The Diffiety Institute Preprint Series

Preprint DIPS–8/2004 November 3, 2004

Revised March 16, 2005

Ricci flat 4-metrics with bidimensional null orbits.

Part II. The Abelian case.

by

D. Catalano Ferraioli and A. M. Vinogradov

Available via INTERNET:
http://diffiety.ac.ru; http://diffiety.org

The Diffiety Institute

RICCI FLAT 4-METRICS WITH BIDIMENSIONAL NULL ORBITS.
PART II. THE ABELIAN CASE.

D. CATALANO FERRAIOLI AND A. M. VINOGRADOV

Abstract. Ricci flat pseudo-Riemannian 4-metrics possessing an Abelian Killing algebra with bidimesnional null orbits are studied under the assumption that the distribution orthogonal to the orbits is completely integrable. The results of this paper toghether with those of part I (see [4]) give a full description of Ricci flat metrics with bidimensional null Killing orbits whose distribution orthogonal to the orbits is completely integrable.

1. Introduction

This paper is a direct continuation of [4] and contains an exact description of Ricci flat 4-metrics g such that

(i) g admits a Killing algebra $\mathcal{G} \supseteq \mathcal{A}_2$ ($\mathcal{A}_2$ being an *Abelian bidimensional* Killing algebra) with bidimensional leaves (orbits of $\mathcal{G}$),

(ii) the distribution $\mathcal{D}^{\perp}$ orthogonal to Killing leaves is Frobenius (completely integrable),

(iii) g degenerates when restricted to any Killing leaf.

In [4] the reader will find all necessary preliminaries and, in particular, local normal forms of metrics in question. These metrics are, obviously, either Lorentzian (signature $\pm(-+++)$), or Kleinian (signature $(--++)$) and $\dim \mathcal{G} = 2, 3$ (see [4]). We say that such a metric is of type $(\mathcal{G}, r)$ if $\mathcal{G}$ is its Killing algebra and restrictions of g to orbits of $\mathcal{G}$ are of rank r. It is shown in [4] that types $(\mathcal{A}_2, 1)$,$(\mathcal{A}_2, 0)$,$(\mathcal{H}, 1)$ and $(\mathcal{A}_3, 0)$ are the only possible for metrics that are subject to (i)-(iii) above. Here $\mathcal{A}_i$ stands for an Abelian i-dimensional Lie algebra and $\mathcal{H}$ for the Heisenberg algebra. These four cases are studied in consecutive order and each time we give an exact analytical description of all metrics of the considered type in terms of a suitable local chart. These descriptions are not, however, in one-to-one correspondence with isometry classes of metrics. Elimination of this gauge freedom will be discussed in a separate paper together with construction of global forms and possible singularities.

It is worth stressing that $(\mathcal{A}_2, r)$-type Ricci flat metrics, $r = 0, 1$, are degenerate along Killing leaves analogues of metrics studied by Belinsky and Zakharov [3]. The fact

1991 *Mathematics Subject Classification.* primary 53B30, 53C25, 53C15; secondary 53B50, 83C35.

Key words and phrases. Einstein metrics, Ricci flat metrics, Killing Algebra, null orbits, Kleinian metrics, Lorentzian metrics.

that in our case the Einstein equations can be explicitly solved on the contrary to the ”nondegenerate” case shows that assumption (iii) is rather strong.

The paper is organized as follows.

In section 2, Ricci flat metrics of type $(\mathcal{A}_2, 1)$ are studied. These metrics are divided naturally into two classes, according to whether such a metric admits a null Killing field or not. Lorentzian metrics of the first class are known as pp-waves and were widely studied (see, for instance, [8],[8],[7]). In what concerns Lorentzian metrics of the second class some partial results were obtained by Bampi and Cianci (see [1], [6]). On the contrary Kleinian Ricci flat metrics of $(\mathcal{A}_2, 1)$-type described in section 2, to our knowledge, are

new as well as that of $(\mathcal{A}_2, 0)$-type described in section 3. Finally, in section 4, the results of previous sections are specialized to Killing algebras $\mathcal{A}_3$ and $\mathcal{H}$.

Throughout the paper the following notations and conventions are adopted:

- manifolds, maps, metrics, etc. are assumed to be smooth and manifolds to be connected;
- $\mathcal{D}(M)$ denotes the Lie algebra of all vector fields on a manifold M;
- the term *metric* refers to a non-degenerate symmetric $(0, 2)$- tensor field and g stands for a metric on M of the type we are studying;
- the term *Killing algebra* refers to a sub-algebra of the Lie algebra of all Killing vector fields of g;
- *integral submanifolds* of the distribution generated by vector fields of a Killing algebra $\mathcal{G}$ are called *Killing leaves* and the Killing leaf passing through $p \in M$ is denoted by $\mathcal{K}_p$;
- $\mathcal{D}$ denotes the tangent to Killing leaves bidimensional distribution and $\mathcal{D}^{\perp}$ the g-orthogonal to it;
- $\mathcal{A}_2$ and $\mathcal{G}_2$ stand for bidimensional Abelian and nonabelian Killing algebras, respectively;
- $\mathcal{A}_3$ and $\mathcal{H}$ stand for 3-dimensional Abelian and Heisenberg Killing algebras, respectively;
- $(\mathcal{G}, r)$ refers to the case of metrics of rank $r = 0, 1$ along 2-dimensional leaves of the Killing algebra $\mathcal{G}$;
- components of Ricci tensor $ric\,(X, Y) := tr\,(W \longmapsto R\,(W, X)\,Y)$ with respect to a frame field $\{e_i\}$ are denoted by $R_{x_i x_j}$ or $R_{(i)(j)}$ according to whether the frame field is holonomic (i.e., $e_i = \partial_{x_i}$ with respect to a local chart (x_i)) or not.

2. Ricci flat metrics of type $(\mathcal{A}_2, 1)$

In this case there exists a 1-dimensional distribution $\mathfrak{C}$ tangent to Killing leaves that associates to a point $p \in M$ the kernel $\mathfrak{C}_p$ of the tensor $g|_{\mathcal{K}_p}$ on $\mathcal{K}_p$ at p. The distribution $\mathfrak{C}$ and any vector field C belonging to it, will be called *characteristic* (see [4], section 2, for more details). It may happen that one of Killing fields is also characteristic but, generally, it is not so. We shall distinguish between these two cases.

Recall that (see [4], proposition 3) in a local *adapted chart* (x, y, u, v) a metric g of type $(\mathcal{A}_2, 1)$ reads

$$(1)\qquad \begin{aligned} g = \ & \varepsilon_0 \left(\beta dx - \alpha dy\right)^2 + 2adxdv + 2bdydv \\ & +\varepsilon_1 du^2 + 2pdudv + qdv^2 \end{aligned}$$

with $\varepsilon_0, \varepsilon_1 = \pm 1$ and $\alpha, \beta, a, b, r, p, q$ arbitrary functions in (u, v). In such a chart $\mathcal{A}_2$ is generated by $\{\partial_x, \partial_y\}$ and assumption (ii) is satisfied if there exists a function f such that

$$(2)\qquad \begin{cases} \alpha_u = f\alpha, \\ \beta_u = f\beta. \end{cases}$$

2.1. **Metrics possessing a characteristic Killing vector field.** In this case one can assume ∂_y to be such a field, i.e., that $\alpha = 0$ and compute the components of Ricci tensor for (1) in terms of the non-holonomic basis (see [4])

$$(3)\qquad \mathfrak{e}_1 = m^{-1}\left(b\partial_x - a\partial_y\right), \quad \mathfrak{e}_2 = m^{-1}C, \quad \mathfrak{e}_3 = \partial_u, \quad \mathfrak{e}_4 = \partial_v$$

with $m = \beta b \neq 0$.

Recall that in a non-holonomic frame field $\{e_i\}$, the components $R_{(i)(j)}$ of Ricci tensor ric are

$$R_{(i)(j)} = e_{[j}(\gamma^h_{h]i}) + \gamma^h_{[j|k|}\gamma^k_{h]i} - c^k_{jh}\gamma^h_{ki}$$

where the γ^l_{ij}'s are the Christoffel's symbols

$$\begin{aligned} \gamma^l_{ij} = \ & \frac{1}{2}g^{lh}\left(-e_h(g_{ij}) + e_i(g_{hj}) + e_j(g_{hi})\right) \\ & -\frac{1}{2}(c^l_{ji} + g^{kl}g_{hi}c^h_{jk} + g^{kl}g_{hj}c^h_{ik}) \end{aligned}$$

and the c^h_{ij}'s are such that $[e_i, e_j] = c^h_{ij}e_h$.

First, we have

$$R_{(1)(1)} = \frac{\varepsilon_0\varepsilon_1(fm)_u}{m}$$

so that $R_{(1)(1)} = 0$ iff

$$(4)\qquad (fm)_u = 0.$$

In view of (4) the component $R_{(2)(4)}$ takes the following simple form

$$R_{(2)(4)} = \frac{m_{uu}}{2\varepsilon_1 m}$$

and

$$(5)\qquad R_{(2)(4)} = 0 \Longleftrightarrow m_{uu} = 0.$$

Then, taking into account (4) and (5), one gets

$$R_{(3)(3)} = -\frac{m_u^2 + 2fmm_u - 3f^2m^2}{2m^2}$$

and $R_{(3)(3)} = 0 \Longleftrightarrow (m_u - fm)(m_u + 3fm) = 0$, i.e.,

$$m_u = \tau f m \tag{6}$$

with $\tau = -3, 1$.

In this case (4) gives

$$f_u = -\tau f^2. \tag{7}$$

Thus, taking into account (2), (6), (7) and the definition of m one gets:

$$\begin{cases} f(u,v) = \dfrac{k}{\tau k u + h}, \\ \beta(u,v) = \beta_0 (\tau k u + h)^{1/\tau}, \\ b(u,v) = b_0 (\tau k u + h)^{\frac{\tau-1}{\tau}} \end{cases} \tag{8}$$

with $k = 0, 1$ and h, β_0, b_0 being some functions in v.

In particular, $f = 0 \Longleftrightarrow k = 0$ and in this case $\beta = \beta(v)$, $b = b(v)$.

At this point, in virtue of (8), the only nonvanishing components of the Ricci tensor are

$$\begin{cases} R_{(1)(4)} = \dfrac{a_{uu} - fa_u - 2(1-\tau)f^2 a}{2\varepsilon_1 \beta} \\ R_{(3)(4)} = \dfrac{(1+\tau)f_v\beta + (3-\tau)f\beta_v}{2\beta} \\ R_{(4)(4)} = \dfrac{1}{2\varepsilon_1}\left[q_{uu} + (2-\tau) f q_u + (\tau-1)^2 f^2 q + Q\right] \end{cases} \tag{9}$$

with

$$\begin{aligned} Q = \;& 2\frac{b_v}{b}(p_u + fp) + 2(\tau-1)pf_v - 2fp_v - 2p_{uv} \\ & -2\varepsilon_1 \frac{b_v\beta_v}{b\beta} + 2\varepsilon_1\frac{\beta_{vv}}{\beta} - \varepsilon_0\left[(\tau-1)\frac{fa}{\beta} - \frac{a_u}{\beta}\right]^2. \end{aligned} \tag{10}$$

In the rest of this section we shall solve the corresponding Einstein equations.

The case $f \equiv 0$. In this case $k = 0$ (see (8)), so that $b = b(v)$, $\beta = \beta(v)$ and $m_v/m - \beta_v/\beta = b'/b$. So, components (9) vanish iff

$$a_{uu} = 0, \quad q_{uu} + Q = 0$$

and Q takes a simpler form:

$$Q = 2\frac{b'}{b}p_u - 2p_{uv} + 2\varepsilon_1\left(\frac{\beta''}{\beta} - \frac{b'\beta'}{b\beta}\right) - \varepsilon_0\frac{a_u^2}{\beta^2}.$$

These equations are readily solved:

$$\begin{aligned} a &= a_1 u + a_2, \\ q &= -\int \left(\int Q du + q_1 \right) du + q_0 \\ &= \left[\varepsilon_1 \left(\frac{b'\beta'}{b\beta} - \frac{\beta''}{\beta} \right) + \varepsilon_0 \frac{a_1^2}{2\beta^2} \right] u^2 - 2\frac{b'}{b} P + 2P_v + q_1 u + q_0 \end{aligned}$$

with a_1, a_2, q_0, q_1 arbitrary functions in v and $P = \int p du$.

Therefore, rearranging arbitrary functions, the general form of the metric is

$$\begin{aligned} g = \ \pm & \Bigg[\beta^2 dx^2 + 2\left(a_1 u + a_2\right) dx dv + 2b dy dv + 2P_u du dv \\ & + \varepsilon du^2 + \left\{ \left[\varepsilon \left(\frac{b'\beta'}{b\beta} - \frac{\beta''}{\beta} \right) + \frac{a_1^2}{2\beta^2} \right] u^2 + q_1 u + q_0 \right. \\ & \left. - 2\frac{b'}{b} P + 2P_v \right\} dv^2 \Bigg] \end{aligned} \tag{11}$$

with $\varepsilon = \pm 1$ and a_1, a_2, β, b being some functions in v, P a function in (u, v).

A much simpler expression for metric (11), one gets by passing to coordinates

$$\begin{cases} x' = u\mathcal{S}_\varepsilon \left(\int \frac{a_1}{2\beta} dv \right) + \left(\beta x + \int \frac{a_2}{\beta} dv \right) \mathcal{C}_\varepsilon \left(\int \frac{a_1}{2\beta} dv \right) \\ y' = -2by - a_1 u x + \beta\beta' x^2 - 2P - \frac{a_1}{\beta} u \int \frac{a_2}{\beta} dv \\ u' = -\varepsilon u \mathcal{C}_\varepsilon \left(\int \frac{a_1}{2\beta} dv \right) + \left(\beta x + \int \frac{a_2}{\beta} dv \right) \mathcal{S}_\varepsilon \left(\int \frac{a_1}{2\beta} dv \right) \\ v' = \frac{v}{2} \end{cases}$$

where $\mathcal{S}_\varepsilon(t) := \dfrac{e^{\sqrt{-\varepsilon}t} - e^{-\sqrt{-\varepsilon}t}}{2}$ and $\mathcal{C}_\varepsilon(t) := \dfrac{e^{\sqrt{-\varepsilon}t} + e^{-\sqrt{-\varepsilon}t}}{2}$, is the following:

$$g = \pm \left(dx'^2 + \varepsilon du'^2 - 2dy' dv' + H(x', u', v') dv'^2 \right) \tag{12}$$

This form is preserved under transformations

$$\begin{cases} \bar{x} = x' \mathcal{C}_\varepsilon(\phi) + u' \mathcal{S}_\varepsilon(\phi) + \eta \\ \bar{y} = \lambda \left[y' + x' \left(\dot{\eta} \mathcal{C}_\varepsilon(\phi) - \dot{\xi} \mathcal{S}_\varepsilon(\phi) \right) + u' \left(\dot{\eta} \mathcal{S}_\varepsilon(\phi) + \varepsilon \dot{\xi} \mathcal{C}_\varepsilon(\phi) \right) + V \right] \\ \bar{u} = -x' \mathcal{S}_\varepsilon(\phi) + \varepsilon u' \mathcal{C}_\varepsilon(\phi) + \varepsilon \xi \\ \bar{v} = \lambda^{-1} \left(v + \mu \right) \end{cases} \tag{13}$$

with $\lambda, \mu, \phi \in \mathbb{R}$, η, ξ and V arbitrary functions of v'. Here the dot stands for the derivation $\dfrac{d}{dv'}$.

Throughout a coordinate transformation of the form (13), the function H in the metric (12) is changed into

$$\begin{aligned}\bar{H} = \ &\lambda^2 \Big\{ H + 2x' \left(\ddot{\eta} \mathcal{C}_\varepsilon(\phi) - \ddot{\xi} \mathcal{S}_\varepsilon(\phi) \right) + 2u' \left(\ddot{\eta} \mathcal{S}_\varepsilon(\phi) + \varepsilon \ddot{\xi} \mathcal{C}_\varepsilon(\phi) \right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad + 2\dot{V} - \dot{\eta}^2 - \varepsilon \dot{\xi}^2 \Big\}\end{aligned}$$

Hence, the arbitrariness of $\lambda, \mu, \phi, \eta, \xi$ and V allows one to introduce coordinates $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ in which (12) reads

$$g = \pm \left(d\tilde{x}^2 + \varepsilon d\tilde{u}^2 - 2d\tilde{y}d\tilde{v} + \left[A_1 \left(\tilde{x}^2 - \varepsilon \tilde{u}^2 \right) + A_2 \tilde{x}\tilde{u} \right] d\tilde{v}^2 \right) \tag{14}$$

with A_1, A_2 arbitrary functions in $\tilde{v}$.

The case $f \neq 0$. In virtue of (8) components (9) vanish iff

$$\begin{aligned}&\beta_0 = const., \qquad a = \varepsilon_0 \left(\frac{a_1}{f^{\frac{2}{3}(2+\tau)}} + a_2 f^{\frac{1}{3}(\tau-1)} \right) \\ &q = \frac{\varepsilon_0}{f^{\frac{1}{3}(1-\tau)}} \left(q_0 + q_1 \ln|f| + \varepsilon_0 \frac{a_1^2}{\beta_0^2 f^{\tau+1}} - \frac{b_0'}{b_0} P + P_v \right) \\ &\qquad\qquad\qquad\qquad + \frac{\varepsilon \varepsilon_0 (1-\tau)}{4} \left(\frac{h'}{\tau} \right)^2\end{aligned}$$

with a_1, a_2, q_0, q_1 arbitrary functions in v and

$$P = 2 \int \left(\varepsilon_0 p - \frac{\varepsilon_1 h'}{\tau} \right) f^{\frac{1}{3}(1-\tau)} du.$$

Therefore, in an adapted chart the metric g looks as

$$
\begin{aligned}
g = \pm \Bigg\{ & \frac{\beta_0^2}{f^{\frac{2}{\tau}}} dx^2 + 2\left[\frac{a_1}{f^{\frac{2}{3}(2+\tau)}} + a_2 f^{\frac{1}{3}(\tau-1)}\right] dxdv + 2\frac{b_0}{f^{\frac{\tau-1}{\tau}}} dydv \\
& + \left(\frac{P_u}{f^{\frac{1-\tau}{3}}} + \frac{2\varepsilon h'}{\tau}\right) dudv + \varepsilon du^2 + \Bigg[\frac{(1-\tau)\,\varepsilon}{4}\left(\frac{h'}{\tau}\right)^2 \qquad (15) \\
& + \frac{\left(q_0 + q_1 \ln|f| + \frac{a_1^2}{\beta_0^2 f^{\tau+1}} - \frac{b_0'}{b_0}P + P_v\right)}{f^{\frac{1-\tau}{3}}}\Bigg] dv^2 \Bigg\}
\end{aligned}
$$

with $\varepsilon = \pm 1$, β_0 being a constant, a_1, a_2, h and b_0 some functions in v and P a function in (u, v).

In order to simplify expression (15) we shall distinguish sub-cases $\tau = 1$ and $\tau = -3$.

Sub-case $\tau = 1$. In the chart (x', y', u', v') defined by

$$
\begin{cases}
x' = (u+h)\mathcal{S}_\varepsilon\left(\beta_0 x + \dfrac{\int a_1 dv}{\beta_0}\right) \\
y' = -\left(y + \dfrac{a_2}{b_0}x + \dfrac{P}{2b_0}\right) \\
u' = (u+h)\mathcal{C}_\varepsilon\left(\beta_0 x + \dfrac{\int a_1 dv}{\beta_0}\right) \\
v' = \int b_0 dv
\end{cases}
$$

metric (15) takes the form (12). Then, similarly to the case $f = 0$, a suitable transformation of the form (13) brings (15) to

$$
g = \pm\left(d\tilde{x}^2 + \varepsilon d\tilde{u}^2 - 2d\tilde{y}d\tilde{v} + \left(A_1 \ln|\tilde{x}^2 + \varepsilon\tilde{u}^2| + A_2 S\right) d\tilde{v}^2\right) \qquad (16)
$$

with A_1, A_2 arbitrary functions in $\tilde{v}$ and

$$S = \begin{cases} \arcsin\left(\dfrac{\tilde{x}}{\sqrt{\tilde{u}^2+\tilde{x}^2}}\right) & \text{if } \varepsilon = 1 \\ \operatorname{arcsinh}\left(\dfrac{\tilde{x}}{\sqrt{\tilde{u}^2-\tilde{x}^2}}\right) & \text{if } \varepsilon = -1. \end{cases}$$

Sub-case $\tau = -3$. In the chart $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ defined by

$$\begin{cases} \bar{x} = \dfrac{1}{\sqrt[3]{4}}\left(\beta_0 x + \dfrac{\int a_1 dv}{\beta_0}\right) \\ \bar{y} = 8\sqrt[3]{\dfrac{2}{3}}\,(2b_0 y + 2a_2 x + P) \\ \bar{u} = \dfrac{1}{4\sqrt[3]{4}}(-3u+h)^{4/3} \\ \bar{v} = \dfrac{1}{2}\sqrt[3]{\dfrac{3}{4}}v \end{cases}$$

metric (15) takes the form

$$g = \pm\left[\frac{1}{\sqrt{\bar{u}}}\left(d\bar{x}^2 + \varepsilon d\bar{u}^2\right) + 2\bar{u}d\bar{y}d\bar{v} + \bar{u}\left(\mathcal{A}_1\bar{x} + \mathcal{A}_2\bar{y} + \mathcal{A}_3\ln(\bar{u}) + \mathcal{A}_4\right)d\bar{v}^2\right]$$

with $\mathcal{A}_i$'s arbitrary functions in $\bar{v}$.

Since this form is preserved under transformations $(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \longmapsto (\bar{x}, \alpha_1\bar{y} + \alpha_2, \bar{u}, \int \alpha_1 d\bar{v})$ with α_1 and α_2 arbitrary functions in $\bar{v}$, one can pass to coordinates $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ in terms of which the metric reads

$$g = \pm\left\{\frac{1}{\sqrt{\tilde{u}}}\left(d\tilde{x}^2 + \varepsilon d\tilde{u}^2\right) - 2\tilde{u}d\tilde{v}\left[d\tilde{y} + \left(A_1\tilde{x} + A_2\ln(\tilde{u})\right)d\tilde{v}\right]\right\} \tag{17}$$

with A_i's arbitrary functions in $\tilde{v}$.

The coordinates $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$, defined above in the cases $f = 0$ and $f \neq 0$, will be called *improved adapted*.

So, we have got the following result

Proposition 1. *In an adapted chart a Ricci-flat metric of type* $(\mathcal{A}_2, 1)$ *for which one of the Killing fields is characteristic has one of forms (11) or (15), and vice versa. In an improved adapted chart* $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ *such a metric takes form (14) for* $f = 0$ *and (16) or (17) for* $\tau = 1$ *or* -3*, respectively.*

Now observe that in the Lorentzian case $\varepsilon = 1$ the metrics described above belong to the following wider class of Ricci flat metrics:

$$g = \frac{1}{\sqrt{\tilde{u}}}\left(d\tilde{x}^2 + d\tilde{u}^2\right) - 2\tilde{u}d\tilde{v}\left(d\tilde{y} + M(\tilde{x},\tilde{u},\tilde{v})d\tilde{v}\right) \quad \text{with } \tilde{u}M_{\tilde{x}\tilde{x}} + \left(\tilde{u}M_{\tilde{u}}\right)_{\tilde{u}} = 0 \tag{18}$$

and

$$g = d\tilde{x}^2 + d\tilde{u}^2 - 2d\tilde{y}d\tilde{v} + 2H(\tilde{x},\tilde{u},\tilde{v})d\tilde{v}^2 \quad \text{with } H_{\tilde{x}\tilde{x}} + H_{\tilde{u}\tilde{u}} = 0 \tag{19}$$

which completely describe Lorentzian metrics admitting a null Killing vector field (see, for instance, [9], ch. 24 and the bibliography in it). Metrics (19) are well known and physically interpreted as plane-fronted gravitational waves with parallel rays, i.e., *pp-waves* (see [5], [9], [8]). When $H = A(\tilde{v})\left(\tilde{x}^2 - \tilde{u}^2\right) + B(\tilde{v})\tilde{x}\tilde{u}$, the pp-wave is called a plane wave.

Therefore, in the Lorentzian case $\varepsilon = 1$, metrics (16) are pp-waves and (14) are plane waves.

2.2. **Metrics without characteristic Killing vector fields.** Put $\alpha = \varepsilon_2 e^{\psi}$ with $\varepsilon_2 = \pm 1$. Then (2) gives $f = \psi_u$, $\beta = \beta_0 e^{\psi}$ with $\beta_0 = \beta_0(v)$ and the characteristic vector field reads $C = \alpha\left(\partial_x + \varepsilon_2\beta_0\partial_y\right)$. This shows that β_0 is not a constant: otherwise $\partial_x + \varepsilon_2\beta_0\partial_y$ would be a characteristic Killing vector field.

In the considered case, it is convenient to modify the adapted coordinates (x, y, u, v) as follows:

$$\begin{cases} x' = x + \displaystyle\int \frac{p}{a + \varepsilon_2\beta_0 b}du, \\ y' = \beta_0 x - \varepsilon_2 y, \\ u' = u, \\ v' = \beta_0. \end{cases} \tag{20}$$

Note that the restriction of the characteristic vector field $\partial_x + \varepsilon_2\beta_0\partial_y$ to the Killing leaf $\{u = u_0, v = v_0\}$ coincides with that of the Killing field $\partial_x + \lambda\partial_y$, $\lambda = \varepsilon_2\beta_0(v_0)$. So the function $\beta_0(v)$ tells which Killing field is characteristic along a given Killing leaf and acquires in this way an intrinsic meaning. This is the motivation to pass to the new coordinate v'.

In the chart (20) g takes the form

$$g = \pm\left\{2adx'dv' + \alpha^2 dy'\left[dy' + 2\left(A - x'\right)dv'\right] + \varepsilon_1 du'^2 + \left[\alpha^2\left(A - x'\right)^2 + q\right]dv'^2\right\} \tag{21}$$

with rearranged α, a, q and A arbitrary function in (u', v').

Chart (20) is good enough in the sense that allows to solve equation $ric(g) = 0$. However, the occurring technical difficulties simplify much by taking function a from (21) as a new coordinate function instead of u'. But this can be done if $a_{u'} \neq 0$. Assuming the contrary one finds that $R_{x'y'} = \alpha/(2a) \neq 0$. So, $a_{u'} \neq 0$ for Ricci flat metrics of form (21) and we can pass to the chart

$$\left\{ \begin{aligned} \bar{x} &= x' - \varepsilon_1 \int \frac{a_{v'}}{a a_{u'}} du', \\ \bar{y} &= y', \\ \bar{u} &= a, \\ \bar{v} &= v'. \end{aligned} \right. \tag{22}$$

In chart (22) the metric g reads

$$g = \pm \Big\{ 2\bar{u} d\bar{x} d\bar{v} + \alpha^2 d\bar{y} \left[d\bar{y} + 2\left(A - \bar{x}\right) d\bar{v} \right] + B d\bar{u}^2 + \left[\alpha^2 \left(A - \bar{x}\right)^2 + q \right] d\bar{v}^2 \Big\} \tag{23}$$

with rearranged α, A, q and B arbitrary nonvanishing function in $(\bar{u}, \bar{v})$. Chart (22) will be called *improved adapted.*

In an improved chart equation $ric(g) = 0$ can be easily solved. It is convenient to subdivide the integration procedure into the following **3 steps.**

Step 1. A direct computation gives

$$R_{\bar{x}\bar{v}} = \frac{2\alpha^3 B^2 - 2\bar{u}\alpha_{\bar{u}} B + \bar{u}\alpha B_{\bar{u}}}{4\alpha B^2 \bar{u}},$$

$$R_{\bar{y}\bar{y}} = \frac{-\alpha\left(\alpha^3 B^2 + 2\bar{u}\alpha_{\bar{u}} B - \bar{u}^2 \alpha_{\bar{u}} B_{\bar{u}} + 2\bar{u}^2 \alpha_{\bar{u}\bar{u}} B\right)}{2\bar{u}^2 B^2},$$

$$R_{\bar{u}\bar{u}} = \frac{\alpha B + \bar{u}\alpha B_{\bar{u}} - 2\bar{u}^2 \alpha_{\bar{u}\bar{u}} B + \bar{u}^2 \alpha_{\bar{u}} B_{\bar{u}}}{2\alpha B \bar{u}^2}.$$

Therefore $R_{\bar{x}\bar{v}} = 0$ iff

$$B_{\bar{u}} = \frac{2B\left(\bar{u}\alpha_{\bar{u}} - \alpha^3 B\right)}{\bar{u}\alpha} \tag{24}$$

and by substituting (24) in $R_{\bar{y}\bar{y}} + R_{\bar{u}\bar{u}} = 0$ one gets

$$B = \frac{4\bar{u}\alpha_{\bar{u}} + \alpha}{\alpha^3}. \tag{25}$$

In view of (25), (24) gives

$$\alpha^2 + 8\bar{u}\alpha\alpha_{\bar{u}} + 2\left(\alpha\alpha_{\bar{u}\bar{u}} + 3\alpha_{\bar{u}}^2\right)\bar{u}^2 = 0 \tag{26}$$

and $R_{\bar{u}\bar{u}}$ vanishes. Now, by substituting $\alpha = \varepsilon_2 e^{\psi}$ into (26), one gets

$$\left(\psi_{\bar{u}} + \frac{1}{2\bar{u}}\right)_{\bar{u}} + 4\left(\psi_{\bar{u}} + \frac{1}{2\bar{u}}\right)^2 = 0$$

and subsequently

$$\alpha = \varepsilon_2 \left(\frac{\alpha_1 + \bar{u}\alpha_0}{\bar{u}^2}\right)^{1/4} \tag{27}$$

with α_0, α_1 arbitrary functions in $\bar{v}$. Finally, (25) and (27) give

$$B = -\frac{\alpha_1 \bar{u}}{(\alpha_1 + \bar{u}\alpha_0)^{3/2}}. \tag{28}$$

Note that $\alpha_1 \neq 0$, since otherwise B would vanish.

Step 2. Taking into account (27) and (28) one finds that

$$R_{\bar{u}\bar{v}} = -\frac{4(\alpha_1 + \bar{u}\alpha_0)^{3/2}\alpha_1 A_{\bar{u}} - \bar{u}(2\alpha_1 + \bar{u}\alpha_0)\alpha_1'}{8\bar{u}^2(\alpha_1 + \bar{u}\alpha_0)\alpha_1}$$

hence $R_{\bar{u}\bar{v}} = 0$ iff

$$\begin{aligned} A &= -\frac{\alpha_1'}{\alpha_1}\int \frac{\bar{u}\alpha_{\bar{u}}}{\alpha^3} d\bar{u} + A_1 \\ &= \begin{cases} \dfrac{\alpha_1'\left(4\alpha_1^2 + 2\bar{u}\alpha_1\alpha_0 + \bar{u}^2\alpha_0^2\right)}{6\alpha_1\alpha_0^2\sqrt{\alpha_1 + \alpha_0\bar{u}}} + A_1, & \text{if } \alpha_0 \neq 0 \\ \dfrac{\alpha_1'\bar{u}^2}{4\alpha_1^{3/2}} + A_1, & \text{if } \alpha_0 = 0 \end{cases} \end{aligned} \tag{29}$$

with A_1 arbitrary function in $\bar{v}$.

Step 3. In view of (27), (28) and (29) the only non vanishing component of Ricci tensor is

$$R_{\bar{v}\bar{v}} = -\frac{1}{2B}\left(q_{\bar{u}\bar{u}} - \frac{2\alpha_1 + \bar{u}\alpha_0}{\bar{u}(\alpha_1 + \bar{u}\alpha_0)} q_{\bar{u}} + \frac{2\alpha_1 + \bar{u}\alpha_0}{\bar{u}^2(\alpha_1 + \bar{u}\alpha_0)} q + \rho_0\right)$$

with

$$\begin{aligned} \rho_0 = \frac{\bar{u}^2}{(\alpha_1 + \bar{u}\alpha_0)^{5/2}}(\alpha_1\alpha_0'' - \alpha_0\alpha_1'') + \frac{\bar{u}^2}{16(\alpha_1 + \bar{u}\alpha_0)^{7/2}}\Bigg[& -36\bar{u}\alpha_1\alpha_0'^2 \\ & -24(2\alpha_1 - \bar{u}\alpha_0)\alpha_0'\alpha_1' + \alpha_0\left(\bar{u}^2\alpha_0^2 + 13\bar{u}\alpha_0\alpha_1 + 48\alpha_1^2\right)\frac{\alpha_1'^2}{\alpha_1^2}\Bigg]. \end{aligned}$$

if $\alpha_0 \neq 0$ and $\rho_0 = 0$ if $\alpha_0 = 0$.

Now, recall that the general solution of

$$y'' + p(x)y' + q(x)y + r(x) = 0$$

can be written in the form

$$y = k_1 y_1 + k_2 y_2 + y_1 \int \frac{r y_2}{W} dx - y_2 \int \frac{r y_1}{W} dx$$

with k_1, k_2 arbitrary constants, y_1, y_2 independent solutions of the associated homogeneous equation and $W = y_1 y_2' - y_2 y_1'$.

Therefore, by observing that $\bar{u}$ and $\bar{u}\ln(\alpha_1 + \bar{u}\alpha_0)$ are two independent solutions of the homogeneous equation associated to $R_{\bar{v}\bar{v}} = 0$, the general solution of $R_{\bar{v}\bar{v}} = 0$ can be written in the form

$$q = c_1 u + u \int \left(\frac{c_2 - \int \frac{\rho_0 (a_1 + u a_0)}{u} du}{a_1 + u a_0} \right) du \tag{30}$$

with c_1, c_2 arbitrary functions in $\bar{v}$.

By summing up the above results one gets the following

Proposition 2. *In an improved adapted chart $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ Ricci flat metrics of type $(\mathcal{A}_2, 1)$ not possessing characteristic Killing vector fields have the following form*

$$g = \pm \left\{ 2\bar{u} d\bar{x} d\bar{v} + \left(\frac{\alpha_1 + \bar{u}\alpha_0}{\bar{u}^2} \right)^{1/2} \left[d\bar{y}^2 + 2\left(A - \bar{x} \right) d\bar{y} d\bar{v} \right] \right. \\ \left. - \frac{\alpha_1 \bar{u}}{(\alpha_1 + \bar{u}\alpha_0)^{3/2}} d\bar{u}^2 + \left[\left(\frac{\alpha_1 + \bar{u}\alpha_0}{\bar{u}^2} \right)^{1/2} \left(A - \bar{x} \right)^2 + q \right] d\bar{v}^2 \right\} \tag{31}$$

with α_0, α_1 arbitrary functions in $\bar{v}$ and A, q given by (29) and (30), respectively .

In the Lorentzian case (31) generalizes Bampi and Cianci metric (see [1])

$$g = p^2 \frac{\sqrt{cu - p^2}}{u} (v dx + dy)^2 + 2u dx dv + \frac{u}{(cu - p^2)^{3/2}} du^2 \\ + 2uh \ln(cu - p^2) dv^2 \tag{32}$$

where p, c are arbitrary constants and h is an arbitrary function in v. In fact, the coordinate transformation $\{\bar{x} = x, \bar{y} = vx + y, \bar{u} = u, \bar{v} = v\}$, brings (32) to the form

$$g = 2\bar{u} d\bar{x} d\bar{v} + p^2 \frac{\sqrt{c\bar{u} - p^2}}{\bar{u}} (d\bar{y}^2 - 2\bar{x} d\bar{y} d\bar{v}) + \frac{\bar{u}}{(c\bar{u} - p^2)^{3/2}} d\bar{u}^2 \\ + \left[p^2 \frac{\sqrt{c\bar{u} - p^2}}{\bar{u}} \bar{x}^2 + 2\bar{u} h \ln(c\bar{u} - p^2) \right] d\bar{v}^2$$

which is a particular case of (31).

3. Metrics of type $(\mathcal{A}_2, 0)$

Recall that (see [4], proposition 4) in a local *adapted chart* (x, y, u, v) a metric g of type $(\mathcal{A}_2, 0)$ reads

$$g = 2dxdu + 2adxdv + 2bdydv + rdu^2 + 2pdudv + qdv^2 \tag{33}$$

with a, b, r, p, q arbitrary functions in (u, v).

Put $b = \varepsilon_1 e^{\phi}$ with $\varepsilon_1 = \pm 1$. A straightforward computation shows that the only non vanishing components of the Ricci tensor are:

$$\begin{aligned} R_{uu} &= -\frac{\phi_u^2}{2} - \phi_{uu} \\ R_{uv} &= -\frac{1}{2}\left[\phi_{uv} + a\left(\phi_{uu} + \phi_u^2\right) - a_{uu}\right] \\ R_{vv} &= -\frac{1}{2}\left[2\left(-a_{uv} + a_u\phi_v + a\phi_{uv}\right) + \left(a\phi_u - a_u\right)^2\right]. \end{aligned} \tag{34}$$

We shall distinguish between two cases: $\phi_u \equiv 0$ and $\phi_u \neq 0$.

The case $\phi_u = 0$. In this case $R_{uv} = R_{vv} = 0$ iff

$$a_{uu} = 0, \quad 2\left(-a_{uv} + a_u\phi_v\right) + a_u^2 = 0.$$

so that

$$a = -\frac{\tau e^{\phi} u}{\displaystyle\int e^{\phi} dv + c} + a_0$$

with c arbitrary constant, $a_0 = a_0(v)$ and $\tau = 0, 2$.

Therefore, by passing to the adapted coordinates $\left\{\bar{x} = x, \bar{y} = \varepsilon_1 y, \bar{u} = u, \bar{v} = \displaystyle\int e^{\phi} dv + c\right\}$ and rearranging arbitrary functions one gets g in the form

$$g = 2d\bar{x}d\bar{u} + 2\left(-\frac{\tau\bar{u}}{\bar{v}} + a_0\right)d\bar{x}d\bar{v} + 2d\bar{y}d\bar{v} + rd\bar{u}^2 + 2pd\bar{u}d\bar{v} + qd\bar{v}^2 \tag{35}$$

with a_0 arbitrary function in $\bar{v}$, r, p, q arbitrary function in $(\bar{u}, \bar{v})$, $c \in \mathbb{R}$ and $\tau = 0, 2$. In particular, when $\tau = 0$ the transformation $\bar{u} \longmapsto \bar{u} + \int a_0 d\bar{v}$ brings (35) to the form

$$g = 2d\bar{x}d\bar{u} + 2d\bar{y}d\bar{v} + rd\bar{u}^2 + 2pd\bar{u}d\bar{v} + qd\bar{v}^2 \tag{36}$$

The case $\phi_u \neq 0$. In this case $R_{uu} = 0$ iff $\phi = \phi_1 + 2\ln(u + \phi_2)$ with $\phi_i = \phi_i(v)$, $i = 1, 2$. Actually both ϕ_1 and ϕ_2 are inessential since they can be gauged out by passing to the adapted chart $\left\{x, y, u + \phi_2, \displaystyle\int e^{\phi_1} dv\right\}$. Therefore,

$$\phi = 2\ln|u|.$$

and, in view of $R_{uv}=0$ and $R_{vv}=0$, one eventually gets

$$a = cu^2$$

with $c \in \mathbb{R}$. The rescaling $\{\bar{x}=x, \bar{y}=\varepsilon_1 y+sx, \bar{u}=u, \bar{v}=v\}$ or $\{\bar{x}=x, \bar{y}=\varepsilon_1 y/c+sx, \bar{v}=cv\}$ (if $c\neq 0$) brings g to the form

$$g = 2d\bar{x}d\bar{u} + 2\bar{u}^2 d\bar{y}d\bar{v} + rd\bar{u}^2 + 2pd\bar{u}d\bar{v} + qd\bar{v}^2 \tag{37}$$

with r,p,q arbitrary function in $(\bar{u},\bar{v})$ and $s=0,1$.

By summing up these results one gets the following

Proposition 3. *In adapted coordinates, a Ricci flat metric of type $(\mathcal{A}_2,0)$ can be brought either to the form (35) ((36) if $\tau=0$), or to the form (37), and conversely.*

4. Ricci flat metrics with $\dim\mathcal{G}=3$

In this section we shall find Ricci-flat metrics of type $(\mathcal{G},r)$ with $\dim\mathcal{G}>2$. To this end we shall use the results of section 3 of [4], corollaries 2 and 3, and compute the components of the Ricci tensor in the (holonomic) frame $(\partial_x,\partial_y,\partial_u,\partial_v)$.

4.1. **The case $(\mathcal{H},1)$.** In view of corollary 2 of [4], coefficients of metric (1) are $\alpha=0, a=0$ and $b=-\varepsilon_0\beta^2$. Then it is easy to see that the only non-vanishing components of the Ricci tensor are R_{xx}, R_{uu} and R_{vv}. In particular $R_{xx}=-\varepsilon_0\varepsilon_1(2\beta_u^2+\beta\beta_{uu})$ and $R_{uu}=-3\beta_{uu}/\beta$ so that corresponding Einstein's equations reduce to $\beta_u=0$. Now (2) implies that $f=0$ and we get

Proposition 4. *For any Ricci flat metric of type $(\mathcal{H},1)$ the derived algebra $[\mathcal{H},\mathcal{H}]$ is generated by a characteristic (or null) Killing vector field and the distribution $\mathcal{D}^{\perp}$ is completely integrable.*

As a consequence Ricci flat metrics of type $(\mathcal{H},1)$ belong to class (11). The exact expression for them is

$$g = \pm\left[\beta^2\left(dx^2-2dydv\right)+\varepsilon du^2+2P_u dudv+qdv^2\right] \tag{38}$$

with

$$q=\varepsilon\left[2\left(\frac{\beta'}{\beta}\right)^2-\frac{\beta''}{\beta}\right]u^2-4\left(\frac{\beta'}{\beta}\right)P+2P_v+q_1u+q_0.$$

In terms of an improved adapted chart $(\tilde{x},\tilde{y},\tilde{u},\tilde{v})$ such a metric looks as

$$g=\pm\left(d\tilde{x}^2+\varepsilon d\tilde{u}^2-2d\tilde{y}d\tilde{v}+\left(\tilde{x}^2-\varepsilon\tilde{u}^2\right)Ad\tilde{v}^2\right) \tag{39}$$

where A is an arbitrary function in v and $\varepsilon=\pm1$. Thus, we have

Corollary 1. *A Ricci flat metric of type $(\mathcal{H},1)$ has form (38) in adapted chart and form (39) in an improved adapted chart, and conversely.*

In the Lorentzian case, Ricci flat metric of type $(\mathcal{H},1)$ are plane waves (see [2]).

4.2. **The case $(\mathcal{A}_3, 0)$.** As above taking into account corollary 3 of [4] we see that for metrics (33) $a = \eta_v/\eta_u$ and $b = -1/\eta_u$. Then it is straightforward from (34) that in this case the most general Ricci flat metric reads

$$g = 2dxdu + 2\left[c\frac{(u+h_2)^2}{h_1} + h_2'\right] dxdv + 2\frac{(u+h_2)^2}{h_1}dydv$$
$$+rdu^2 + 2pdudv + qdv^2$$

h_1, h_2 being arbitrary functions in v, r, p, q arbitrary functions in (u, v) and $c \in \mathbb{R}$.

Actually h_1 and h_2 are inessential, since they can be gauged out by passing to the adapted chart $\left\{\bar{x} = x, \bar{y} = y, \bar{u} = u + h_2, \bar{v} = \int \frac{1}{h_1}dv\right\}$, and one can easily rewrite such a metric in the form (37).

Thus, one gets

Proposition 5. *A Ricci flat metric of type $(\mathcal{A}_3, 0)$ has form (37) in a suitable adapted chart, and conversely.*

References

[1] Bampi F. and Cianci R. (1979). *Generalized axisymmetric spacetimes.* Commun. Math. Phys. 70, 69.

[2] Barnes, A. (1979). *On space-times admitting a three-parameter isometry group with two-dimensional null orbits.* J. Phys. A 12, 1493.

[3] V.A. Belinsky and V.E. Zakharov. (1978). *Integration of the Einstein equations by means of the inverse scattering problem technique and construction of the exact soliton solutions.* Soviet Phys. JETP 75, 1953.

[4] D. Catalano Ferraioli and A. M. Vinogradov. (2004). *Ricci flat 4-metrics with bidimensional null orbits. Part I. General aspects and nonabelian case.* DIPS 7/2004

[5] Ehlers, J. and Kundt, W. (1962). *In Gravitation: An Introduction to Current Research.* L. Witten, ed, (Wiley, New York).

[6] G. C. Joly, M. A. H. MacCallum and S. T. C. Siklos. (1992). *Comment on the 'generalized axisymmetric spacetimes' of Bampi and Cianci.* Class. Quantum Grav. 9, 795.

[7] Kundt, W.(1961). *The plane-fronted Gravitational Waves.* Zeitschrift für Physik 163, 77.

[8] Schimming, R. (1974). *Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie.* Math. Nachr. 59, 129.

[9] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt. (2003).*Exact Solutions of Einstein's Field Equations.* Second Edition. Cambridge University Press.

Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy.

E-mail address: dcatalan@unisa.it

Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA),Italy, Istituto Nazionale di Fisica Nucleare, sezione di Napoli-Salerno, and Diffiety Institute, Russia.

E-mail address: vinograd@unisa.it