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M. Marvan, A. M. Vinogradov, V. A. Yumaguzhin

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Differential invariants of generic hyperbolic Monge–Ampère equations

M. MARVAN, A. M. VINOGRADOV, V. A. YUMAGUZHIN

Abstract. In this paper, we construct differential invariants of generic hyperbolic Monge–Ampère equations with respect to contact transformations. We give a solution of the equivalence problem for these equations.

Introduction. In this paper we look for differential invariants of classical Monge–Ampère equations of hyperbolic type with respect to the group of contact transformations by interpreting them as a geometrical structure on 5-dimensional contact manifolds. We limit ourselves to the case of generic equations and solve the equivalence problem for them by the approach developed by the second author, see [10],[1].

In spite of more than 200 years of history of Monge–Ampère equations and numerous publications dedicated to them it would be an exaggeration to say that their nature is completely understood. Establishing the existence and uniqueness theorems for this class of equations (see [4, 3] for local aspects and [9] for global ones) was an important achievement of the classical theory. For a modern exposition of elements of the classical theory, say, of Monge's method of integration, see [5, 6] and Morimoto [7].

Differential invariants of non-generic equations and some applications will be given in subsequent papers.

Below, all manifolds and maps are supposed to be smooth. By $[f]_p^k$, $k = 0, 1, 2, \ldots$ denote the k-jet of a map f at a point p. As usually, $\mathbb{R}^n$ stands for the n-dimensional arithmetic space, $\mathbb{R}$ being the real numbers, and $\mathbb{RP}^n$ for the n-dimensional projective space.

Bundle of Monge–Ampère equations. Consider the trivial bundle

$$\tau : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$$

and the manifold M of all 1-jets of its sections. Natural coordinates in M are denoted by (x, y, z, z_x, z_y). Classically Monge–Ampère equations are defined to be equations of the form

$$N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0\,, \qquad (1)$$

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where the coefficients N, A, B, C, D are functions of x, y, z, z_x, z_y. Geometrically such an equation may be seen as the section

$$p \mapsto \big[N(p) : A(p) : B(p) : C(p) : D(p)\big]$$

of the trivial bundle

$$\pi' : \mathbb{R}\mathrm{P}^4 \times M \to M\,.$$

Equation (1) is called hyperbolic if

$$B^2 - 4AC + 4ND > 0\,,$$

see [2]. This condition defines the open subset $E \subset \mathbb{R}\mathrm{P}^4 \times M$. Put

$$\pi = \pi'\big|_E : E \to M\,.$$

Recall that M has a canonical contact structure, see [8]. Contact transformations of M preserve the class of hyperbolic Monge–Ampère equations. In particular, such a transformation φ is lifted to a diffeomorphism $\varphi^{(0)}$ of E such that $\varphi \circ \pi = \varphi^{(0)} \circ \pi$. Let

$$\pi_k : J^k\pi \to M\,, \quad \pi_k : [S]^k_p \mapsto p\,,$$

be the bundle of all k-jets of sections S of π, $k = 0, 1, 2, \dots$ $\varphi^{(0)}$ can be lifted to the diffeomorphism $\varphi^{(k)}$ of $J^k\pi$ by the formula

$$\varphi^{(k)}(\,[S]^k_p\,) = \big[\,\varphi^{(0)} \circ S \circ \varphi^{-1}\,\big]^k_{\varphi(p)}\,.$$

Let Γ be the pseudogroup of all contact transformations of M. It acts on $J^k\pi$'s by lifted diffeomorphisms.

Proposition 1. (1) *Γ acts transitively on $J^0\pi$ and $J^1\pi$.*
(2) *Generic orbits of Γ on $J^2\pi$ are of codimension 2, and on $J^3\pi$ of codimension 29.*

Let I be a tensor on $J^k\pi$, say, a function, vector field or a differential form. It is a *differential invariant of order k* if $\varphi^{(k)}$ preserves I for any $\varphi \in \Gamma$. Functions that are differential invariants are also called *scalar differential invariants*, see [1].

Corollary 1. (1) *The algebra of scalar differential invariants on $J^2\pi$ is generated by 2 functionally independent invariants.*
(2) *The algebra of scalar differential invariants on $J^3\pi$ is generated by 29 functionally independent invariants.*

Let $\mathcal{E}$ be equation (1), S the corresponding section of π, and I a differential invariant of order k. Denote by $I_{\mathcal{E}}$ the restriction of I to the image of j_kS. Since j_kS is a diffeomorphism on its image, $I_{\mathcal{E}}$ may be viewed as a tensor on M. Below we describe differential invariants I in terms of their restrictions $I_{\mathcal{E}}$, i.e., as tensors on M. The subscript $\mathcal{E}$ will be usually omitted.

Skew-orthogonal distributions associated with a hyperbolic Monge–Ampère equation. Let

$$\tau_k : J^k\tau \to \mathbb{R}^2\,, \quad \tau_k : [S]^k_x \mapsto x\,,$$

be the bundle of all k-jets of sections of τ. A natural projection

$$\tau_{k,l} : J^k\tau \longrightarrow J^l\tau,\ k \geq l\,,$$

sends $[S]^k_x$ to $[S]^l_x$. Sections j_kS of τ_k, $k \geq 0$, associated with a section S of τ, are defined as

$$j_kS : x \mapsto [S]^k_x\,.$$

Put $L^k_S = \operatorname{Im} j_kS$. Let $[S]^2_x = q \in J^2\tau$. Recall that q is naturally identified with the tangent space K_q to L^1_S at $[S]^1_x$. The canonical contact 1-form on $M = J^1\tau$ is denoted by U_1 and the distribution of contact hyperplanes on M by

$$\mathcal{C} : p \mapsto \mathcal{C}_p\,.$$

Below, equation (1) is considered to be a submanifold $\mathcal{E}$ of $J^2\tau$, as usual. Let P be a one-dimensional subspace of $\mathcal{C}_p$, $p \in M$, such that $(\tau_1)_*P \neq 0$. Put

$$l(P) = \{\, q \in (\tau_{2,1})^{-1}(p) \,\big|\, P \subset K_q \,\}\,.$$

Denote by $\mathcal{Q}_p$ the span of all one-dimensional subspaces P of $\mathcal{C}_p$ such that $\tau_*P \neq 0$ and $l(P)$ is tangent to $\mathcal{E}$ at least at one point.

Proposition 2. *Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation. Then $\mathcal{Q}_p$ is the union of two-dimensional subspaces $\mathcal{D}^1_p$ and $\mathcal{D}^2_p$ such that*

(1) $\mathcal{C}_p = \mathcal{D}^1_p \oplus \mathcal{D}^2_p$,
(2) $\mathcal{D}^1_p$ *and* $\mathcal{D}^2_p$ *are skew-orthogonal with respect to the symplectic form* $dU_1\big|_p$.

In this way $\mathcal{E}$ determines a pair of 2-dimensional skew-orthogonal subdistributions

$$\mathcal{D}^1 : p \mapsto \mathcal{D}^1_p \quad \text{and} \quad \mathcal{D}^2 : p \mapsto \mathcal{D}^2_p$$

of the contact distribution $\mathcal{C}$.

Proposition 3. *Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation and $q \in J^2\tau$. Then $q \in \mathcal{E}$ iff one of the following equivalent conditions holds:*

(1) $\dim K_q \cap \mathcal{D}^1_p = 1$,
(2) $\dim K_q \cap \mathcal{D}^2_p = 1$.

In its turn, a pair of 2-dimensional skew-orthogonal subdistributions $\mathcal{D}^1$ and $\mathcal{D}^2$ of $\mathcal{C}$ determines a submanifold $\mathcal{E} = \{q \in J^2\tau \mid \dim K_q \cap \mathcal{D}^i = 1,\ i = 1, 2\} \in J^2\pi$ which is a hyperbolic Monge–Ampère equation.

So, there exists a natural bijection between hyperbolic Monge–Ampère equations and pairs of 2-dimensional skew-orthogonal non-lagrangian subdistributions $\mathcal{D}^1$, $\mathcal{D}^2$ of $\mathcal{C}$ on M. In particular, the equivalence problem for hyperbolic Monge–Ampère equations with respect to contact transformations is equivalent to that for such pairs of subdistributions.

Projections. For a distribution $\mathcal{D}$ on M denote by $\mathcal{D}^{(1)}$ the distribution generated by all vector fields $X, Y \in \mathcal{D}$ and their commutators $[X, Y]$. Put $\mathcal{D}^{(2)} = (\mathcal{D}^{(1)})^{(1)}$.

For a hyperbolic Monge–Ampère equation $\mathcal{E}$ we have

$$\dim(\mathcal{D}^1)^{(1)} = \dim(\mathcal{D}^2)^{(1)} = 3 \quad \text{and} \quad D^3 = (\mathcal{D}^1)^{(2)} \cap (\mathcal{D}^2)^{(2)}$$

is a one-dimensional distribution not belonging to $\mathcal{C}$. In this way one gets the decomposition

$$T(M) = \mathcal{D}^1 \oplus \mathcal{D}^2 \oplus \mathcal{D}^3$$

and the corresponding projections

$$\mathcal{P}_i : T(M) \to \mathcal{D}^i,\ i = 1, 2, 3, \quad \mathcal{P}_j^{(1)} : T(M) \to \mathcal{D}^j \oplus \mathcal{D}^3,\ j = 1, 2\,.$$

Interpreted as vector-valued 1-forms these projections have the following description. Let $X_1, \ldots, X_5$ be vector fields on M such that

$$\mathcal{D}^1 = \langle X_1, X_2 \rangle\,, \quad \mathcal{D}^2 = \langle X_3, X_4 \rangle\,, \quad \text{and} \quad \mathcal{D}^3 = \langle X_5 \rangle\,.$$

Denote by $\omega^1, \ldots, \omega^5$ the dual 1-forms, i.e., $X_j \lrcorner\, \omega^i = \delta^i_j$, $i, j = 1, \ldots, 5$. Then

$$\begin{aligned} &\mathcal{P}_1 = \omega^1 \otimes X_1 + \omega^2 \otimes X_2\,, \quad &&\mathcal{P}_2 = \omega^3 \otimes X_3 + \omega^4 \otimes X_4\,, \\ &\mathcal{P}_3 = \omega^5 \otimes X_5\,, &&\mathcal{P}_j^{(1)} = \mathcal{P}_j + \mathcal{P}_3\,, \quad j = 1, 2\,. \end{aligned}$$

These vector-valued differential 1-forms are differential invariants of $\mathcal{E}$ with respect to contact transformations.

For a generic $\mathcal{E}$,

$$\dim(\mathcal{D}^1)^{(2)} = \dim(\mathcal{D}^2)^{(2)} = 5\,.$$

Curvature forms. Let $P : TM \to TM$ be a projection. Then the *curvature* $\mathcal{R}$ *of* P is a 2-form on M defined by the formula

$$\mathcal{R}(X, Y) = (\mathrm{id}_{TM} - P)(\,[P(X), P(Y)]\,)\,,$$

where id_{TM} is the identity map. The curvatures of projections $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_1^{(1)}$, $\mathcal{P}_2^{(1)}$ are:

$$\begin{aligned} &\mathcal{R}_1 = \omega^1 \wedge \omega^2 \otimes X_5\,, \quad \mathcal{R}_2 = \omega^3 \wedge \omega^4 \otimes X_5\,, \\ &\mathcal{R}_1^1 = (b^3_{15}\omega^1 + b^3_{25}\omega^2) \wedge \omega^5 \otimes X_3 + (b^4_{15}\omega^1 + b^4_{25}\omega^2) \wedge \omega^5 \otimes X_4\,, \\ &\mathcal{R}_2^1 = (b^1_{35}\omega^3 + b^1_{45}\omega^4) \wedge \omega^5 \otimes X_1 + (b^2_{35}\omega^3 + b^2_{45}\omega^4) \wedge \omega^5 \otimes X_2 \end{aligned}$$

respectively, with functions b^i_{jk} coming from relations

$$[X_j, X_k] = \sum_{i=1}^{5} b^i_{jk} X_i\,.$$

Clearly, these curvatures are differential invariants of $\mathcal{E}$.

Scalar invariants on $J^2\pi$. From invariant 5-forms:

$$\tfrac{1}{2}\big(\mathcal{R}_2^1 \lrcorner \mathcal{R}_1\big) \lrcorner \big(\mathcal{R}_2^1 \lrcorner \mathcal{R}_1\big) = \lambda_1 \cdot \omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5\,,$$
$$\tfrac{1}{2}\big(\mathcal{R}_1^1 \lrcorner \mathcal{R}_2\big) \lrcorner \big(\mathcal{R}_1^1 \lrcorner \mathcal{R}_2\big) = \lambda_2 \cdot \omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5\,,$$
$$\big(\mathcal{R}_2^1 \lrcorner \mathcal{R}_1\big) \lrcorner \big(\mathcal{R}_1^1 \lrcorner \mathcal{R}_2\big) = \lambda_{12} \cdot \omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5$$

with

$$\lambda_1 = b_{35}^2 b_{45}^1 - b_{35}^1 b_{45}^2\,, \quad \lambda_2 = b_{15}^4 b_{25}^3 - b_{15}^3 b_{25}^4\,,$$
$$\lambda_{12} = b_{15}^3 b_{35}^1 + b_{15}^4 b_{45}^1 + b_{25}^3 b_{35}^2 + b_{25}^4 b_{45}^2$$

one immediately derives scalar differential invariants

$$I^1 = \lambda_{12}/\lambda_1\,, \quad I^2 = \lambda_{12}/\lambda_2$$

on $J^2\pi$ due to the fact that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ for a generic $\mathcal{E}$.

Theorem 1. *As smooth function algebra, the algebra of scalar differential invariants on $J^2\pi$ is generated by I^1 and I^2.*

The complete parallelism. Consider the invariant 1-forms:

$$\Omega^1 = \mathcal{P}_1 \lrcorner dI^1 = X_1(I^1)\omega^1 + X_2(I^1)\omega^2\,,$$
$$\Omega^2 = \mathcal{P}_1 \lrcorner dI^2 = X_1(I^2)\omega^1 + X_2(I^2)\omega^2\,,$$
$$\Omega^3 = \mathcal{P}_2 \lrcorner dI^1 = X_3(I^1)\omega^3 + X_4(I^1)\omega^4\,,$$
$$\Omega^4 = \mathcal{P}_2 \lrcorner dI^2 = X_3(I^2)\omega^3 + X_4(I^2)\omega^4\,,$$
$$\Omega^5 = \mathcal{P}_3 \lrcorner dI^1 = X_5(I^1)\omega^5\,, \quad \tilde{\Omega}^5 = \mathcal{P}_1 \lrcorner dI^2 = X_5(I^2)\omega^5\,.$$

If $\mathcal{E}$ is generic, then

$$X_5(I^1) \neq 0\,, \quad X_5(I^2) \neq 0\,,$$
$$\Delta_1 = \begin{vmatrix} X_1(I^1) & X_2(I^1) \\ X_1(I^2) & X_2(I^2) \end{vmatrix} \neq 0\,, \quad \Delta_2 = \begin{vmatrix} X_3(I^1) & X_4(I^1) \\ X_3(I^2) & X_4(I^2) \end{vmatrix} \neq 0\,.$$

So, $\{\Omega^1, \ldots, \Omega^4, \Omega^5\}$ and $\{\Omega^1, \ldots, \Omega^4, \tilde{\Omega}^5\}$ are invariant coframes on M. Each of them defines an invariant complete parallelism on M.

Scalar invariants on $J^3\pi$. Coefficient of proportionality

$$I^3 = X_5(I^1)/X_5(I^2)$$

between invariant 1-forms Ω^5 and $\tilde{\Omega}^5$ is a scalar differential invariant on $J^3\pi$. By applying natural operations of linear algebra and tensor analysis to already obtained differential invariants one gets numerous new scalar differential invariants of $\mathcal{E}$. For instance, further invariants I^4 and I^5 are obtained as factors connecting invariant 2-forms $\mathcal{R}_1 \lrcorner dI^1$, $\mathcal{R}_2 \lrcorner dI^1$ and $\Omega^i \wedge \Omega^j$ on $J^3\pi$:

$$\mathcal{R}_1 \lrcorner dI^1 = I^4 \Omega^1 \wedge \Omega^2, \quad \mathcal{R}_2 \lrcorner dI^1 = I^5 \Omega^3 \wedge \Omega^4.$$

More exactly,

$$I^4 = \Delta_1/X_5(I^1)\,, \quad I^5 = \Delta_2/X_5(I^1)\,.$$

Theorem 2. *Scalar differential invariants $I^1, \ldots, I^5$ on $J^3\pi$ are functionally independent.*

The equivalence problem. For a generic hyperbolic Monge–Ampère equation $\mathcal{E}$ functions $I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}}$ form a natural *invariant* chart on M. Put

$$\Omega^i_{\mathcal{E}} = \sum_{j=1}^{5} \Omega^i_j(I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}}) dI^j_{\mathcal{E}}\,, \quad i = 1, \ldots, 5\,.$$

Theorem 3. *The equivalence class of a generic equation $\mathcal{E}$ with respect to contact transformations is uniquely determined by the functions $\Omega^i_j(I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}})$, $i, j = 1, \ldots, 5$.*

Invariant operators. Invariant vector fields Y_j are defined by duality relations $Y_j \rfloor \Omega^i = \delta^i_j$, $i, j = 1, \ldots, 5$. Consider invariant endomorphisms of vector bundles

$$\Box_1 = Y_5 \rfloor \mathcal{R}^1_1 : TM \to \mathcal{D}^2 \quad \text{and} \quad \Box_2 = Y_5 \rfloor \mathcal{R}^1_2 : TM \to \mathcal{D}^1$$

and their compositions

$$\nabla_1 = \Box_2\big|_{\mathcal{D}^2} \circ \Box_1\big|_{\mathcal{D}^1} \colon \mathcal{D}^1 \to \mathcal{D}^1 \quad \text{and} \quad \nabla_2 = \Box_1\big|_{\mathcal{D}^1} \circ \Box_2\big|_{\mathcal{D}^2} \colon \mathcal{D}^2 \to \mathcal{D}^2$$

which are invariant as well.

Theorem 4. *The endomorphism ∇_i of $\mathcal{D}_i$, $i = 1, 2$, has two, one, or zero different real eigenvalues, if $I > 0$, $I = 0$, $I < 0$, respectively, where $I = 4 - I^1 I^2$.*

Thus, generic hyperbolic Monge–Ampère equations are subdivided locally into three subclasses according to the sign of I.

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Mathematical Institute of the Silesian University at Opava, Na Rybnicku 1, 746 01 Opava, Czech Republic.
E-mail address: `Michal.Marvan@math.slu.cz`

University of Salerno, Dipartimento di Matematica ed Informatica, Facolta' di Scienze Matematiche, Fisiche e Naturali, Universita' degli Studi di Salerno, Via Ponte del Melillo, 84084 Fisciano (SA), Italy, and Diffiety Institute, Russia.
E-mail address: `vinograd@unisa.it`

Program Systems Institute of RAS, 152020, Pereslavl'-Zalesskiy, m.Botik, Russia.
E-mail address: `yuma@diffiety.botik.ru`