,1}^w &= x(v + 3w^2) - 6p_1w - 5p_3, & \bar{\psi}_{5,2}^w &= tuw + 4p_1w + 4p_3, \\
\bar{\psi}_{5,1}^u &= xu - 4p_2, & \bar{\psi}_{5,2}^u &= \frac{1}{2}(t(2v + w^2) + 6p_2), \\
\bar{\psi}_{5,1}^v &= xw - 3p_1, & \bar{\psi}_{5,2}^v &= tu + 2p_1,
\end{aligned}$$

two at level 9,

$$\begin{aligned}
\bar{\psi}_{9,1}^w &= \frac{1}{11}(x(2v^2 + 12vw^2 + 11w^4) + 8p_2uw + 20p_3(v + 3w^2) \\
& - 112p_5w - 63p_7), \\
\bar{\psi}_{9,1}^u &= \frac{4}{11}(xuv + p_2(2v + w^2) + 5p_3 - 10p_6), \\
\bar{\psi}_{9,1}^v &= \frac{2}{11}(x(u^2 + 2vw + 2w^3) + 4p_2u + 10p_3w - 28p_5),
\end{aligned}$$

$$\begin{aligned}\bar{\psi}_{9,2}^w &= \frac{1}{3}(tu(-2u^2 + 6vw + 3w^3) - 6p_2uw - 16p_3(v + 3w^2) + 96p_5w + 56p_7), \\ \bar{\psi}_{9,2}^u &= \frac{1}{12}(3t(-8u^2w + 4v^2 + 4vw^2 + w^4) - 12p_2(2v + w^2) - 64p_3u + 140p_6), \\ \bar{\psi}_{9,2}^v &= \frac{1}{3}(3tu(2v + w^2) - 6p_2u - 16p_3w + 48p_5),\end{aligned}$$

etc.

2.3.4. *Distribution of cosymertries.* We present the distribution of cosymertries described above in Table 2.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	ψ_2	ψ_3	ψ_4		ψ_6	ψ_7	ψ_8		ψ_{10}	ψ_{11}	ψ_{12}		ψ_{14}	ψ_{15}	ψ_{16}	
								$\psi_{9,1}$				$\psi_{13,1}$				$\psi_{17,1}$
												$\psi_{13,2}$				$\psi_{17,2}$
																$\psi_{17,3}$
$\bar{\psi}_{1,1}$				$\bar{\psi}_{5,1}$				$\bar{\psi}_{9,1}$				$\bar{\psi}_{13,1}$				$\bar{\psi}_{17,1}$
				$\bar{\psi}_{5,2}$				$\bar{\psi}_{9,2}$				$\bar{\psi}_{13,2}$				$\bar{\psi}_{17,2}$

TABLE 2. Distribution of cosymertries along gradings

2.4. **Nonlocal vectors.** We construct nonlocal vectors on the $\ell_{\mathcal{E}}^*$ -covering (see Remark 9) associated with symmetries $\varphi_{-4}, \dots, \varphi_7$ and denote them by ρ^{-4}, \dots, ρ^7 , respectively. Thus, by definition, we have

$$\tilde{D}_x(\rho^i) = \rho^w \varphi_i^w + \rho^u \varphi_i^u + \rho^v \varphi_i^v, \quad i = -4, 1, 2, 3, 5, 6, 7.$$

The corresponding t -components are given by the following relations:

$$\tilde{D}_t(\rho^i) = \rho^w \varphi_i^u + \rho^u (\varphi_i^v + \varphi_i^w w) - \rho^v (3\varphi_i^u w + \varphi_i^w u).$$

Here ρ^w, ρ^u, ρ^v are the coordinates in the $\ell_{\mathcal{E}}^*$ -covering (see equations (61) below).

2.5. **Nonlocal forms.** We also constructed nonlocal forms associated to the cosymertries ψ_2, ψ_3, ψ_4 , and ψ_6, ψ_7, ψ_8 . We denote these forms by $\omega^2, \dots, \omega^8$ and have by definition

$$\tilde{D}_x(\omega^i) = \psi_i^w \omega^w + \psi_i^u \omega^u + \psi_i^v \omega^v, \quad i = 2, 3, 4, 6, 7, 8,$$

where $\omega^w, \omega^u, \omega^v$ are the coordinates in the $\ell_{\mathcal{E}}$ -covering (see equations (60)).

The t -components of the nonlocal forms under consideration are given by the equations

$$\tilde{D}_t(\omega^i) = (\psi_i^u w - \psi_i^v u) \omega^w + (\psi_i^w - 3\psi_i^v w) \omega^u + \psi_i^u \omega^v.$$

3. THE MAIN RESULTS

We begin with the defining equations for the $\ell_{\mathcal{E}}$ - and $\ell_{\mathcal{E}}^*$ -coverings of the dB-equation. In the first case we have

$$\begin{aligned}\omega_t^w &= \omega_x^u, \\ \omega_t^u &= w\omega_x^w + w_x\omega^w + \omega_x^v, \\ \omega_t^v &= -u\omega_x^w - w_x\omega^u - 3w\omega_x^u - 3u_x\omega^w,\end{aligned}\tag{60}$$

while the equations of the $\ell_{\mathcal{E}}^*$ -covering are

$$\begin{aligned}\rho_t^w &= w\rho_x^u - u\rho_x^v + 2u_x\rho^v, \\ \rho_t^u &= \rho_x^w - 3w\rho_x^v - 2w_x\rho^v, \\ \rho_t^v &= \rho_x^u.\end{aligned}\tag{61}$$

The total derivatives on the $\ell_{\mathcal{E}}$ -covering are of the form

$$\begin{aligned}\tilde{D}_x &= D_x + \sum_{i \geq 0} \left(\omega_{i+1}^w \frac{\partial}{\partial \omega_i^w} + \omega_{i+1}^u \frac{\partial}{\partial \omega_i^u} + \omega_{i+1}^v \frac{\partial}{\partial \omega_i^v} \right), \\ \tilde{D}_t &= D_t + \sum_{i \geq 0} \left(\omega_{i+1}^u \frac{\partial}{\partial \omega_i^w} + \tilde{D}_x^i (w\omega_1^w + w_1\omega^w + \omega_1^v) \frac{\partial}{\partial \omega_i^u} \right. \\ &\quad \left. - \tilde{D}_x^i (u\omega_1^w + w_1\omega^u + 3w\omega_1^u + 3u_1\omega^w) \frac{\partial}{\partial \omega_i^v} \right),\end{aligned}\tag{62}$$

where D_x and D_t are total derivatives on the dB-equation, while on the $\ell_{\mathcal{E}}^*$ -covering they are presented by

$$\begin{aligned}\tilde{D}_x &= D_x + \sum_{i \geq 0} \left(\rho_{i+1}^w \frac{\partial}{\partial \rho_i^w} + \rho_{i+1}^u \frac{\partial}{\partial \rho_i^u} + \rho_{i+1}^v \frac{\partial}{\partial \rho_i^v} \right), \\ \tilde{D}_t &= D_t + \sum_{\geq 0} \left(\tilde{D}_x (w\rho_1^u - u\rho_1^v + 2u_1\rho^v) \frac{\partial}{\partial \rho_i^w} \right. \\ &\quad \left. + \tilde{D}_x (\rho_1^w - 3w\rho_1^v - 2w_1\rho^v) \frac{\partial}{\partial \rho_i^u} + \rho_{i+1}^u \frac{\partial}{\partial \rho_i^v} \right).\end{aligned}\tag{63}$$

3.1. Recursion operators for symmetries. We construct recursion operators $\mathcal{R} = (\mathcal{R}^w, \mathcal{R}^u, \mathcal{R}^v)$ for symmetries of the dB-equation solving the system

$$\begin{aligned}\tilde{D}_t(\mathcal{R}^w) &= \tilde{D}_x(\mathcal{R}^u), \\ \tilde{D}_t(\mathcal{R}^u) &= w\tilde{D}_x(\mathcal{R}^w) + w_1\mathcal{R}^w + \tilde{D}_x(\mathcal{R}^v), \\ \tilde{D}_t(\mathcal{R}^v) &= -u\tilde{D}_x(\mathcal{R}^w) - 3u_1\mathcal{R}^w - 3w\tilde{D}_x(\mathcal{R}^u) - w_1\mathcal{R}^u,\end{aligned}$$

where the total derivatives are defined by formulas (62). Recall that we look for solutions linear in ω_i 's and admit their dependency on nonlocal forms. Besides the trivial solution

$$\mathcal{R}_0^w = \omega^w, \quad \mathcal{R}_0^u = \omega^u, \quad \mathcal{R}_0^v = \omega^v$$

that corresponds to the identity operator, we found the following two nontrivial solutions of grading 4:

$$\begin{aligned} \mathcal{R}_{4,1}^w &= (2\omega_4 w_1 + \omega^3 u_1 + \omega^2(2w_1 w + v_1))/2, \\ \mathcal{R}_{4,1}^u &= (2\omega^4 u_1 + \omega^3(w_1 w + v_1) - \omega^2(w_1 u + u_1 w))/2, \\ \mathcal{R}_{4,1}^v &= (2\omega^4 v_1 + \omega^3(-w_1 u - 3u_1 w) + \omega^2(-3w_1 w^2 - u_1 u - 2v_1 w))/2, \\ \mathcal{R}_{4,2}^w &= (-4\omega^4 w_1 - \omega^3 u_1 + 2\omega^v w + 3\omega^u u + 4\omega^w(v + 2w^2))/8, \\ \mathcal{R}_{4,2}^u &= (-4\omega^4 u_1 - \omega^3(w_1 w + v_1) + 3\omega^v u + 2\omega^u(2v + w^2) + \omega^w u w)/8, \\ \mathcal{R}_{4,2}^v &= (-4\omega^4 v_1 + \omega^3(w_1 u + 3u_1 w) + 4\omega^v v - 11\omega^u u w + 3\omega^w(-u^2 - 2w^3))/8. \end{aligned}$$

Remark 15. We also found three solutions at level 8, but they are too cumbersome to present them here.

The operators corresponding to the above solutions are

$$\mathcal{R}_{4,1} = \frac{1}{2} \langle 2\varphi_1, \varphi_2, \varphi_3 \mid D_x^{-1} \mid \psi_4, \psi_3, \psi_2 \rangle \quad (64)$$

and

$$\begin{aligned} \mathcal{R}_{4,2} &= -\frac{1}{8} \langle 4\varphi_1, \varphi_2 \mid D_x^{-1} \mid \psi_4, \psi_3 \rangle \\ &\quad + \frac{1}{8} \begin{pmatrix} 4(v + 2w^2) & 3u & 2w \\ uw & 2(2v + w^2) & 3u \\ -3(u^2 + 2w^3) & -11uw & 4v \end{pmatrix} \quad (65) \end{aligned}$$

(here and below we use the notation introduced in Remark 10). The explicit expressions for symmetries φ_i and cosymmetries ψ_j are given in Subsections 2.2.1 and 2.3.1.

3.2. Recursion operators for cosymmetries. Recursion operators for cosymmetries are of the form $\mathcal{R}^* = (\mathcal{R}^{*,u}, \mathcal{R}^{*,w}, \mathcal{R}^{*,v})$ and can be constructed from the equation

$$\begin{aligned} \tilde{D}_t(\mathcal{R}^{*,w}) &= w\tilde{D}_x(\mathcal{R}^{*,u}) - u\tilde{D}_x(\mathcal{R}^{*,v}) + 2u_1\mathcal{R}^{*,v}, \\ \tilde{D}_t(\mathcal{R}^{*,u}) &= \tilde{D}_x(\mathcal{R}^{*,w}) - 3w\tilde{D}_x(\mathcal{R}^{*,v}) - 2w_1\mathcal{R}^{*,v}, \\ \tilde{D}_t(\mathcal{R}^{*,v}) &= \tilde{D}_x(\mathcal{R}^{*,u}), \end{aligned}$$

where the total derivatives are given by formulas (63). The first two nontrivial solutions (both of grading 4) are

$$\begin{aligned} R_{4,1}^{*,w} &= (\rho^3 + 2\rho^1 w)/2, \\ R_{4,1}^{*,u} &= \rho^2/2, \end{aligned}$$

$$\begin{aligned}
R_{4,1}^{*,v} &= \rho^1/2, \\
R_{4,2}^{*,w} &= (-2\rho^3 + 3\rho^v(-2w^3 - u^2) + \rho^u wu + 4\rho^w(2w^2 + v))/8, \\
R_{4,2}^{*,u} &= (-\rho^2 - 11\rho^v wu + 2\rho^u(w^2 + 2v) + 3\rho^w u)/8, \\
R_{4,2}^{*,v} &= (4\rho^v v + 3\rho^u u + 2\rho^w w)/8.
\end{aligned}$$

The operators corresponding to these solutions are

$$\mathcal{R}_{4,1}^* = \frac{1}{2} \langle 2\psi_4, \psi_3, \psi_2 \mid D_x^{-1} \mid \varphi_1, \varphi_2, \varphi_3 \rangle \quad (66)$$

and

$$\begin{aligned}
\mathcal{R}_{4,2}^* &= -\frac{1}{8} \langle \psi_3, 2\psi_2 \mid D_x^{-1} \mid \varphi_2, \varphi_3 \rangle \\
&\quad + \frac{1}{8} \begin{pmatrix} 4(2w^2 + v) & wu & -3(2w^3 + u^2) \\ 3u & 2(w^2 + 2v) & -11wu \\ 2w & 3u & 4v \end{pmatrix}. \quad (67)
\end{aligned}$$

3.3. Hamiltonian structures. The first step to construct Hamiltonian structures $\mathcal{H} = (\mathcal{H}^w, \mathcal{H}^u, \mathcal{H}^v)$ is to solve the equation

$$\begin{aligned}
\tilde{D}_t(\mathcal{H}^w) &= \tilde{D}_x(\mathcal{H}^u), \\
\tilde{D}_t(\mathcal{H}^u) &= w\tilde{D}_x(\mathcal{H}^w) + w_1\mathcal{H}^w + \tilde{D}_x(\mathcal{H}^v), \\
\tilde{D}_t(\mathcal{H}^v) &= -u\tilde{D}_x(\mathcal{H}^w) - 3u_1\mathcal{H}^w - 3w\tilde{D}_x(\mathcal{H}^u) - w_1\mathcal{H}^u
\end{aligned}$$

with total derivatives defined by formulas (63). The first three solutions are

$$\begin{aligned}
\mathcal{H}_{0,1}^w &= \rho_1^v, \\
\mathcal{H}_{0,1}^u &= \rho_1^u, \\
\mathcal{H}_{0,1}^v &= -4\rho_1^v w + \rho_1^w - 2\rho^v w_1, \\
\mathcal{H}_{4,1}^w &= (4\rho_1^v v + 3\rho_1^u u + 2\rho_1^w w + 2\rho^v v_1 + \rho^u u_1)/2, \\
\mathcal{H}_{4,1}^u &= (-11\rho_1^v wu + 2\rho_1^u(w^2 + 2v) + 3\rho_1^w u - 2\rho^v(wu_1 + 4uw_1) \\
&\quad + \rho^u(ww_1 + v_1))/2, \\
\mathcal{H}_{4,1}^v &= (-\rho_1^v(6w^3 + 16wv + 3u^2) - 11\rho_1^u wu + 4\rho_1^w v \\
&\quad - 2\rho^v(3w^2 w_1 + 2wv_1 + uu_1 + 4vw_1) \\
&\quad + \rho^u(-3wu_1 - uw_1))/2, \\
\mathcal{H}_{4,2}^w &= \rho^v v_1 + \rho^u u_1 + \rho^w w_1, \\
\mathcal{H}_{4,2}^u &= \rho^v(3wu_1 + uw_1) + \rho^u(ww_1 + v_1) + \rho^w u_1, \\
\mathcal{H}_{4,2}^v &= \rho^v(-3w^2 w_1 - 4wv_1 - uu_1) + \rho^u(-3wu_1 - uw_1) + \rho^w v_1.
\end{aligned}$$

The operators corresponding to these solutions are

$$\mathcal{H}_{0,1} = \mathcal{H}_1 = \begin{pmatrix} 0 & 0 & D_x \\ 0 & D_x & 0 \\ D_x & 0 & -4D_x w - 2w_1 \end{pmatrix}$$

and

$$\mathcal{H}_{4,1} = \frac{1}{2} \begin{pmatrix} 2wD_x & 3uD_x + u_1 & 2(2vD_x + v_1) \\ 3uD_x & 2(w^2 + 2v)D_x + ww_1 + v_1 & -11wuD_x - 2(wu_1 + 4uw_1) \\ 4vD_x & -11wuD_x - 3wu_1 - uw_1 & h_{2,2}^1 D_x + h_{2,2}^0 \end{pmatrix},$$

$$\mathcal{H}_{4,2} = \begin{pmatrix} w_1 & u_1 & v_1 \\ u_1 & ww_1 + v_1 & -3wu_1 - uw_1 \\ v_1 & -3wu_1 - uw_1 & -3w^2w_1 - 4wv_1 - uu_1 \end{pmatrix},$$

where

$$h_{2,2}^1 = -(6w^3 + 16wv + 3u^2), \quad h_{2,2}^0 = -2(3w^2w_1 + 2wv_1 + uu_1 + 4wv_1).$$

The operator \mathcal{H}_1 is skew-adjoint and by Theorem 5 is a Hamiltonian structure for the dB-equation, but neither of the last two operators is Hamiltonian. Nevertheless, their linear combination

$$\mathcal{H}_2 = \mathcal{H}_{4,1} + \frac{1}{2}\mathcal{H}_{4,2}$$

is skew-adjoint and consequently Hamiltonian. Again, by Theorem 5 the structures \mathcal{H}_1 and \mathcal{H}_2 are compatible.

3.4. Symplectic structures. To find symplectic structures $\mathcal{S} = (\mathcal{S}^w, \mathcal{S}^u, \mathcal{S}^v)$, we solve the equation

$$\begin{aligned} \tilde{D}_t(\mathcal{S}^w) &= w\tilde{D}_x(\mathcal{S}^u) - u\tilde{D}_x(\mathcal{S}^v) + 2u_1\mathcal{S}^v, \\ \tilde{D}_t(\mathcal{S}^u) &= \tilde{D}_x(\mathcal{S}^w) - 3w\tilde{D}_x(\mathcal{S}^v) - 2w_1\mathcal{S}^v, \\ \tilde{D}_t(\mathcal{S}^v) &= \tilde{D}_x(\mathcal{S}^u), \end{aligned}$$

where the total derivatives are given by formulas (62). We obtained one solution

$$\begin{aligned} \mathcal{S}_{0,1}^w &= \omega^4 + \omega^2 w, \\ \mathcal{S}_{0,1}^u &= \omega^3/2, \\ \mathcal{S}_{0,1}^v &= \omega^2/2 \end{aligned}$$

of grading 0 and to solutions of grading 4:

$$\begin{aligned} \mathcal{S}_{4,1}^w &= -14\omega^8 - 8\omega^6 w + 4\omega^4(3w^2 + v) + \omega^3 w u, \\ \mathcal{S}_{4,1}^u &= (-10\omega^7 + 8\omega^4 u + \omega^3(w^2 + 2v))/2, \\ \mathcal{S}_{4,1}^v &= -4\omega^6 + 4\omega^4 w + \omega^3 u, \\ \mathcal{S}_{4,2}^w &= (70\omega^8 + 36\omega^6 w - 12\omega^4(3w^2 + v) + \omega^2(14w^3 + 12wv - 3u^2))/14, \end{aligned}$$

$$\begin{aligned}\mathcal{S}_{4,2}^u &= (12\omega^7 - 6\omega^4u - 3\omega^2wu)/7, \\ \mathcal{S}_{4,2}^v &= (9\omega^6 - 6\omega^4w + 3\omega^2(w^2 + v))/7.\end{aligned}$$

The operator

$$\mathcal{S}_1 = \mathcal{S}_{0,1} = \langle \psi_4, \frac{1}{2}\psi_3, \psi_2 \mid D_x^{-1} \mid \psi_2, \psi_3, \psi_4 \rangle$$

corresponds to the first solution, while to the second and the third ones the operators

$$\mathcal{S}_{4,1} = \langle \psi_7, 4\psi_6, -8\psi_4, -5\psi_3, -14\psi_2 \mid D_x^{-1} \mid \psi_3, \psi_4, \psi_6, \psi_7, \psi_8 \rangle$$

and

$$\mathcal{S}_{4,2} = \langle \psi_8, -\frac{6}{7}\psi_6, \frac{18}{7}\psi_4, \frac{12}{7}\psi_3, 5\psi_2 \mid D_x^{-1} \mid \psi_2, \psi_4, \psi_6, \psi_7, \psi_8 \rangle$$

correspond.

The operator \mathcal{S}_1 is skew-adjoint and hence is a symplectic structure for the dB-equation. In addition, the operator

$$\mathcal{S}_2 = 2\mathcal{S}_{4,1} + 7\mathcal{S}_{4,2}$$

is a symplectic structure as well.

4. INTERRELATIONS

We shall now present the basic algebraic relations between the above described structures. They include description of the following compositions:

$$\mathcal{R} \circ \mathcal{R}, \mathcal{R} \circ \mathcal{H}, \mathcal{S} \circ \mathcal{R}, \mathcal{H} \circ \mathcal{S}, \mathcal{S} \circ \mathcal{H}, \mathcal{H} \circ \mathcal{R}^*, \mathcal{R}^* \circ \mathcal{S}, \mathcal{R}^* \circ \mathcal{R}^*,$$

together with the action of the operators \mathcal{R} on symmetries and of \mathcal{R}^* on cosymmetries and as well as commutator relations between symmetries.

Of course, some of these relations are deducible from the other ones, but we prefer explicit presentation.

4.1. $\mathcal{R} \circ \mathcal{R}$. Compositions of the first two recursion operators look as follows

$$\begin{aligned}R_{4,1} \circ R_{4,1} &= \frac{1}{4}R_{8,1} + \frac{7}{12}R_{8,2}, \\ R_{4,1} \circ R_{4,2} &= -\frac{1}{16}R_{8,1}, \\ R_{4,2} \circ R_{4,1} &= \frac{3}{16}R_{8,1} + \frac{7}{12}R_{8,2}, \\ R_{4,2} \circ R_{4,2} &= -\frac{3}{64}R_{8,1} + \frac{13}{16}R_{8,3}.\end{aligned}$$

So we have

$$R_{4,2} \circ R_{4,1} = R_{4,1} \circ R_{4,2} + R_{4,1} \circ R_{4,1},$$

or

$$[R_{4,2}, R_{4,1}] = R_{4,1}^2. \tag{68}$$

Actually, this relation determines the entire structure of all algebraic invariants related to the dB-equation.

4.2. $\mathcal{R} \circ \mathcal{H}$. We computed the following compositions:

$$\begin{aligned}
R_{4,1} \circ H_{0,1} &= \frac{1}{2}H_{4,2}, \\
R_{4,1} \circ H_{4,1} &= -H_{8,1} + H_{8,2} + 2H_{8,3}, \\
R_{4,1} \circ H_{4,2} &= H_{8,1}, \\
R_{4,2} \circ H_{0,1} &= \frac{1}{4}H_{4,1} - \frac{1}{4}H_{4,2}, \\
R_{4,2} \circ H_{4,1} &= -\frac{1}{2}H_{8,2} - \frac{1}{2}H_{8,3}, \\
R_{4,2} \circ H_{4,2} &= H_{8,1} + \frac{1}{2}H_{8,2} + \frac{1}{H_{8,3}}, \\
R_{8,1} \circ H_{0,1} &= 8H_{8,1} - 4H_{8,2} - 8H_{8,3}, \\
R_{8,2} \circ H_{0,1} &= -\frac{18}{7}H_{8,1} + \frac{12}{7}H_{8,2} + \frac{24}{7}H_{8,3}, \\
R_{8,3} \circ H_{0,1} &= \frac{6}{13}H_{8,1} - \frac{7}{13}H_{8,2} - \frac{12}{13}H_{8,3}.
\end{aligned}$$

etc. These relations imply, in particular, the equality

$$(3\mathcal{R}_{4,1} + 4\mathcal{R}_{4,2}) \circ \mathcal{H}_{0,1} = \mathcal{H}_{4,1} + \frac{1}{2}\mathcal{H}_{4,2}.$$

Denoting $3\mathcal{R}_{4,1} + 4\mathcal{R}_{4,2}$ by \mathcal{R} , we have

$$\mathcal{R} \circ \mathcal{H}_1 = \mathcal{H}_2, \tag{69}$$

where \mathcal{H}_1 and \mathcal{H}_2 are the Hamiltonian structures introduced in Subsection 3.3.

Proposition 3. *The operator*

$$\mathcal{R} = 3\mathcal{R}_{4,1} + 4\mathcal{R}_{4,2} \tag{70}$$

satisfies the Nijenhuis conditions (see Remark 5) and generates an infinite family of compatible Hamiltonian structures

$$\mathcal{H}_n = \mathcal{R}^{n-1} \circ \mathcal{H}_1$$

for the dB-equation.

4.3. $\mathcal{S} \circ \mathcal{R}$. These compositions look as follows

$$\begin{aligned}
S_{0,1} \circ R_{4,1} &= \frac{1}{4}S_{4,1} + \frac{7}{12}S_{4,2}, \\
S_{0,1} \circ R_{4,2} &= -\frac{1}{16}S_{4,1},
\end{aligned}$$

$$\begin{aligned}
S_{0,1} \circ R_{8,1} &= \frac{11}{2}S_{8,1} + \frac{1}{4}S_{8,2}, \\
S_{0,1} \circ R_{8,2} &= -\frac{33}{28}S_{8,1} + \frac{9}{28}S_{8,3}, \\
S_{0,1} \circ R_{8,3} &= \frac{11}{104}S_{8,1}, \\
S_{4,1} \circ R_{4,1} &= -\frac{11}{2}S_{8,1} - \frac{3}{4}S_{8,2} - 3S_{8,3}, \\
S_{4,1} \circ R_{4,2} &= \frac{11}{4}S_{8,1} + \frac{3}{16}S_{8,2}, \\
S_{4,2} \circ R_{4,1} &= \frac{99}{28}S_{8,1} + \frac{3}{7}S_{8,2} + \frac{45}{28}S_{8,3}, \\
S_{4,2} \circ R_{4,2} &= -\frac{99}{56}S_{8,1} - \frac{3}{28}S_{8,2},
\end{aligned}$$

etc.

4.4. $\mathcal{H} \circ \mathcal{S}$. These compositions look as follows

$$\begin{aligned}
H_{0,1} \circ S_{0,1} &= \frac{1}{2}R_{0,1}, \\
H_{0,1} \circ S_{4,1} &= -8R_{4,2}, \\
H_{0,1} \circ S_{4,2} &= \frac{6}{7}R_{4,1} + \frac{24}{7}R_{4,2}, \\
H_{0,1} \circ S_{8,1} &= \frac{52}{11}R_{8,3}, \\
H_{0,1} \circ S_{8,2} &= 2R_{8,1} - 104R_{8,3}, \\
H_{0,1} \circ S_{8,3} &= R_{8,1} + \frac{14}{9}R_{8,2} + \frac{52}{3}R_{8,3}, \\
H_{4,1} \circ S_{0,1} &= R_{4,1} + 2R_{4,2}, \\
H_{4,1} \circ S_{4,1} &= \frac{5}{2}R_{8,1} - 26R_{8,3}, \\
H_{4,1} \circ S_{4,2} &= 3R_{8,2} + \frac{78}{7}R_{8,3}, \\
H_{4,2} \circ S_{0,1} &= R_{4,1}, \\
H_{4,2} \circ S_{4,1} &= R_{8,1}, \\
H_{4,2} \circ S_{4,2} &= R_{8,1} + R_{8,2}, \\
H_{8,1} \circ S_{0,1} &= \frac{1}{4}R_{8,1} + \frac{7}{12}R_{8,2}, \\
H_{8,2} \circ S_{0,1} &= -\frac{3}{4}R_{8,1} - \frac{7}{2}R_{8,2} - \frac{13}{2}R_{8,3},
\end{aligned}$$

$$H_{8,3} \circ S_{0,1} = \frac{9}{16}R_{8,1} + \frac{7}{3}R_{8,2} + \frac{13}{4}R_{8,3},$$

etc.

4.5. $\mathcal{S} \circ \mathcal{H}$. Similar to the identities of the previous subsection, we have

$$\begin{aligned} S_{0,1} \circ H_{0,1} &= \frac{1}{2}R_{0,1}^*, \\ S_{0,1} \circ H_{4,1} &= -R_{4,1}^* + 2R_{4,2}^*, \\ S_{0,1} \circ H_{4,2} &= R_{4,1}^*, \\ S_{0,1} \circ H_{8,1} &= \frac{1}{4}R_{8,1}^* + \frac{7}{12}R_{8,2}^*, \\ S_{0,1} \circ H_{8,2} &= -\frac{1}{4}R_{8,1}^* - \frac{13}{2}R_{8,3}^*, \\ S_{0,1} \circ H_{8,3} &= \frac{1}{16}R_{8,1}^* + \frac{13}{4}R_{8,3}^*, \\ S_{4,1} \circ H_{0,1} &= 8R_{4,1}^* - 8R_{4,2}^*, \\ S_{4,1} \circ H_{4,1} &= -\frac{3}{2}R_{8,1}^* - 26R_{8,3}^*, \\ S_{4,1} \circ H_{4,2} &= R_{8,1}^*, \\ S_{4,2} \circ H_{0,1} &= -\frac{18}{7}R_{4,1}^* + \frac{24}{7}R_{4,2}^*, \\ S_{4,2} \circ H_{4,1} &= -R_{8,2}^* + \frac{78}{7}R_{8,3}^*, \\ S_{4,2} \circ H_{4,2} &= R_{8,2}^*, \\ S_{8,1} \circ H_{0,1} &= \frac{20}{11}R_{8,1}^* + \frac{28}{11}R_{8,2}^* + \frac{52}{11}R_{8,3}^*, \\ S_{8,2} \circ H_{0,1} &= -30R_{8,1}^* - \frac{112}{3}R_{8,2}^* - 104R_{8,3}^*, \\ S_{8,3} \circ H_{0,1} &= 4R_{8,1}^* + \frac{14}{3}R_{8,2}^* + \frac{52}{3}R_{8,3}^* \end{aligned}$$

etc.

4.6. $\mathcal{H} \circ \mathcal{R}^*$. We computed the following compositions of this type

$$\begin{aligned} H_{0,1} \circ R_{4,1}^* &= \frac{1}{2}H_{4,2}, \\ H_{0,1} \circ R_{4,2}^* &= \frac{1}{4}H_{4,1} + \frac{1}{4}H_{4,2}, \\ H_{0,1} \circ R_{8,1}^* &= -4H_{8,2} - 8H_{8,3}, \end{aligned}$$

$$\begin{aligned}
H_{0,1} \circ R_{8,2}^* &= \frac{6}{7}H_{8,1} + \frac{12}{7}H_{8,2} + \frac{24}{7}H_{8,3}, \\
H_{0,1} \circ R_{8,3}^* &= \frac{1}{13}H_{8,2} + \frac{4}{13}H_{8,3}, \\
H_{4,1} \circ R_{4,1}^* &= H_{8,1} + H_{8,2} + H_{8,3}, \\
H_{4,1} \circ R_{4,2}^* &= \frac{1}{2}H_{8,2} + \frac{3}{2}H_{8,3}, \\
H_{4,2} \circ R_{4,1}^* &= H_{8,1}, \\
H_{4,2} \circ R_{4,2}^* &= \frac{1}{2}H_{8,2} + H_{8,3},
\end{aligned}$$

which are in a sense dual to the results of Subsection 4.6.

4.7. $\mathcal{R}^* \circ \mathcal{S}$. Computing compositions of recursion operators with operators \mathcal{S} we obtain

$$\begin{aligned}
R_{4,1}^* \circ S_{0,1} &= \frac{1}{4}S_{4,1} + \frac{7}{12}S_{4,2}, \\
R_{4,1}^* \circ S_{4,1} &= \frac{11}{2}S_{8,1} + \frac{1}{4}S_{8,2}, \\
R_{4,1}^* \circ S_{4,2} &= -\frac{33}{28}S_{8,1} + \frac{9}{28}S_{8,3}, \\
R_{4,2}^* \circ S_{0,1} &= \frac{3}{16}S_{4,1} + \frac{7}{12}S_{4,2}, \\
R_{4,2}^* \circ S_{4,1} &= \frac{33}{4}S_{8,1} + \frac{7}{16}S_{8,2}, \\
R_{4,2}^* \circ S_{4,2} &= -\frac{99}{56}S_{8,1} + \frac{9}{14}S_{8,3}, \\
R_{8,1}^* \circ S_{0,1} &= -\frac{11}{2}S_{8,1} - \frac{3}{4}S_{8,2} - \frac{3}{S_{8,3}}, \\
R_{8,2}^* \circ S_{0,1} &= \frac{99}{28}S_{8,1} + \frac{3}{7}S_{8,2} + \frac{45}{28}S_{8,3}, \\
R_{8,3}^* \circ S_{0,1} &= \frac{33}{104}S_{8,1} + \frac{3}{52}S_{8,2} + \frac{15}{52}S_{8,3},
\end{aligned}$$

etc. In particular, we have the relation

$$(4\mathcal{R}_{4,2}^* - \mathcal{R}_{4,1}^*) \circ \mathcal{S}_1 = \frac{1}{4}\mathcal{S}_2,$$

where

$$\mathcal{S}_1 = \mathcal{S}_{0,1}, \quad \mathcal{S}_2 = 2\mathcal{S}_{4,1} + 7\mathcal{S}_{4,2}$$

are the symplectic structures introduced in Subsection 3.4. Moreover, we have the following result:

Proposition 4. *The operator*

$$\mathcal{R}^* = 4(4\mathcal{R}_{4,2}^* - \mathcal{R}_{4,1}^*) \quad (71)$$

generates an infinite family of symplectic structures for the dB-equation by

$$\mathcal{S}_n = (\mathcal{R}^*)^{n-1} \circ \mathcal{S}_1.$$

4.8. $\mathcal{R}^* \circ \mathcal{R}^*$. The first relations we obtained are

$$\begin{aligned} R_{4,1}^* \circ R_{4,1}^* &= \frac{1}{4}R_{8,1}^* + \frac{7}{12}R_{8,2}^*, \\ R_{4,1}^* \circ R_{4,2}^* &= -\frac{1}{16}R_{8,1}^*, \\ R_{4,2}^* \circ R_{4,1}^* &= \frac{3}{16}R_{8,1}^* + \frac{7}{12}R_{8,2}^*, \\ R_{4,2}^* \circ R_{4,2}^* &= -\frac{3}{64}R_{8,1}^* + \frac{13}{16}R_{8,3}^*. \end{aligned}$$

Hence, the relation

$$R_{4,2}^* \circ R_{4,1}^* = R_{4,1}^* \circ R_{4,2}^* + R_{4,1}^* \circ R_{4,1}^*$$

takes place, or

$$[R_{4,2}^*, R_{4,1}^*] = R_{4,1}^* \circ R_{4,1}^* \quad (72)$$

that is similar to the one we had for the operators \mathcal{R} (see equation (68)).

4.9. Action of \mathcal{R} on symmetries.

4.9.1. *Action on local symmetries.* The action of the operator $R_{4,1}$ is as follows

$$\begin{aligned} R_{4,1}(\varphi_1) &= \frac{1}{2}\varphi_5, & R_{4,1}(\varphi_2) &= \frac{1}{4}\varphi_6, & R_{4,1}(\varphi_3) &= \frac{1}{12}\varphi_7, \\ R_{4,1}(\varphi_5) &= \frac{1}{8}\varphi_9, & R_{4,1}(\varphi_6) &= \frac{1}{24}\varphi_{10}, & R_{4,1}(\varphi_7) &= \frac{1}{4}\varphi_{11}, \\ R_{4,1}(\varphi_9) &= \frac{1}{60}\varphi_{13}, & R_{4,1}(\varphi_{10}) &= \frac{1}{4}\varphi_{14}, & R_{4,1}(\varphi_{11}) &= \frac{1}{28}\varphi_{15} \end{aligned}$$

and is related with the action of $R_{4,2}$ in the following way

$$R_{4,1}(\varphi_i) = \frac{4}{i+1}R_{4,2}(\varphi_i). \quad (73)$$

4.9.2. *Action on (x, t) -independent nonlocal symmetries.* Here we have

$$\begin{aligned} R_{4,1}(\varphi_{4,1}) &= -\frac{1}{104}\varphi_{8,2}, \\ R_{4,1}(\varphi_{8,1}) &= 385\varphi_{12,1} + 21\varphi_{12,2}, \\ R_{4,1}(\varphi_{8,2}) &= 1430\varphi_{12,1} + 156\varphi_{12,2} + 234\varphi_{12,3} \end{aligned}$$

and

$$\begin{aligned} R_{4,2}(\varphi_{4,1}) &= \frac{1}{56}\varphi_{8,1} - \frac{1}{104}\varphi_{8,2} \\ R_{4,2}(\varphi_{8,1}) &= \frac{1155}{2}\varphi_{12,1} + \frac{147}{4}\varphi_{12,2} \\ R_{4,2}(\varphi_{8,2}) &= 2145\varphi_{12,1} + 273\varphi_{12,2} + 468\varphi_{12,3} \end{aligned}$$

etc.

4.9.3. *Action on (x, t) -dependent nonlocal symmetries.* We have

$$\begin{aligned} R_{4,1}(\bar{\varphi}_{0,1}) &= \frac{1}{2}\bar{\varphi}_{4,1} \\ R_{4,1}(\bar{\varphi}_{0,2}) &= \frac{1}{4}\bar{\varphi}_{4,2} \\ R_{4,1}(\bar{\varphi}_{4,1}) &= -\frac{1}{208}\varphi_{8,2} + \frac{1}{8}\bar{\varphi}_{8,1} \\ R_{4,1}(\bar{\varphi}_{4,2}) &= \frac{1}{156}\varphi_{8,2} + \frac{1}{24}\bar{\varphi}_{8,2} \\ R_{4,1}(\bar{\varphi}_{8,1}) &= 2\varphi_{12,2} + \frac{55}{2}\varphi_{12,1} + \frac{1}{60}\bar{\varphi}_{12,1} \\ R_{4,1}(\bar{\varphi}_{8,2}) &= -6\varphi_{12,2} - 88\varphi_{12,1} + \frac{1}{4}\bar{\varphi}_{12,2} \end{aligned}$$

and

$$\begin{aligned} R_{4,2}(\bar{\varphi}_{0,1}) &= -\frac{1}{8}\varphi_{4,1} + \frac{1}{4}\bar{\varphi}_{4,1}, \\ R_{4,2}(\bar{\varphi}_{0,2}) &= \frac{1}{8}\varphi_{4,1} + \frac{3}{16}\bar{\varphi}_{4,2}, \\ R_{4,2}(\bar{\varphi}_{4,1}) &= -\frac{1}{208}\varphi_{8,2} + \frac{3}{16}\bar{\varphi}_{8,1}, \\ R_{4,2}(\bar{\varphi}_{4,2}) &= \frac{1}{156}\varphi_{8,2} + \frac{7}{96}\bar{\varphi}_{8,2}, \\ R_{4,2}(\bar{\varphi}_{8,1}) &= \frac{7}{2}\varphi_{12,2} + \frac{165}{4}\varphi_{12,1} + \frac{1}{24}\bar{\varphi}_{12,1}, \\ R_{4,2}(\bar{\varphi}_{8,2}) &= -\frac{21}{2}\varphi_{12,2} - 132\varphi_{12,1} + \frac{11}{16}\bar{\varphi}_{12,2}. \end{aligned}$$

4.10. **Action of \mathcal{R}^* on cosymmetries.** In a similar way, we have the following actions of \mathcal{R}^* :

4.10.1. *Action on local cosymmetries.*

$$R_{4,1}^*(\psi_2) = \frac{1}{2}\psi_6, \quad R_{4,1}^*(\psi_3) = \frac{1}{2}\psi_7, \quad R_{4,1}^*(\psi_4) = \frac{7}{12}\psi_8,$$

$$\begin{aligned}
R_{4,1}^*(\psi_6) &= \frac{11}{8}\psi_{10}, & R_{4,1}^*(\psi_7) &= \frac{1}{4}\psi_{11}, & R_{4,1}^*(\psi_8) &= \frac{1}{56}\psi_{12}, \\
R_{4,1}^*(\psi_{10}) &= \frac{91}{330}\psi_{14}, & R_{4,1}^*(\psi_{11}) &= \frac{1}{8}\psi_{15}, & R_{4,1}^*(\psi_{12}) &= \frac{57}{14}\psi_{16}
\end{aligned}$$

and

$$R_{4,2}^*(\psi_i) = \frac{i}{4}R_{4,1}^*(\psi_i). \quad (74)$$

4.10.2. *Action on (x, t) -independent nonlocal cosymmetries.*

$$\begin{aligned}
R_{4,1}^*(\psi_{9,1}) &= \frac{3}{14}\psi_{13,1} + \frac{9}{28}\psi_{13,2}, \\
R_{4,1}^*(\psi_{13,1}) &= \frac{91}{18}\psi_{17,1} + \frac{1}{8}\psi_{17,2}, \\
R_{4,1}^*(\psi_{13,2}) &= -\frac{91}{54}\psi_{17,1} + \frac{19}{84}\psi_{17,3}, \\
R_{4,2}^*(\psi_{9,1}) &= \frac{3}{8}\psi_{13,1} + \frac{9}{14}\psi_{13,2}, \\
R_{4,2}^*(\psi_{13,1}) &= \frac{455}{36}\psi_{17,1} + \frac{11}{32}\psi_{17,2}, \\
R_{4,2}^*(\psi_{13,2}) &= -\frac{455}{108}\psi_{17,1} + \frac{19}{28}\psi_{17,3}.
\end{aligned}$$

4.10.3. *Action on (x, t) -dependent nonlocal cosymmetries.*

$$\begin{aligned}
R_{4,1}^* \circ \bar{\psi}_{1,1} &= \frac{1}{2}\bar{\psi}_{5,1} + \frac{1}{2}\bar{\psi}_{5,2}, \\
R_{4,1}^* \circ \bar{\psi}_{5,1} &= -\frac{7}{2}\psi_{9,1} + \frac{11}{8}\bar{\psi}_{9,1}, \\
R_{4,1}^* \circ \bar{\psi}_{5,2} &= \frac{7}{3}\psi_{9,1} + \frac{1}{4}\bar{\psi}_{9,2}, \\
R_{4,1}^* \circ \bar{\psi}_{9,1} &= \frac{2}{11}\psi_{13,1} + \frac{91}{330}\bar{\psi}_{13,1}, \\
R_{4,1}^* \circ \bar{\psi}_{9,2} &= -\frac{1}{2}\psi_{13,1} + \frac{1}{8}\bar{\psi}_{13,2}, \\
R_{4,2}^* \circ \bar{\psi}_{1,1} &= \frac{1}{4}\bar{\psi}_{5,1} + \frac{3}{8}\bar{\psi}_{5,2}, \\
R_{4,2}^* \circ \bar{\psi}_{5,1} &= -\frac{7}{2}\psi_{9,1} + \frac{33}{16}\bar{\psi}_{9,1}, \\
R_{4,2}^* \circ \bar{\psi}_{5,2} &= \frac{7}{3}\psi_{9,1} + \frac{7}{16}\bar{\psi}_{9,2}, \\
R_{4,2}^* \circ \bar{\psi}_{9,1} &= \frac{7}{22}\psi_{13,1} + \frac{91}{132}\bar{\psi}_{13,1},
\end{aligned}$$

$$R_{4,2}^* \circ \bar{\psi}_{9,2} = -\frac{7}{8}\psi_{13,1} + \frac{11}{32}\bar{\psi}_{13,2}.$$

4.11. Commutators.

4.11.1. $[\bullet, \varphi_i]$ -commutators. First of all we have the relation

$$[\varphi_i, \varphi_j] = 0,$$

and besides these relations we obtain:

$$\begin{aligned} [\varphi_{4,1}, \varphi_1] &= 0, & [\varphi_{8,1}, \varphi_1] &= 0, & [\varphi_{8,2}, \varphi_1] &= 0, \\ [\varphi_{4,1}, \varphi_2] &= 0, & [\varphi_{8,1}, \varphi_2] &= 0, & [\varphi_{8,2}, \varphi_2] &= 0, \\ [\varphi_{4,1}, \varphi_3] &= \frac{1}{3}\varphi_7, & [\varphi_{8,1}, \varphi_3] &= \frac{14}{3}\varphi_{11}, & [\varphi_{8,2}, \varphi_3] &= -\frac{26}{3}\varphi_{11}, \\ [\varphi_{4,1}, \varphi_5] &= 3\varphi_9, & [\varphi_{8,1}, \varphi_5] &= \frac{21}{5}\varphi_{13}, & [\varphi_{8,2}, \varphi_5] &= -\frac{26}{5}\varphi_{13}, \\ [\varphi_{4,1}, \varphi_6] &= \frac{5}{3}\varphi_{10}, & [\varphi_{8,1}, \varphi_6] &= \frac{245}{6}\varphi_{14}, & [\varphi_{8,2}, \varphi_6] &= -\frac{130}{3}\varphi_{14}, \\ [\varphi_{4,1}, \varphi_7] &= 15\varphi_{11}, & [\varphi_{8,1}, \varphi_7] &= 60\varphi_{15}, & [\varphi_{8,2}, \varphi_7] &= -\frac{390}{7}\varphi_{15}, \\ [\varphi_{4,1}, \varphi_9] &= \frac{28}{15}\varphi_{13}, \\ [\varphi_{4,1}, \varphi_{10}] &= 36\varphi_{14}, \\ [\varphi_{4,1}, \varphi_{11}] &= \frac{45}{7}\varphi_{15}, \end{aligned}$$

and

$$\begin{aligned} [\bar{\varphi}_{0,1}, \varphi_1] &= \varphi_1, & [\bar{\varphi}_{0,2}, \varphi_1] &= 0, \\ [\bar{\varphi}_{0,1}, \varphi_2] &= 0, & [\bar{\varphi}_{0,2}, \varphi_2] &= \varphi_2, \\ [\bar{\varphi}_{0,1}, \varphi_3] &= -\varphi_3, & [\bar{\varphi}_{0,2}, \varphi_3] &= 2\varphi_3, \\ [\bar{\varphi}_{0,1}, \varphi_5] &= -3\varphi_5, & [\bar{\varphi}_{0,2}, \varphi_5] &= 4\varphi_5, \\ [\bar{\varphi}_{0,1}, \varphi_6] &= -4\varphi_6, & [\bar{\varphi}_{0,2}, \varphi_6] &= 5\varphi_6, \\ [\bar{\varphi}_{0,1}, \varphi_7] &= -5\varphi_7, & [\bar{\varphi}_{0,2}, \varphi_7] &= 6\varphi_7, \\ [\bar{\varphi}_{4,1}, \varphi_1] &= \varphi_5, & [\bar{\varphi}_{4,2}, \varphi_1] &= 0, \\ [\bar{\varphi}_{4,1}, \varphi_2] &= 0, & [\bar{\varphi}_{4,2}, \varphi_2] &= \varphi_6, \\ [\bar{\varphi}_{4,1}, \varphi_3] &= -\frac{1}{6}\varphi_7, & [\bar{\varphi}_{4,2}, \varphi_3] &= \frac{2}{3}\varphi_7, \\ [\bar{\varphi}_{4,1}, \varphi_5] &= -\frac{3}{4}\varphi_9, & [\bar{\varphi}_{4,2}, \varphi_5] &= 2\varphi_9, \\ [\bar{\varphi}_{4,1}, \varphi_6] &= -\frac{1}{3}\varphi_{10}, & [\bar{\varphi}_{4,2}, \varphi_6] &= \frac{5}{6}\varphi_{10}, \end{aligned}$$

$$\begin{aligned}
[\bar{\varphi}_{4,1}, \varphi_7] &= -\frac{5}{2}\varphi_{11}, & [\bar{\varphi}_{4,2}, \varphi_7] &= \frac{6}{\varphi_{11}}, \\
[\bar{\varphi}_{8,1}, \varphi_1] &= \varphi_9, & [\bar{\varphi}_{8,2}, \varphi_1] &= 0, \\
[\bar{\varphi}_{8,1}, \varphi_2] &= 0, & [\bar{\varphi}_{8,2}, \varphi_2] &= \varphi_{10}, \\
[\bar{\varphi}_{8,1}, \varphi_3] &= -\frac{2}{3}\varphi_{11}, & [\bar{\varphi}_{8,2}, \varphi_3] &= \frac{16}{3}\varphi_{11}, \\
[\bar{\varphi}_{8,1}, \varphi_5] &= -\frac{3}{10}\varphi_{13}, & [\bar{\varphi}_{8,2}, \varphi_5] &= \frac{8}{5}\varphi_{13}, \\
[\bar{\varphi}_{8,1}, \varphi_6] &= -\frac{7}{3}\varphi_{14}, & [\bar{\varphi}_{8,2}, \varphi_6] &= \frac{35}{3}\varphi_{14}, \\
[\bar{\varphi}_{8,1}, \varphi_7] &= -\frac{20}{7}\varphi_{15}, & [\bar{\varphi}_{8,2}, \varphi_7] &= \frac{96}{7}\varphi_{15}.
\end{aligned}$$

4.11.2. $[\bullet, \varphi_{i,j}]$ -commutators. We computed the following ones

$$\begin{aligned}
[\varphi_{4,1}, \varphi_{8,1}] &= 9240\varphi_{12,1} + 840\varphi_{12,2} + 504\varphi_{12,3}, \\
[\varphi_{4,1}, \varphi_{8,2}] &= 34320\varphi_{12,1} + 6240\varphi_{12,2} + 13104\varphi_{12,3}
\end{aligned}$$

and

$$\begin{aligned}
[\bar{\varphi}_{0,1}, \varphi_{4,1}] &= -4\varphi_{4,1}, \\
[\bar{\varphi}_{0,1}, \varphi_{8,1}] &= -8\varphi_{8,1}, \\
[\bar{\varphi}_{0,1}, \varphi_{8,2}] &= -8\varphi_{8,2}, \\
[\bar{\varphi}_{0,2}, \varphi_{4,1}] &= 4\varphi_{4,1}, \\
[\bar{\varphi}_{0,2}, \varphi_{8,1}] &= 8\varphi_{8,1}, \\
[\bar{\varphi}_{0,2}, \varphi_{8,2}] &= 8\varphi_{8,2}, \\
[\bar{\varphi}_{4,1}, \varphi_{4,1}] &= \frac{1}{26}\varphi_{8,2} - 3\bar{\varphi}_{8,1}, \\
[\bar{\varphi}_{4,1}, \varphi_{8,1}] &= 252\varphi_{12,3} - 168\varphi_{12,2} - 2310\varphi_{12,1} - \frac{21}{5}\bar{\varphi}_{12,1}, \\
[\bar{\varphi}_{4,1}, \varphi_{8,2}] &= -2808\varphi_{12,3} - 1248\varphi_{12,2} - 8580\varphi_{12,1} + \frac{26}{5}\bar{\varphi}_{12,1}, \\
[\bar{\varphi}_{4,2}, \varphi_{4,1}] &= -\frac{4}{39}\varphi_{8,2} - \frac{5}{3}\bar{\varphi}_{8,2}, \\
[\bar{\varphi}_{4,2}, \varphi_{8,1}] &= -336\varphi_{12,3} + 420\varphi_{12,2} + 6160\varphi_{12,1} - \frac{245}{6}\bar{\varphi}_{12,2}, \\
[\bar{\varphi}_{4,2}, \varphi_{8,2}] &= 6240\varphi_{12,3} + 3120\varphi_{12,2} + 22880\varphi_{12,1} + \frac{130}{3}\bar{\varphi}_{12,2}, \\
[\bar{\varphi}_{8,1}, \varphi_{4,1}] &= 72\varphi_{12,3} - 80\varphi_{12,2} - 880\varphi_{12,1} - \frac{28}{15}\bar{\varphi}_{12,1}, \\
[\bar{\varphi}_{8,2}, \varphi_{4,1}] &= -576\varphi_{12,3} + 96\varphi_{12,2} + 2112\varphi_{12,1} - 36\bar{\varphi}_{12,2}.
\end{aligned}$$

4.11.3. $[\bullet, \bar{\varphi}_{i,j}]$ -commutators. We computed the following commutators of this type

$$\begin{aligned}
[\bar{\varphi}_{0,1}, \bar{\varphi}_{0,2}] &= 0, \\
[\bar{\varphi}_{0,1}, \bar{\varphi}_{4,1}] &= -4\bar{\varphi}_{4,1}, \\
[\bar{\varphi}_{0,1}, \bar{\varphi}_{4,2}] &= -4\bar{\varphi}_{4,2}, \\
[\bar{\varphi}_{0,1}, \bar{\varphi}_{8,1}] &= -8\bar{\varphi}_{8,1}, \\
[\bar{\varphi}_{0,1}, \bar{\varphi}_{8,2}] &= -8\bar{\varphi}_{8,2}, \\
[\bar{\varphi}_{0,2}, \bar{\varphi}_{4,1}] &= 4\bar{\varphi}_{4,1}, \\
[\bar{\varphi}_{0,2}, \bar{\varphi}_{4,2}] &= 4\bar{\varphi}_{4,2}, \\
[\bar{\varphi}_{0,2}, \bar{\varphi}_{8,1}] &= 8\bar{\varphi}_{8,1}, \\
[\bar{\varphi}_{0,2}, \bar{\varphi}_{8,2}] &= 8\bar{\varphi}_{8,2}, \\
[\bar{\varphi}_{4,1}, \bar{\varphi}_{4,2}] &= \frac{1}{39}\varphi_{8,2} - 2\bar{\varphi}_{8,1} - \frac{1}{3}\bar{\varphi}_{8,2}, \\
[\bar{\varphi}_{4,1}, \bar{\varphi}_{8,1}] &= -36\varphi_{12,3} - 16\varphi_{12,2} - 110\varphi_{12,1} + \frac{1}{15}\bar{\varphi}_{12,1}, \\
[\bar{\varphi}_{4,1}, \bar{\varphi}_{8,2}] &= 288\varphi_{12,3} + 96\varphi_{12,2} + 528\varphi_{12,1} - \frac{8}{5}\bar{\varphi}_{12,1} - 4\bar{\varphi}_{12,2}, \\
[\bar{\varphi}_{4,2}, \bar{\varphi}_{8,1}] &= 48\varphi_{12,3} + 40\varphi_{12,2} + 352\varphi_{12,1} + \frac{8}{15}\bar{\varphi}_{12,1} + \frac{7}{3}\bar{\varphi}_{12,2}, \\
[\bar{\varphi}_{4,2}, \bar{\varphi}_{8,2}] &= -384\varphi_{12,3} - 192\varphi_{12,2} - 1408\varphi_{12,1} - \frac{8}{3}\bar{\varphi}_{12,2}.
\end{aligned}$$

5. CONCLUSIONS

We studied the dispersionless Boussinesq equation

$$\begin{aligned}
w_t &= u_x, \\
u_t &= ww_x + v_x, \\
v_t &= -uw_x - 3wu_x
\end{aligned}$$

and found out that it possesses three families of commuting local symmetries, Φ_1 , Φ_2 and Φ_3 . The seed symmetries for these families are φ_1 , φ_2 and φ_3 presented in Subsection 2.2.1. In addition, the equation has a family Φ_4 of nonlocal (x, t) -independent symmetries generated by the symmetry $\varphi_{4,1}$ (Subsection 2.2.2) and two families, $\bar{\Phi}_1$ and $\bar{\Phi}_2$, of nonlocal (x, t) -dependent symmetries generated by $\bar{\varphi}_{0,1}$ and $\bar{\varphi}_{0,2}$ (Subsection 2.2.3). All nonlocal symmetries act, by the commutator, as hereditary symmetries for the local ones.

We also constructed an infinite algebra $\text{Rec}(\mathcal{E})$ of recursion operators for symmetries. This is an associative noncommutative algebra with two generators $\mathcal{R}_{4,1}$ and $\mathcal{R}_{4,2}$ and one relation (see Subsection 3.1 and equation (68)). The operators produce the above mentioned families from the corresponding seed symmetries. This algebra contains an

Abelian subalgebra of operators satisfying the Nijenhuis condition. The subalgebra is generated by operator (70). Existence of this operator is the reason for commutativity of the families Φ_1 , Φ_2 and Φ_3 .

In complete parallel, one observes five families of cosymmetries and an algebra of the corresponding recursion operator that generate these families. This similarity is due to existence of operators $\mathcal{H}: \text{sym}^*(\mathcal{E}) \rightarrow \text{sym}(\mathcal{E})$ and $\mathcal{S}: \text{sym}(\mathcal{E}) \rightarrow \text{sym}^*(\mathcal{E})$ that relate symmetries and cosymmetries to each other and are described in Subsections 3.3 and 3.4, resp.

The operators \mathcal{H} form a left module over the algebra $\text{Rec}(\mathcal{E})$ with one generator $\mathcal{H}_{0,1}$. This operator is a Hamiltonian structure for the dB-equation and by the action of recursion operator (70) one obtains an infinite family of compatible Hamiltonian structures of which the first two are local. In a similar way, the operators \mathcal{S} form a left module over the algebra of recursion operators for cosymmetries, while the recursion operator given by equation (71) generates an infinite family of symplectic structures, all of them being nonlocal.

On the other hand, the dB-equation \mathcal{E} admits the potential y defined by

$$y_x = w, \quad y_t = u$$

identical to the nonlocal variable p_1 (see Subsection 2.1). The corresponding covering equation \mathcal{E}_1 is of the form

$$y_{tt} = y_x y_{xx} + v_t, \quad v_t = -y_t y_{xx} - 3y_x y_{xt}.$$

The potential z defined by

$$z_x = u, \quad z_t = v + \frac{1}{2}w^2,$$

that corresponds to the nonlocal variable p_2 leads to the covering equation \mathcal{E}_2

$$w_t = z_{xx}, \quad z_{tt} - 2ww_t = -z_x w_x - 3wz_{xx}.$$

Finally, the equation \mathcal{E}'

$$y_{ttt} + 2y_x y_{xxt} + y_t y_{xtt} + 3y_{xx} y_{xt} = 0.$$

This means that we have the following covering structure

$$\begin{array}{ccc} \mathcal{E} & \longleftarrow & \mathcal{E}_1 \\ \uparrow & & \downarrow \\ \mathcal{E}_2 & \longrightarrow & \mathcal{E}' \end{array}$$

that relates equations \mathcal{E} and \mathcal{E}' . From this structure it follows that an auto-Bäcklund transformation can be constructed for the dB-equation. Existence of this transformation may be closely related to the algebraic structures described in the preceding sections.

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