The Diffiety Institute Preprint Series

Preprint DIPS–7/2004 **September 2, 2004**

Revised March 16, 2005

Ricci flat 4-metrics with bidimensional null orbits.

Part I. General aspects and nonabelian case.

by

D. Catalano Ferraioli and A. M. Vinogradov

Available via INTERNET:
http://diffiety.ac.ru; http://diffiety.org

The Diffiety Institute

RICCI FLAT 4-METRICS WITH BIDIMENSIONAL NULL ORBITS.
PART I. GENERAL ASPECTS AND NONABELIAN CASE.

D. CATALANO FERRAIOLI AND A. M. VINOGRADOV

Abstract. Pseudo-Riemannian 4-metrics with bidimensional null Killing orbits are studied. Both Lorentzian and Kleinian (or neutral) cases, are treated simultaneously. Under the assumption that the distribution orthogonal to the orbits is completely integrable a complete exact description of Ricci flat metrics admitting a bidimensional nonabelian Killing algebra is found.

1. Introduction

In this and the forthcoming paper [4] we give an exact description of Ricci flat 4-metrics g under assumptions that

(i) g admits a Killing algebra $\mathcal{G}$ with bidimensional leaves (orbits of $\mathcal{G}$),

(ii) the distribution $\mathcal{D}^{\perp}$ orthogonal to Killing leaves is Frobenius (completely integrable),

(iii) g degenerates when restricted to any Killing leaf.

Condition (i) implies that $\dim \mathcal{G} \geq 2$ and, moreover, that $\mathcal{G}$ contains a (not necessarily proper) 2-dimensional subalgebra. There exists only one, up to isomorphism, 2-dimensional nonabelian Lie algebra $\mathcal{G}_2$ and the case in which $\mathcal{G} \supseteq \mathcal{G}_2$ is studied in this paper. The case $\mathcal{G} \supseteq \mathcal{A}_2$, $\mathcal{A}_2$ being an Abelian 2-dimensional algebra, is the subject of the forthcoming paper [4].

Condition (iii) subjects such a metric to be either of signature $\pm(-+++)$, or $(--++)$. Due to their importance in general relativity Lorentzian, i.e., of signature $(-+++)$, Ricci flat metrics were intensively studied for decades. In particular, almost exhaustive classification of this kind of metrics admitting at least two Killing fields was done. See, for instance, [13] and [18]. On the contrary, relatively few results were obtained for Ricci flat Kleinian (or neutral), i.e., of signature $(--++)$, metrics. We stress from the very beginning that while new results of this and forthcoming paper [4] concern mainly Kleinian metrics our approach makes no distinction between these two cases.

Ricci flat manifolds of Kleinian signature possess a number of interesting geometrical properties and undoubtedly deserve attention in their own right. Some topological aspects of these manifolds were studied for the first time in [9] and [10] and subsequently

1991 *Mathematics Subject Classification.* primary 53B30, 53C25, 53C15; secondary 53B50, 83C35.

Key words and phrases. Einstein metrics, Ricci flat metrics, Killing Algebra, null orbits, Kleinian metrics, Lorentzian metrics.

in [12]. In recent years geometry of these manifolds has seen a revival of interest. In part, this is due to the emergence of some new applications in physics. For example the 'no boundary' proposal of Hartle and Hawking [7] has suggested the idea that the signature of the space-time metric may have changed in the early universe. So, assuming that particles are free to move between Lorentzian and Kleinian regions some surprising physical phenomena, like time travelling, would be observable (see [1] and [14]). Some other examples of Kleinian geometry in physics occur in the theory of heterotic $N = 2$ string (see [11] and [2]) for which the target space is four dimensional.

The problem of describing Ricci flat metrics under assumptions (i) and (ii) that are, on the contrary to (iii), nondegenerate along the orbits of $\mathcal{G}$ was already studied in a number of works. For instance, Belinsky and Zakharov [3] proved that in the case of an Abelian bidimensional algebra $\mathcal{G} = \mathcal{A}_2$ vacuum Einstein equations $ric\,(g) = 0$ are integrable by means of the inverse scattering problem method (see also [5] and [6]). In the case of a nonabelian bidimensional algebra $\mathcal{G} = \mathcal{G}_2$, instead, it has been proved (see [15], [16], [17]) that the corresponding Ricci flat metrics are parametrized by solutions of a linear equation in two variables. So, this paper together with [4] completes the study of Ricci flat metrics that are subject to (i) and (ii).

Assumption (iii) naturally splits into two subcases according to whether the rank of g restricted to Killing leaves is one or zero. Hence, in order to distinguish various cases occurring in the sequel, we use the notation $(\mathcal{G}, r)$ when referring to a metric satisfying assumptions (i)-(iii) and of rank $r = 0, 1$ along leaves of $\mathcal{G}$.

In this paper we use an approach similar to that of [15]. Namely, we first find normal forms of $(\mathcal{G}, r)$-type metrics that are subject to (i)-(iii) and then analyze the corresponding reduction of the Einstein equations. Condition (i) implies that (see section 3 below) $\dim \mathcal{G} = 2$ or 3. If $\dim \mathcal{G} = 3$, then a further analysis shows that $\mathcal{G}$ is either the Abelian algebra $\mathcal{A}_3$, or the Heisenberg algebra $\mathcal{H}$ according to whether $r = 0$ or $r = 1$, respectively. This is a generalization of a result by Petrov (see [13]) concerning Lorentzian metrics.

It is interesting that the zero rank assumption for Ricci flat metrics turns out to be incompatible with assumptions (i) and (ii) for $\mathcal{G} = \mathcal{G}_2$ in contrast to the abelian case.(see [4]). Moreover Ricci flat metrics of type $(\mathcal{G}_2, 1)$ are Kleinian (see proposition (7) below).

In this paper we discuss only local forms of Ricci flat metrics in question. The global aspects will be discussed separately.

The paper is organized as follows.

In section 2, necessary definitions and preliminaries used throughout the paper are collected. Then we construct for $\mathcal{G} = \mathcal{A}_2, \mathcal{G}_2$ some special charts, *adapted* to assumptions (i)-(iii). These charts are key in the subsequent analysis of the problem. In section 3, 3-dimensional Killing algebras are described by generalizing to Kleinian metrics a result by Petrov (see [13]). In section 4 we introduce a non-holonomic frame field in which the Ricci tensor takes a particularly simple form in the case $(\mathcal{G}, 1)$. We also prove that the zero rank assumption for Ricci flat metrics is incompatible with assumptions (i) and (ii) when $\mathcal{G} = \mathcal{G}_2$. In sections 5 we solve explicitly the equation $ric(g) = 0$ in the case $(\mathcal{G}_2, r)$.

Throughout the paper the following notations and conventions are adopted:

- manifolds, maps, metrics, etc. are assumed to be smooth and manifolds to be connected;
- $\mathcal{D}(M)$ denotes the Lie algebra of all vector fields on a manifold M;
- the term *metric* refers to a non-degenerate symmetric $(0, 2)$- tensor field and g stands for a metric on M of the type we are studying;
- the term *Killing algebra* refers to a sub-algebra of the Lie algebra of all Killing vector fields of g;
- *integral submanifolds* of the distribution generated by vector fields of a Killing algebra $\mathcal{G}$ are called *Killing leaves* and the Killing leaf passing through $p \in M$ is denoted by $\mathcal{K}_p$;
- $\mathcal{D}$ denotes the tangent to Killing leaves bidimensional distribution and $\mathcal{D}^\perp$ the g-orthogonal to it;
- $\mathcal{A}_2$ and $\mathcal{G}_2$ stand for bidimensional Abelian and nonabelian Killing algebras, respectively;
- $\mathcal{A}_3$ and $\mathcal{H}$ stand for 3-dimensional Abelian and Heisenberg Killing algebras, respectively;
- $(\mathcal{G}, r)$ refers to the case of metrics of rank $r = 0, 1$ along 2-dimensional leaves of the Killing algebra $\mathcal{G}$.

2. Adapted coordinates

When $\dim \mathcal{G} = 2$ it is convenient to treat both cases, abelian and non, simultaneously. To this end we consider the bidimensional Lie algebra $\mathcal{G}$ generated by e_1 and e_2 such that $[e_1, e_2] = se_2$, $s \in \mathbb{R}$. Obviously, $\mathcal{G}$ is isomorphic to $\mathcal{G}_2$ if $s \neq 0$ and to $\mathcal{A}_2$ otherwise.

Assume that $\mathcal{G}$ is realized as an algebra of Killing vector fields of a metric g on a manifold M, i.e., there exists a representation of $\mathcal{G}$, say $\rho : \mathcal{G} \to \mathcal{D}(M)$, such that $\rho(\xi)$ is a Killing vector field for g, $\forall \xi \in \mathcal{G}$.

The distribution $\mathcal{D}$ generated by vector fields $\rho(\xi)$, $\xi \in \mathcal{G}$, is completely integrable and of dimension two in view of the following elementary fact [15].

Lemma 1. *Let $X \neq 0$ be a vector field on M. If X and fX, $f \in C^\infty(M)$, are both Killing vector fields of a metric on M, then f is a constant.*

In other words, ρ is automatically faithful, i.e., $\ker \rho = 0$, so that $\mathcal{G}$ and $\rho(\mathcal{G})$ are identified naturally. In what follows we shall consider $\mathcal{G}$ as a sub-algebra of the algebra $D(M)$ of vector fields on M and put $X = \rho(e_1)$, $Y = \rho(e_2)$, $X, Y \in D(M)$.

Metrics g's we are considering in this paper are assumed becoming degenerate when restricted to any Killing leaf. In other words, the restricted tensor $g|_{\mathcal{K}_p}$ for a generic $p \in M$ may be either of rank 1, or of rank 0. These two cases are studied separately.

2.1. **Metrics of type $(\mathcal{G}, 1), \dim \mathcal{G} = 2$.** In this case there exists a 1-dimensional tangent to Killing leaves distribution that associates to a point $p \in M$ the kernel $\mathfrak{C}_p$ of the tensor $g|_{\mathcal{K}_p}$ on $\mathcal{K}_p$ at p. This distribution, called *characteristic*, will be denoted by $\mathfrak{C} = \{\mathfrak{C}_p\}$ and we shall refer to $\mathfrak{C}_p$ as the *characteristic direction* at p.

A vector field C belonging to the characteristic distribution will be also called *characteristic*. In such an instance we write $C \in \mathfrak{C}$ and similarly for other distributions we shall deal with.

The 3-dimensional distribution $\mathfrak{C}^\perp$, g-orthogonal to $\mathfrak{C}$, obviously contains the 2-dimensional distribution $\mathcal{D}^\perp$, g-orthogonal to $\mathcal{D}$. In its turn $\mathcal{D}^\perp$ contains the characteristic distribution $\mathfrak{C}$.

Observe that the intersection of distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ is exactly the characteristic distribution (i.e., $\mathcal{D}_p \cap \mathcal{D}_p^\perp = \mathfrak{C}_p, \forall p \in M$). So, a vector field $U \in \mathcal{D}^\perp$ which is transversal to $\mathcal{D}$ generates together with a characteristic field $\mathfrak{C}$ the distribution $\mathcal{D}^\perp$.

Obviously, U is determined uniquely up to a transformation

$$U \longrightarrow U' = \lambda U + \mu C \tag{1}$$

for some $\lambda, \mu \in C^\infty(M)$, λ being everywhere non zero.

While $\mathfrak{C}$ is the intersection of distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, the distribution $\mathfrak{C}^\perp$ is the span of them in the sense that $\mathfrak{C}_p^\perp = \mathrm{span}(\mathcal{D}_{\mathrm{p}}, \mathcal{D}_{\mathrm{p}}^\perp)$, $\forall p \in M$. Therefore, $\mathfrak{C}^\perp$ is generated by X, Y and U. The following lemma is needed to prove integrability of $\mathfrak{C}^\perp$.

Lemma 2. *Let $C \in \mathfrak{C}$ and $K \in \mathcal{G}$, then*

$$[K, C] = fC \tag{2}$$

for some $f \in C^\infty(M)$.

Proof. Recall the well-known formula for the Lie derivative of g along a vector field X_1:

$$L_{X_1}(g)(X_2, X_3) = X_1\left(g\left(X_2, X_3\right)\right) - g\left([X_1, X_2], X_3\right) - g\left(X_2, [X_1, X_3]\right).$$

If $X_1, X_2 \in \mathcal{D}$ and $X_3 \in \mathfrak{C}$, then $g\left(X_2, X_3\right) = g\left([X_1, X_2], X_3\right) = 0$, since $[X_1, X_2] \in \mathcal{D}$ for $X_1, X_2 \in \mathcal{D}$. Therefore, if X_1 is a Killing vector field,i.e., $L_{X_1}(g) = 0$, then

$$g\left(X_2, [X_1, X_3]\right) = 0$$

for any $X_2 \in \mathcal{D}$ and, therefore, $[X_1, X_3]$ is a characteristic vector field. □

Proposition 1. *The distribution $\mathfrak{C}^\perp$ is completely integrable.*

Proof. Since $\mathfrak{C}^\perp$ is generated by fields X, Y and U it suffices to show that their commutators belong to $\mathfrak{C}^\perp$ as well. Trivially, $[X, Y] = sY \in \mathcal{D} \subset \mathfrak{C}^\perp$.

Let $C \in \mathfrak{C}$. Then

$$0 = L_X(g)(U, C) = L_X\left(g\left(U, C\right)\right) - g\left([X, U], C\right) - g\left(U, [X, C]\right).$$

Here $g(U, C) = 0$ by hypothesis while $g\left(U, [X, C]\right) = 0$ in view of lemma 2, so that $g\left(C, [X, U]\right) = 0 \Longleftrightarrow [X, U] \in \mathfrak{C}^\perp$. Similarly for $[Y, U]$. □

By the Frobenius theorem there exists a unique 3-dimensional integral submanifold of the distribution $\mathfrak{C}^{\perp}$ passing through a point $a \in M$. Denote it by P_a.

After these preliminaries we are ready to construct an adapted local chart in a neighborhood of a point $a \in M$. To this end choose a curve $\gamma(v)$, with v running through an interval $I \subset \mathbb{R}$, that passes through a and is transversal to all submanifolds $P_{\gamma(v)}$'s. Next consider the local flow $\{\varphi_u\}$ generated by a nowhere vanishing field $U \in \mathcal{D}^{\perp}$ defined in a neighborhood of a and transversal to Killing leaves. Then $(u,v) \longmapsto \varphi_u(\gamma(v))$ with (u,v) ranging in a "small" domain $\mathcal{U}$ in $\mathbb{R}^2$ is a parametric surface Σ in M. By construction, Σ is transversal to Killing leaves.

The local chart $\psi(x,y,u,v) = (A_x \circ B_y \circ \varphi_u)(\gamma(v))$ with $\{A_x\}$and $\{B_y\}$ being local flows generated by X and Y, respectively, and (x,y,u,v) ranging over a suitable "small" domain in $\mathbb{R}^4$ is called *almost adapted.*

The following properties of an almost adapted chart will be used further.

Proposition 2. *For any* almost adapted *chart* (x,y,u,v) *it holds:*

(a) $X = \partial_x$, $Y = e^{sx}\partial_y$;
(b) *vector fields* ∂_u *and* ∂_v *are invariant with respect to flows* $\{A_x\}$ *and* $\{B_y\}$;
(c) $\partial_u \in \mathcal{D}^{\perp}$.

Proof. Denote by $F(Z) \in D(N)$ the image of a vector field on $Z \in D(M)$ under a diffeomorphism $F : M \to N$. Recall that

$$F(Z) = \left(F^{-1}\right)^* \circ X \circ F^*$$

(as derivation of the algebra $C^\infty(N)$).

(a) If $f \in C^\infty(M)$, then

$$\begin{aligned}
\frac{\partial}{\partial x} f\left(x,y,u,v\right) &= \frac{\partial}{\partial x} f\left(\left(A_x \circ B_y \circ \varphi_u\right)\left(\gamma\left(v\right)\right)\right) \\
&= \frac{\partial}{\partial x}\left[\left(\left(A_x \circ B_y\right)^*\left(f\right)\right)\left(\varphi_u\left(\gamma\left(v\right)\right)\right)\right] \\
&= \frac{\partial}{\partial x}\left[\left(\left(B_y^* \circ A_x^*\right)\left(f\right)\right)\left(\varphi_u\left(\gamma\left(v\right)\right)\right)\right] \\
&= \left(\left(B_y^* \circ \frac{d}{dx}\left(A_x^*\right)\right)\left(f\right)\right)\left(\varphi_u\left(\gamma\left(v\right)\right)\right).
\end{aligned}$$

Since $\frac{d}{dx} \circ A_x^* = A_x^* \circ X$, one proceeds as

$$\begin{aligned}
\left(\left(B_y^* \circ A_x^* \circ X\right)\left(f\right)\right)\left(\varphi_u\left(\gamma\left(v\right)\right)\right) &= X\left(f\right)\left(\left(A_x \circ B_y \circ \varphi_u\right)\left(\gamma\left(v\right)\right)\right) \\
&= X(f)(x,y,u,v).
\end{aligned}$$

This proves that $\frac{\partial}{\partial x} = X$.

Similarly,

$$\begin{aligned}\frac{\partial}{\partial y} f\,(x,y,u,v) &= \left(\left(B_y^* \circ Y \circ A_x^*\right)(f)\right)(\varphi_u(\gamma(v)))\\ &= \left(\left(B_y^* \circ A_x^* \circ A_x(Y)\right)(f)\right)(\varphi_u(\gamma(v)))\\ &= A_x(Y)(f)(x,y,u,v),\end{aligned}$$

so that $\dfrac{\partial}{\partial y} = A_x(Y)$. On the other hand,

$$\frac{d}{dt}A_t(Y) = \frac{d}{dt}\left(A_{-t}^* \circ Y \circ A_t^*\right) = -X \circ A_t(Y) + A_t(Y) \circ X = [A_t(Y), X],$$

that is

$$\frac{d}{dt}A_t(Y) = [A_t(Y), X]. \tag{3}$$

But $A_t(Y)$ is tangent to Killing leaves and, therefore, $A_t(Y) = \alpha(t)X + \beta(t)Y$. In these terms equation (3) reads as

$$\dot{\alpha} = 0, \qquad \dot{\beta} = -s\beta.$$

Moreover, from $A_0(Y) = Y$ one sees that $\alpha(0) = 0$ and $\beta(0) = 1$. This shows that $\alpha = 0$ and $\beta = e^{-st}$, that is $A_t(Y) = e^{-st}Y$, and therefore $Y = e^{sx}\partial_y$.

(b) Straightforwardly from (a), since ∂_u and ∂_v commute, obviously, with X and Y.

(c)For a given $p \in \Sigma$ the curve $\lambda_p : u \mapsto \varphi_u(p)$ is, by construction, g-orthogonal to Killing leaves. Since $A_x \circ B_y$ is a local isometry of g the curve $A_x \circ B_y \circ \lambda_p$ is also orthogonal to Killing leaves. It remains to observe that (see (b)) these curves are trajectories of ∂_u. □

Now we have to fix some peculiarities of components of g in an almost adapted chart (x, y, u, v).

First of all, by Proposition 2(a), vector fields ∂_x, and ∂_y are tangent to Killing leaves. Hence, $g(\partial_x, \partial_u) = g(\partial_y, \partial_u) = 0$ by Proposition 2(c). Moreover, for any Killing leaf $\mathcal{K}$ the tensor $g|_{\mathcal{K}}$ being of rank one has the form

$$g|_{\mathcal{K}} = \pm(c_2dx - c_1dy)^2,$$

c_1, c_2 being some functions in (x, y, u, v). So, the vector field

$$C := c_1\partial_x + c_2\partial_y$$

is characteristic. Therefore, the matrix (g_{ij}) has the following form

$$(g_{ij}) = \begin{pmatrix} \mathbb{F} & \mathbb{T} \\ \mathbb{T}^T & \mathbb{S} \end{pmatrix}$$

with

$$\mathbb{F} := \varepsilon_0 \begin{pmatrix} c_2^2 & -c_1c_2 \\ -c_1c_2 & c_1^2 \end{pmatrix}, \quad \mathbb{T} := \begin{pmatrix} 0 & \mathfrak{t}_1 \\ 0 & \mathfrak{t}_2 \end{pmatrix}, \quad \mathbb{S} = \begin{pmatrix} r & p \\ p & q \end{pmatrix}.$$

Here $\varepsilon_0 = \pm 1$ and $\mathfrak{t}_1, \mathfrak{t}_2, p, q$ are some functions in (x, y, u, v).

By observing that $\det(g_{ij}) = -\varepsilon_0 r \left(c_1\mathfrak{t}_1 + c_2\mathfrak{t}_2\right)^2$ one gets that $r = g(\partial_u, \partial_u)$ and $c_1\mathfrak{t}_1 + c_2\mathfrak{t}_2$ are everywhere nonzero. In particular, this allows one to construct an almost adapted chart with $r = \pm 1$. In fact, since U and ∂_u both belong to $\mathcal{D}^\perp$, it results from (1) that $U = \lambda\partial_u + \mu C$. Hence

$$g(\partial_u, \partial_u) = \frac{g(U,U)}{\lambda^2}.$$

It suffices now to normalize U to $g(U,U) = \pm\lambda^2$.

An almost adapted chart (x, y, u, v) will be called *adapted* if $r = g(\partial_u, \partial_u) = \pm 1$. So, the above considerations prove existence of adapted charts.

The fact that $X = \partial_x$ is a Killing vector field implies, obviously, that the components of g in an almost adapted chart do not depend on x. Moreover, a direct simple computation shows that Killing property of $Y = e^{sx}\partial_y$ implies in its turn that $c_1, \mathfrak{t}_2, p$ and q are arbitrary functions in (u, v) and if we respectively denote c_1 and $\mathfrak{t}_2$ simply by α and b then

$$\begin{cases} c_2 = sy\alpha(u,v) + \beta(u,v) \\ \mathfrak{t}_1 = -syb(u,v) + a(u,v). \end{cases}$$

In particular, the characteristic vector field C now reads

$$C := \alpha\partial_x + (sy\alpha + \beta)\partial_y \tag{4}$$

and $[C, X] = [C, Y] = 0$.

The distribution $\mathcal{D}^\perp$ is generated by fields ∂_u and C. So, it is completely integrable if and only if

$$[\partial_u, C] = h\partial_u + fC,$$

that is

$$\alpha_u\partial_x + (sy\alpha_u + \beta_u)\partial_y = h\partial_u + f\alpha\partial_x + f(sy\alpha + \beta)\partial_y$$

for some $h, f \in C^\infty(M)$. This, obviously, implies $h = 0$.

We sum up the obtained results in the following proposition.

Proposition 3. *Let g be a metric subject to conditions (i), (ii) of the introduction and of rank one along Killing leaves. Then in an adapted chart (x, y, u, v) it has the form*

$$\begin{aligned} g = \varepsilon_0 \left((sy\alpha + \beta)\, dx - \alpha dy\right)^2 + 2\left(-syb + a\right) dxdv + 2bdydv \\ + \varepsilon_1 du^2 + 2pdudv + qdv^2 \end{aligned} \tag{5}$$

where $\varepsilon_0, \varepsilon_1 = \pm 1$ and $\alpha, \beta, a, b, p, q$ are functions in (u, v).

The distribution $\mathcal{D}^\perp$ is integrable if and only if

$$\begin{cases} \alpha_u = f\alpha, \\ \beta_u = f\beta \end{cases} \tag{6}$$

and the field (4) and ∂_u satisfy the commutation relation

$$[\partial_u, C] = fC$$

for a function $f \in C^\infty(M)$.

2.2. **Metrics of type** $(\mathcal{G}, 0), \dim \mathcal{G} = 2$**.** First, note that this case is characterized by the fact that $\mathcal{D}^\perp = \mathcal{D}$. So, assumption (ii) of the introduction is satisfied automatically and any 1-dimensional distribution $\mathcal{C}$ tangent to Killing leaves may be "declared" as a characteristic one. If, moreover, $\mathcal{C}$ is preserved by the algebra $\mathcal{G}$, i.e., $[X, Z], [Y, Z] \in \mathcal{C}$, $\forall\, Z \in \mathcal{C}$, then one obtains a geometrically privileged chart just by substituting $\mathcal{C}$ for $\mathfrak{C}$ in the construction of the previous section. The only difference coming out in the considered context is that the component $r = g(\partial_u, \partial_u)$ is not necessarily nowhere vanishing and, so, can not be normalized to ± 1 in such a case.

Observe further that the vector field sY generates the derived algebra of $\mathcal{G}$. So, with the distribution $\mathcal{Y}$ generated by sY we eliminate the arbitrariness in the choice of $\mathcal{C}$. Thus we call *adapted* a local chart that is constructed according to the scheme of the preceding section in which $\mathfrak{C}$ is replaced by $\mathcal{C}$.

We shall keep the notation (x, y, u, v) for adapted coordinates in the considered context as well. It should also be noted that the analogue of Proposition 3 in which assertion (3) is modified to $\partial_u \in \mathcal{C}$ holds.

In an adopted chart (x, y, u, v) the matrix of g looks as

$$(g_{ij}) = \begin{pmatrix} \mathbb{O} & \mathbb{H} \\ \mathbb{H}^T & \mathbb{B} \end{pmatrix}$$

with

$$\mathbb{H} := \begin{pmatrix} \alpha & \beta \\ 0 & b \end{pmatrix} \quad \mathbb{B} := \begin{pmatrix} r & p \\ p & q \end{pmatrix}$$

$\alpha, \beta, b, p, q, r$ being some functions in (x, y, u, v). Observe that $\det(g_{ij}) = \alpha^2 b^2$ and both functions α and b are nowhere vanishing.

It is easy to see now that if $X = \partial_x$ and $Y = e^{sx}\partial_y$ are Killing vector fields then α, r, p and q are arbitrary functions in (u, v) and

$$\beta = -syb(u, v) + a(u, v).$$

Since the form of the metric is preserved under the transformation $(x, y, u, v) \longmapsto \left(x, y, \int \alpha du, v\right)$, one may assume without loss in generality that in the adapted coordinates $\alpha = 1$. Hence we have.

Proposition 4. *Let g be a metric admitting a bidimensional Killing algebra and vanishing along Killing leaves. Then in an adapted chart (x, y, u, v) it has the form*

$$g = 2dxdu + 2(-syb + a)dxdv + 2bdydv + rdu^2 + 2pdudv + qdv^2 \tag{7}$$

with a, b, p, q, r some functions in (u, v).

3. Metrics of type $(\mathcal{G}, r)$, $\dim \mathcal{G} > 2$

In this section our aim is to prove the following

Proposition 5. *For metrics of type $(\mathcal{G}, r)$ with $\dim \mathcal{G} > 2$, either $\mathcal{G} = \mathcal{A}_3$ and $r = 0$, or $\mathcal{G} = \mathcal{H}$ and $r = 1$.*

which generalizes to Kleinian metrics a result by Petrov (see [13]). Before passing to the proof of 5 we collect some useful facts.

First, recall (see [8])

Lemma 3. *Given a metric g then $[L_X, i_Y](g) = i_{[X,Y]}(g)$ for any vector fields X and Y.*

In other words, $L_X(i_Y(g)) = i_{[X,Y]}(g) + i_Y(L_X(g))$.

Corollary 1. *Let X be a Killing field of a metric g. Then $L_X(i_Y(g)) = i_{[X,Y]}(g)$, for any vector field Y.*

The following lemma that generalizes lemma 1 is a consequence of the formula

$$L_{fX}(g) = fL_X(g) + i_X(g)df \tag{8}$$

where $X \in \mathcal{D}(M)$, $f \in C^\infty(M)$ and the second term in the right hand side is twice the *symmetric product* of two differential 1-forms.

Lemma 4. *Let X_1, X_2 and $f_1X_1 + f_2X_2$, $f_1, f_2 \in C^\infty(M)$, be Killing fields of a metric g, then*

1. *if X_1 and X_2 are $\mathbb{R}$-independent, then either f_1 and f_2 are functionally independent, or f_1 and f_2 are constant;*
2. *if X_1, X_2 and $f_1X_1 + f_2X_2$ are $\mathbb{R}$-independent, then f_1 and f_2 are functionally independent and there exists a non vanishing function λ such that*

$$i_{X_1}(g) = \lambda df_2, \quad i_{X_{2(g)}} = -\lambda df_1. \tag{9}$$

Proof. (1) A consequence of (8) (see [15] (section 7) for more details).

(2) In virtue of (1), f_1 and f_2 are functionally independent. Formula (8) gives $0 = L_{X_3}(g) = i_{X_1}(g)df_1 + i_{X_2}(g)df_2$. Since df_1 and df_2 are independent, this relation implies that df_1 divides $i_{X_2}(g)$ and df_2 divides $i_{X_1}(g)$. □

Lemma 5. *Let X_1, X_2 and $f_1X_1 + f_2X_2$, $f_1, f_2 \in C^\infty(M)$, be $\mathbb{R}$-independent Killing fields of a metric g. Assume that the 2-dimensional distribution $\mathcal{D}$ generated by X_1 and X_2 is completely integrable and g degenerates on its leaves. Then*

1. *if g completely degenerates along the leaves of $\mathcal{D}$ then $\{X_1, X_2, f_1X_1 + f_2X_2\}$ spans a 3-dimensional Abelian Killing algebra;*
2. *if $[X_1, X_2] = f_1X_1 + f_2X_2$, then $\{X_1, X_2, f_1X_1 + f_2X_2\}$ spans a Lie algebra isomorphic to $\mathcal{H}$ and g has rank one on the leaves of $\mathcal{D}$;*

(3) *if* $[X_1, X_2] = sX_2$, $s = 0, 1$, *and* g *has rank one on the leaves of* $\mathcal{D}$ *then* $s = 0$ *and* $\{X_1, X_2,\ f_1X_1 + f_2X_2\}$ *spans a Lie algebra isomorphic to* $\mathcal{H}$ *.*

Proof. (1) The previous lemma shows that f_1 and f_2 are functionally independent. Moreover, as it is easy to see from (9), one has

$$g(X_1, X_1) = \lambda X_1(f_2),$$
$$g(X_1, X_2) = -\lambda X_1(f_1) = \lambda X_2(f_2), \qquad g(X_2, X_2) = -\lambda X_2(f_1). \tag{10}$$

Since g completely degenerates on the leaves of $\mathcal{D}$, i.e., $g(X_i, X_j) = 0$, $i, j = 1, 2$, relations (10) imply $X_i(f_j) = 0$, $i, j = 1, 2$. Then $[X_1, f_1X_1 + f_2X_2] = f_2[X_1, X_2]$ and $[X_2, f_1X_1 + f_2X_2] = f_1[X_2, X_1]$ are two proportional Killing fields and lemma 1 now implies that $[X_1, X_2] = 0$.

(2) Corollary 1 gives

$$L_{X_1}(i_{X_1}(g)) = L_{X_2}(i_{X_2}(g)) = 0,$$
$$L_{X_1}(i_{X_2}(g)) = -L_{X_2}(i_{X_1}(g)) = i_{[X_1,X_2]}(g).$$

In view of (9) and $[X_1, X_2] = f_1X_1 + f_2X_2$ these relations become

$$\begin{aligned} &X_1(\lambda)df_2 + \lambda d(X_1(f_2)) = 0,\\ &X_2(\lambda)df_1 + \lambda d(X_2(f_1)) = 0,\\ &-X_1(\lambda)df_1 - \lambda d(X_1(f_1)) = \lambda\left(f_1df_2 - f_2df_1\right),\\ &X_2(\lambda)df_2 + \lambda d(X_2(f_2)) = \lambda\left(-f_1df_2 + f_2df_1\right). \end{aligned} \tag{11}$$

Since $X_1(f_1) = -X_2(f_2)$ (see (10)) the difference of last two relations (11) is as follows:

$$\left(X_1(\lambda) - 2\lambda f_2\right)df_1 + \left(X_2(\lambda) + 2\lambda f_1\right)df_2 = 0.$$

So, since f_1 and f_2 are functionally independent

$$X_1(\lambda) = 2\lambda f_2, \quad X_2(\lambda) = -2\lambda f_1. \tag{12}$$

Then substituting (12) into (11) one gets

$$\begin{aligned} X_1(f_1) &= -f_1f_2 + c, & X_1(f_2) &= -f_2^2 + c_1,\\ X_2(f_1) &= f_1^2 + c_2, & X_2(f_2) &= f_1f_2 - c \end{aligned} \tag{13}$$

with $c, c_1, c_2 \in \mathbb{R}$.

Hence the following commutation relations hold

$$[X_1, [X_1, X_2]] = cX_1 + c_1X_2, \quad [X_2, [X_1, X_2]] = c_2X_1 - cX_2.$$

On the other hand, g degenerates on Killing leaves of $\mathcal{D}$ iff

$$\begin{vmatrix} X_1(f_1) & X_1(f_2)\\ X_2(f_1) & X_2(f_2) \end{vmatrix} = 0$$

but in view of (13) this is equivalent to $-c_1f_1^2 + 2cf_1f_2 + c_2f_2^2 - (c_1c_2 + c^2) = 0$.

The last relation establishes a functional dependence between f_1 and f_2 if at least one of constants c_1, c_2 and c is different from zero. So, $c_1 = c_2 = c = 0$ and $[X_1, [X_1, X_2]] = [X_2, [X_1, X_2]] = 0$. Thus fields X_1, X_2 and $[X_1, X_2]$ span an algebra isomorphic to $\mathcal{H}$.

(3)The proof is obtained by modifying the previous one. In fact, using the relation $[X_1, X_2] = sX_2$ instead of $[X_1, X_2] = f_1X_1 + f_2X_2$, one easily finds the following analogues of (12) and (13)

$$X_1(\lambda) = 2s\lambda, \qquad X_1(f_1) = -sf_1 + c, \qquad X_1(f_2) = -sf_2 + c_1,$$
$$X_2(\lambda) = 0, \qquad X_2(f_1) = c_2, \qquad X_2(f_2) = sf_1 - c, \tag{14}$$

with $c, c_1, c_2 \in \mathbb{R}$. In this case the degeneracy of g on the leaves of $\mathcal{D}$ gives the conditions $s = 0, c^2 + c_1c_2 = 0$ and, therefore, the following commutation relations:

$$[X_1, X_2] = 0,$$
$$[X_1, f_1X_1 + f_2X_2] = cX_1 + c_1X_2, \qquad [X_2, f_1X_1 + f_2X_2] = c_2X_1 - cX_2.$$

Relations (10) show that g vanishes on leaves of $\mathcal{D}$ iff $X_i(f_j) = 0$, $\forall i, j$. In view of (14) and $s = c^2 + c_1c_2 = 0$ this is equivalent to $c_1 = c_2 = c = 0$. Since in the considered case the restricted to leaves metric is of rank 1, $c_1c_2 \neq 0$. Now it is easy to see that X_1, X_2 and $f_1X_1 + f_2X_2$ span an algebra isomorphic to $\mathcal{H}$. □

Proof of proposition 5. Assume that g degenerates along leaves of the distribution $\mathcal{D}$ defined by X and Y and $\dim \mathcal{G} > 2$. The proof now proceeds in two steps.

Step 1. First observe that $\mathcal{G}$ contains a sub-algebra either isomorphic to $\mathcal{A}_3$, or to $\mathcal{H}$. In fact, if $r = 0$, then according to lemma 5.1, $\mathcal{G} \supseteq \mathcal{A}_3$. If $r = 1$, then two cases can occur:

a)$[X, Y] = f_1X + f_2Y$, with $f_1, f_2 \in C^\infty(M)$ functionally independent;
b)$[X, Y] = k_1X + k_2Y$, with $k_1, k_2 \in \mathbb{R}$.

The case (a) is literally lemma 5.2. In the case (b) X and Y span a bidimensional Lie algebra and in order to apply lemma 5.3 it suffices to pass to a basis of this algebra, say, X_1 and X_2, such that $[X_1, X_2] = sX_2$.

Step 2. We prove that $\dim \mathcal{G} = 3$. Assume that X_1, X_2 and $f_1X_1 + f_2X_2$ are as in lemma 5 and let $\bar{Z} = \bar{f}_1X + \bar{f}_2Y$ be a Killing field $\mathbb{R}$-independent from $\{X_1, X_2, f_1X_1 + f_2X_2\}$. Then (see lemma 4.2)

$$i_X(g) = \lambda df_2 = \bar{\lambda} d\bar{f}_2, \quad i_Y(g) = -\lambda df_1 = -\bar{\lambda} d\bar{f}_1, \qquad \lambda \neq 0, \bar{\lambda} \neq 0.$$

These relations imply that $\bar{f}_1 = \bar{f}_1(f_1), \bar{f}_2 = \bar{f}_2(f_2)$ and $\bar{f}_1' = \bar{f}_2'$. Therefore $f_1' = f_2' = c$ and $\bar{f}_1 = cf_1 + c_1$, $\bar{f}_2 = cf_2 + c_2$, $c, c_1, c_2 \in \mathbb{R}$, and $\bar{Z}$ belongs to the span of X, Y and Z. □

Now we shall describe metrics (5) of type $(\mathcal{H}, 1)$ and metrics (7) of type $(\mathcal{A}_3, 0)$.

Corollary 2. *If g is a metric of type $(\mathcal{H}, 1)$, then there exists an adapted chart (x, y, u, v) such that:*

(1) $\mathcal{H} = \langle\{\partial_x, \partial_y, v\partial_x + x\partial_y\}\rangle$;

(2) *coefficients of the metric (5) are subject to*

$$\alpha = 0, \quad a = 0, \quad b = -\varepsilon_0 \beta^2.$$

Proof. Assume that Killing fields X, Y and $Z = f_1 X + f_2 Y$ form a basis of an algebra $\mathcal{G}$ isomorphic to $\mathcal{H}$ and $[X, Y] = [Y, Z] = 0$, $[X, Z] = Y$. Consider an adapted chart (x, y, u, v) with respect to X and Y. Then $X = \partial_x$, $Y = \partial_y$ and obviously $f_1 = \xi(u, v)$ and $f_2 = x + \eta(u, v)$. Observe that $\xi \neq const$ since, otherwise, $Z - f_1 X = f_2 Y \in \mathcal{G}$ and f_2 must be a constant according to lemma 1. Killing equation $L_Z(g) = 0$ for the metric (5) gives

$$\begin{cases} \alpha = 0, \\ \xi = \xi(v), \quad \eta = \eta(v), \\ a = \varepsilon_0 \eta' \beta^2, \quad b = -\varepsilon_0 \xi' \beta^2. \end{cases}$$

Now, since in the chart $\{\bar{x} = x + \eta, \bar{y} = y, \bar{u} = u, \bar{v} = \xi\}$ the form of metric (5) (with $s = 0$) is preserved, without loss in generality one may assume that $\xi = v$ and $\eta = 0$. In such an adapted chart $Z = v\partial_x + x\partial_y$ and $a = 0, b = -\varepsilon_0 \beta^2$. □

Corollary 3. *If g is a metric of type $(\mathcal{A}_3, 0)$, then there exists an adapted chart (x, y, u, v) such that:*

(1) $\mathcal{A}_3 = \langle \{\partial_x, \partial_y, v\partial_x + \eta(u, v)\partial_y\} \rangle$, *with* $\eta_u \neq 0$*;*
(2) *coefficients of the metric (7) are subject to*

$$a = \frac{\eta_v}{\eta_u}, \quad b = -\frac{1}{\eta_u}. \tag{15}$$

Proof. As before we assume that X, Y and $Z = f_1 X + f_2 Y$ form a basis of a Killing algebra isomorphic to $\mathcal{A}_3$. In an adapted to $\{X, Y\}$ chart (x, y, u, v) one has $X = \partial_x, Y = \partial_y$ and $Z = \xi(u, v)\partial_x + \eta(u, v)\partial_y$. Functions $f_1 = \xi(u, v)$ and $f_2 = \eta(u, v)$ are functionally independent (see lemma 4). Killing equation $L_Z(g) = 0$ for the metric (7) gives

$$\begin{cases} \xi = \xi(v), \\ \xi_v + \eta_u b = 0, \\ \xi_v a + \eta_v b = 0. \end{cases}$$

Now, since in the chart $\{\bar{x} = x, \bar{y} = y, \bar{u} = u, \bar{v} = \xi\}$ the form of metric (7) is preserved, one may assume without loss in generality that $\xi = v$. In such an adapted chart $Z = v\partial_x + \eta(u, v)\partial_y$ and $a = \dfrac{\eta_v}{\eta_u}, b = -\dfrac{1}{\eta_u}$. □

4. Ricci tensor for metrics of type $(\mathcal{G}, r), \dim \mathcal{G} = 2$

Throughout the paper the following definition of the Ricci tensor is used

$$ric(X, Y) := tr(W \mapsto R(W, X)Y).$$

Its components, with respect to the local chart $\{x_1, x_2, x_3, x_4\}$ in question, will be denoted by $R_{x_i x_j}$.

Sometimes it is more convenient to work with components of ric with respect to a non-holonomic frame field $\{e_i\}$. In such cases the components $ric(e_i, e_j)$ of ric will be denoted by $R_{(i)(j)}$. We recall that they read

$$R_{(i)(j)} = e_{[j}(\gamma^h_{h]i}) + \gamma^h_{[j|k|}\gamma^k_{h]i} - c^k_{jh}\gamma^h_{ki} \tag{16}$$

where the γ^l_{ij}'s are the Christoffel's symbols

$$\begin{aligned} \gamma^l_{ij} &= \frac{1}{2}g^{lh}\left(-e_h(g_{ij}) + e_i(g_{hj}) + e_j(g_{hi})\right) \\ &\quad -\frac{1}{2}(c^l_{ji} + g^{kl}g_{hi}c^h_{jk} + g^{kl}g_{hj}c^h_{ik}) \end{aligned} \tag{17}$$

and the c^h_{ij}'s are such that $[e_i, e_j] = c^h_{ij}e_h$.

Now we pass to collect some needed facts concerning the Ricci tensor.

The case $(\mathcal{G}, \mathbf{1}), \dim \mathcal{G} = 2$. Observe that in the basis of 1-forms

$$\begin{aligned} \theta_1 &= (sy\alpha + \beta)dx - \alpha dy \\ \theta_2 &= (-sby + a)dx + bdy \\ \theta_3 &= du \\ \theta_4 &= dv \end{aligned}$$

metric (5) reduces to the following simple form

$$g = \varepsilon_0\theta_1^2 + 2\theta_2\theta_4 + \varepsilon_1\theta_3^2 + 2p\theta_3\theta_4 + q\theta_4^2.$$

So it is more convenient to work with components of the Ricci tensor in the non-holonomic frame of vector fields dual to $\{\theta_i\}$:

$$\mathfrak{e}_1 = m^{-1}\left(b\partial_x + (sby - a)\,\partial_y\right), \quad \mathfrak{e}_2 = m^{-1}C, \quad \mathfrak{e}_3 = \partial_u, \quad \mathfrak{e}_4 = \partial_v \tag{18}$$

with $m = \alpha a + \beta b$.

The fact that $\mathfrak{e}_2$ is proportional to the characteristic vector field C simplifies much the further computations. In fact, it is readily seen that in the frame (18), the only non-vanishing c^k_{ij}'s are

$$\begin{aligned} &c^1_{12} = \frac{\alpha s}{m}, \quad c^2_{12} = -\frac{bs}{m}, && c^1_{13} = \frac{\alpha_u a + \beta_u b}{m}, && c^1_{14} = \frac{\alpha_v a + \beta_v b}{m}, \\ &c^2_{13} = \frac{a_u b - ab_u}{m}, && c^2_{14} = \frac{a_v b - ab_v}{m}, && c^1_{23} = \frac{\alpha\beta_u - \alpha_u\beta}{m}, \\ &c^1_{24} = \frac{\alpha\beta_v - \alpha_v\beta}{m}, && c^2_{23} = \frac{\alpha a_u + \beta b_u}{m}, && c^2_{24} = \frac{\alpha a_v + \beta b_v}{m} \end{aligned}$$

and a straightforward computation based on formulas (16) and (17) shows that

$$R_{(2)(2)} = \frac{s^2\alpha^2}{m^2}. \tag{19}$$

It is worth stressing that the simplicity of the component $R_{(2)(2)}$ is due to both the choice of frame (18) and to condition (ii) above.

The case $(\mathcal{G}, \mathbf{0}), \dim \mathcal{G} = 2$. In this case, a direct computation shows that in an adapted chart

$$R_{xx} = -\frac{s^2}{2}. \tag{20}$$

5. Ricci flat metrics of type $(\mathcal{G}_2, r)$

In this case $s \neq 0$ so that, in view of (19), the component $R_{(2)(2)}$ for a metric of type $(\mathcal{G}_2, 1)$ vanishes iff

$$\alpha = 0 \tag{21}$$

This is equivalent to the fact that the Killing field Y is characteristic (see proposition 3).

So, in view of (21) and (20) one gets the following

Proposition 6.

1. *For any Ricci flat metric of type* $(\mathcal{G}_2, 1)$ *the Killing vector field* Y *is* characteristic*;*
2. *There do not exist Ricci flat metrics of type* $(\mathcal{G}_2, 0)$.

Now we shall show how to solve Einstein equations $ric\,(g) = 0$ for metrics of type $(\mathcal{G}_2, 1)$.

Starting from condition (21), we shall proceed by simplifying and solving all the remaining Einstein's equations.

To simplify the exposition this procedure will be subdivided into following **5 steps**.

Step 1. In view of (21), a direct computation shows that $R_{(1)(3)} = (b_u - 2fb)/\,(2\beta b)$. So, $R_{(1)(3)} = 0$ iff

$$b_u = 2fb. \tag{22}$$

Step 2. Taking into account (22) one finds that $R_{(3)(3)} = 3f_u + 3f^2$, so that $R_{(3)(3)} = 0$ iff

$$f_u = -f^2. \tag{23}$$

From (22) and (23) one easily gets

$$f = \frac{k}{u+\varphi}, \quad b = \psi\,(u+\varphi)^{2k} \tag{24}$$

with $\psi = \pm e^{\nu}$, φ and ν being two arbitrary functions of v and $k = 0, 1$.

Step 3. In view of (21), (22) and (23) components $R_{(1)(1)}$ and $R_{(2)(4)}$ reduce to $R_{(1)(1)} = \varepsilon_0 R_{(2)(4)} = (\varepsilon_1 + 4\varepsilon_0 f^2\beta^2)/\,(2\varepsilon_1\beta^2)$.

Hence, for a Ricci flat metric, we have the relation $4f^2\beta^2 = -\varepsilon_0\varepsilon_1$ which is equivalent to

$$\beta = \pm\frac{u+\varphi}{2}, \quad \varepsilon_0 = -\varepsilon_1, \quad k = 1. \tag{25}$$

Step 4. Now relations (21), (22), (23) and (25) allow to simplify much the original expression for $R_{(3)(4)}$ which becomes

$$R_{(3)(4)} = -2\varepsilon_1(-f^2p - f^2a_u + 2f^3a - \varepsilon_1 f_v).$$

Hence $R_{(3)(4)} = 0$ is equivalent to

$$p = -a_u + 2fa - \varepsilon_1\frac{f_v}{f^2}.$$

Step 5. Finally the remaining non-vanishing component of the Ricci tensor is $R_{(4)(4)}$:

$$R_{(4)(4)} = \frac{\left(q_{uu} - fq_u + (2a_{uu} - 2fa_u)_v - 6\,(a_u - 2fa)\,f_v\right)b}{2\varepsilon_1 b} + \frac{(-2a_{u\,u} + 2fa_u)\,b_v}{2\varepsilon_1 b}$$

and equation $R_{(4)(4)} = 0$ reads

$$q_{uu} - fq_u = (2a_{u\,u} - 2fa_u)\,\frac{b_v}{b} - (2a_{uu} - 2fa_u)_v + 6\,(a_u - 2fa)\,f_v. \tag{26}$$

To solve this equation observe, first, that in view of (24)

$$\frac{b_v}{b} = \frac{\psi'(u+\varphi)^2 + 2\varphi'\psi(u+\varphi)}{\psi(u+\varphi)^2} = \frac{\psi'}{\psi} + 2\frac{\varphi'}{u+\varphi} = \nu' + 2\varphi' f$$

and then that

$$a_{uu} - fa_u = \frac{(fa_u)_u}{f}, \quad (a_{uu} - fa_u)_v = \frac{(fa_{uv})_u}{f} + \varphi' f^2 a_u.$$

This allows to rewrite (26) as

$$(fq_u)_u = 2\nu'(fa_u)_u - 2(fa_{vu})_u - 4\varphi'(f^3a - f^2a_u)_u$$

which is readily integrated

$$q = 2(\nu' a - a_v) + \frac{4a\varphi'}{u+\varphi} + (u+\varphi)^2\sigma_0 + \chi_0$$

with arbitrary $\sigma_0 = \sigma_0(v)$ and $\chi_0 = \chi_0(v)$.

Thus the general solution of the Einstein equations for metrics of type $(\mathcal{G}_2, 1)$ is

$$\begin{aligned} g = & -\frac{\varepsilon_1 (u+\varphi)^2}{4} dx^2 - 2\left(\varepsilon_2 y e^{\nu} (u+\varphi)^2 - a\right) dx dv \\ & +2\varepsilon_2 e^{\nu} (u+\varphi)^2 dy dv + 2\left(\frac{2a}{u+\varphi} - a_u + \varepsilon_1 \varphi'\right) dv du + \\ & +\varepsilon_1 du^2 + \left(2(\nu' a - a_v) + \frac{4a\varphi'}{u+\varphi} + (u+\varphi)^2 \sigma_0 + \chi_0\right) dv^2 \end{aligned} \tag{27}$$

with $\varepsilon_1, \varepsilon_2 = \pm 1$, and arbitrary $a = a(u,v), \varphi = \varphi(v), \nu = \nu(v), \sigma_0 = \sigma_0(v)$ and $\chi_0 = \chi_0(v)$.

In order to simplify the form (27) of the metric we pass to coordinates $(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ defined by

$$\begin{cases} \bar{x} = (u+\phi)\cosh\left(\frac{x}{2}\right) \\ \bar{y} = 2\left(\varepsilon_2 y - \frac{a e^{-\nu}}{(u+\phi)^2}\right) e^{-x} \\ \bar{u} = (u+\phi)\sinh\left(\frac{x}{2}\right) \\ \bar{v} = \int e^{\nu} dv. \end{cases}$$

In terms of these coordinates metric (27) reads

$$g = \varepsilon_1 \left(d\bar{x}^2 - d\bar{u}^2\right) + 2\left(\bar{x}+\bar{u}\right)^2 d\bar{y} d\bar{v} + \left[A\left(\bar{x}^2 - \bar{u}^2\right) + B\right] d\bar{v}^2 \tag{28}$$

with $A = A(\bar{v})$, $B = B(\bar{v})$ arbitrary functions.

Alternatively one may use coordinates $\{\tilde{x} = \bar{x} + \bar{u}, \tilde{y} = \bar{y}, \tilde{u} = \varepsilon_1(\bar{x} - \bar{u}), \tilde{v} = \bar{v}\}$ in which (28) becomes

$$g = d\tilde{x} d\tilde{u} + 2\tilde{x}^2 d\tilde{y} d\tilde{v} + \left[A\tilde{x}\tilde{u} + B\right] d\tilde{v}^2 \tag{29}$$

The chart $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ will be called *improved adapted*.

By summing up the previous results and noting that metric (28) admits a 2-dimensional distribution of null vectors, generated by the fields $\partial_{\tilde{x}}$ and $\partial_{\tilde{y}}$, one gets the following

Proposition 7.

(1) *There do not exist Ricci flat 4-metrics of type $(\mathcal{G}_2, 0)$;*
(2) *Ricci-flat 4-metrics of type $(\mathcal{G}_2, 1)$ are Kleinian and are of form (29), and vice versa.*

Remark 1. *Comparing this result with that concerning metrics of type $(\mathcal{G}_2, 2)$ (i.e., metrics nondegenerating along Killing leaves), see [15], one may observe that both the dimensions and complexity of the class of Ricci flat $(\mathcal{G}, r)$-type 4-metrics grows noticeably with r.*

References

[1] L. J. Alty. (1994). *Kleinian signature change.* Class. Quantum. Grav. 11, 2523.

[2] J. Barret, G. W. Gibbons, M. J. Perry, C. N. Pope and P. Ruback. (1994). *Kleinian geometry and $N = 2$ superstring.* Int. J. Mod. Phys. A 9, 1457.

[3] V.A. Belinsky and V.E. Zakharov. (1978). *Integration of the Einstein equations by means of the inverse scattering problem technique and construction of the exact soliton solutions.* Soviet Phys. JETP 75, 1953.

[4] D. Catalano Ferraioli and A. M. Vinogradov. (2004). *Ricci flat 4-metrics with bidimensional null orbits. Part II. The Abelian case.* DIPS 8/2004.

[5] R. Geroch. (1971). *A method for generating new solutions of Einstein's equation.* J. Math. Phys. 12, 918.

[6] R. Geroch. (1972). *A method for generating new solutions of Einstein's equations. II.* J. Math. Phys. 13, 394.

[7] Hartle J. B. and Hawking S. W. (1983). *Wave function of the universe.* Phys. Rev. D 28, 2960.

[8] S. Kobayashi, K. Nomizu.(1963). *Foundations of Differential Geometry*, Volume I. Interscience Publishers.

[9] Y. Matsushita. (1981). *On Euler characteristics of compact Einstein 4-manifolds of metric signature* $(++--)$. J. Math. Phys. 22, 979

[10] Y. Matsushita (1983). *Thorpe-Hitchin inequality for compact Einstein 4-manifolds of metric signature* $(++--)$ *and the generalized Hirzebruch index formula.* J. Math. Phys. 24, 36.

[11] H. Oguri and C. Vafa. (1991). *$N = 2$ heterotic strings.* Nucl. Phys. B 365, 83.

[12] Peter R. Law. (1991). *Neutral Einstein metrics in four dimensions.* J. Math. Phys. 32, 3039.

[13] Petrov, A. Z. (1969). *Einstein Spaces.* (Oxford: Pergamon).

[14] A. D. Sakharov. (1984). *Cosmological transitions with changes in the signature of the metric.* Sov. Phys. JETP 60, 214.

[15] G. Sparano, G. Vilasi, A. M. Vinogradov. (2002). *Vacuum Einstein metrics with bidimensional Killing leaves. I. Local aspects.* Differential Geometry and its Applications 16, 95.

[16] G. Sparano, G. Vilasi, A. M. Vinogradov. (2002). *Vacuum Einstein metrics with bidimensional Killing leaves. II. Global aspects.* Differential Geometry and its Applications 17, 15.

[17] G. Sparano, G. Vilasi, A. M. Vinogradov. (2001). *Gravitational Fields with a non-Abelian, bidimensional Lie algebra of symmetries.* Phys. Lett. B 513, 142

[18] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt. (2003).*Exact Solutions of Einstein's Field Equations.* Second Edition. Cambridge University Press.

Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy

E-mail address: `dcatalan@unisa.it`

Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA),Italy, Istituto Nazionale di Fisica Nucleare, sezione di Napoli-Salerno, and Diffiety Institute, Russia.

E-mail address: `vinograd@unisa.it`

E-mail address: `vinograd@unisa.it`