The Diffiety Institute Preprint Series

Preprint DIPS–5/2004 June 18, 2004

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DECOMPOSITIONS OF THE LOOP ALGEBRA OVER $so(4)$ AND INTEGRABLE MODELS OF THE CHIRAL EQUATION TYPE

O. V. EFIMOVSKAYA AND V. V. SOKOLOV

Abstract. Decompositions of the loop algebra over $so(4)$ are considered, and the exactly integrable nonlinear hyperbolic systems of the principal chiral field equation type are analyzed. New example of such system is found and the Lax representation for this example is constructed.

UDC 517.958+512.77

Introduction. In this article, we consider decompositions of the loop algebra $so(4)((\lambda))$ to the direct sum

$$so(4)((\lambda)) = so(4)[[\lambda]] \oplus \mathcal{G} \tag{1}$$

of vector spaces, where $so(4)[[\lambda]]$ denotes the Taylor series Lie subalgebra and $\mathcal{G}$ is some Lie subalgebra. This subalgebra $\mathcal{G}$ is called a *factoring subalgebra*. ¿From the results of the paper [2] it follows that the integrable system of the form

$$u_\xi = [u,\, Sv + vS^t], \qquad v_\eta = [v,\, \bar{S}u + u\bar{S}^t], \tag{2}$$

is assigned to each factoring subalgebra $\mathcal{G}$; here $u,\ v \in so(3)$, S and $\bar{S}$ are some constant matrices and the superscript t denotes transposition. The Lax pair for system (2) is expressed in terms of the projector onto the subalgebra $\mathcal{G}$ due to decomposition (1).

We will describe integrable models of the form (2) within the framework of [2] under the additional assumption that the matrices S and $\bar{S}$ are *diagonal*. In this case system (2) can be rewritten as follows

$$\mathbf{u}_\xi = \Lambda \mathbf{v} \times \mathbf{u}, \qquad \mathbf{v}_\eta = \bar{\Lambda}\, \mathbf{u} \times \mathbf{v}, \tag{3}$$

where

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad \bar{\Lambda} = \mathrm{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3).$$

Here $\mathbf{u} = (u_1,\ u_2,\ u_3)$ and $\mathbf{v} = (v_1,\ v_2,\ v_3)$ are vectors in $\mathbb{R}^3$ and the cross $\times$ denotes vector product. Quite obviously, system (3) is

1991 *Mathematics Subject Classification.* 35Q58, 37K10, 37K30.

Key words and phrases. Factoring subalgebras, Lax representations, integrable models, loop algebra, Cherednik's model.

This paper is submitted for publication in the Issue on Geometry of PDE in: *Fundamental'naya i Prikladnaya Matematika / Jour. of Mathematical Sci.* (2004).

The work was partially supported by the grants RFBR 02-01-00431 and NSh 1716.2003.1.

compatible with the condition $|\mathbf{u}| = |\mathbf{v}| = 1$, which is usually imposed when speaking about applications in physics or geometry.

The transformations

$$u_i \to -u_i, \qquad v_i \to -v_i, \qquad \xi \to -\xi, \qquad \eta \to -\eta$$

do not spoil the integrability of system (3) for any i. They are equivalent to the change $\lambda_i \to -\lambda_i$, $\bar\lambda_i \to -\bar\lambda_i$. In addition, compatible transpositions of diagonal elements within the matrices Λ and $\bar\Lambda$ are also allowed.

Further on, we will show that system (3) admits a Lax representation if the elements of the matrices Λ and $\bar\Lambda$ satisfy the relations

$$\lambda_1\bar\lambda_1(\lambda_3^2 - \lambda_2^2) + \lambda_2\bar\lambda_2(\lambda_1^2 - \lambda_3^2) + \lambda_3\bar\lambda_3(\lambda_2^2 - \lambda_1^2) = 0, \tag{4}$$

$$\lambda_1\bar\lambda_1(\bar\lambda_3^2 - \bar\lambda_2^2) + \lambda_2\bar\lambda_2(\bar\lambda_1^2 - \bar\lambda_3^2) + \lambda_3\bar\lambda_3(\bar\lambda_2^2 - \bar\lambda_1^2) = 0. \tag{5}$$

If all elements λ_i are nonzero, relation (4) implies the equalities

$$\bar\lambda_1 = \kappa_1\lambda_1 + \frac{\kappa_2}{\lambda_1}, \qquad \bar\lambda_2 = \kappa_1\lambda_2 + \frac{\kappa_2}{\lambda_2}, \qquad \bar\lambda_3 = \kappa_1\lambda_3 + \frac{\kappa_2}{\lambda_3}. \tag{6}$$

Relation (5) is equivalent to the equation

$$\kappa_1\kappa_2(\lambda_3^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2) = 0.$$

The case $\kappa_2 = 0$ corresponds to the known integrable model by I. Cherednik ([3]), while the case $\kappa_1 = 0$ was discovered by I. Golubchik and V. Sokolov, see [1, 2]. If $\kappa_i \neq 0$, then without loss of generality we put $\lambda_1 = \lambda_2$. Then we have $\bar\lambda_1 = \bar\lambda_2$, while λ_3, $\bar\lambda_3$ are arbitrary. Possibly, this case is new.

We conjecture that Eq. (4-5) are not only sufficient, but also necessary for exact integrability of the model given in Eq. (3). In principle, one can use the symmetry approach or the Painlevé test to prove this.

1. Factoring subalgebras for the loop algebras over $so(3)$ and $so(4)$

In the sequel, we represent the elements of $so(4)$ as block–diagonal matrices which consist of two blocks, each of them belonging to $so(3)$. By $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ we denote the standard basis in $so(3)$; namely, we put

$$\mathbf{e_1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Obviously, any factoring subalgebra $\mathcal{G}$ in $so(4)((\lambda))$ contains the series

$$
\begin{aligned}
\mathbf{E}_1 &= \frac{1}{\lambda}\begin{pmatrix} \mathbf{e}_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} * & 0 \\ 0 & \bar{\mathbf{c}} \end{pmatrix} + O(\lambda), \\
\mathbf{E}_2 &= \frac{1}{\lambda}\begin{pmatrix} \mathbf{e}_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} * & 0 \\ 0 & \bar{\mathbf{a}} \end{pmatrix} + O(\lambda), \\
\mathbf{E}_3 &= \frac{1}{\lambda}\begin{pmatrix} \mathbf{e}_3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} * & 0 \\ 0 & \bar{\mathbf{b}} \end{pmatrix} + O(\lambda),
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\bar{\mathbf{E}}_1 &= \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{e}_1 \end{pmatrix} + \begin{pmatrix} \mathbf{c} & 0 \\ 0 & * \end{pmatrix} + O(\lambda), \\
\bar{\mathbf{E}}_2 &= \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{e}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{a} & 0 \\ 0 & * \end{pmatrix} + O(\lambda), \\
\bar{\mathbf{E}}_3 &= \frac{1}{\lambda}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{e}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{b} & 0 \\ 0 & * \end{pmatrix} + O(\lambda),
\end{aligned} \tag{8}
$$

where

$$
\mathbf{a} = \sum a_i \mathbf{e}_i, \qquad \mathbf{b} = \sum b_i \mathbf{e}_i, \qquad \mathbf{c} = \sum c_i \mathbf{e}_i, \tag{9}
$$

$$
\bar{\mathbf{a}} = \sum \bar{a}_i \mathbf{e}_i, \qquad \bar{\mathbf{b}} = \sum \bar{b}_i \mathbf{e}_i, \qquad \bar{\mathbf{c}} = \sum \bar{c}_i \mathbf{e}_i \tag{10}
$$

are some elements of $so(3)$.

It is clear that the sum of the Lie subalgebra $\mathcal{G}$, which is generated by series (7-8), and the Taylor series subalgebra coincides with whole algebra $so(4)((\lambda))$. But for generic elements of the form (7), (8) this sum is not direct.

Proposition 1. *For any factoring subalgebra $\mathcal{G}$, the following commutator relations are satisfied*

$$
\begin{pmatrix} [\mathbf{E}_2, \bar{\mathbf{E}}_2] \\ [\mathbf{E}_2, \bar{\mathbf{E}}_3] \\ [\mathbf{E}_2, \bar{\mathbf{E}}_1] \end{pmatrix} = \begin{pmatrix} 0 & a_1 & -a_3 \\ 0 & b_1 & -b_3 \\ 0 & c_1 & -c_3 \end{pmatrix} \begin{pmatrix} \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_1 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{a}_1 & \bar{a}_3 \\ \bar{a}_1 & 0 & -\bar{a}_2 \\ -\bar{a}_3 & \bar{a}_2 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{E}}_2 \\ \bar{\mathbf{E}}_3 \\ \bar{\mathbf{E}}_1 \end{pmatrix}, \tag{11}
$$

$$
\begin{pmatrix} [\mathbf{E}_3, \bar{\mathbf{E}}_2] \\ [\mathbf{E}_3, \bar{\mathbf{E}}_3] \\ [\mathbf{E}_3, \bar{\mathbf{E}}_1] \end{pmatrix} = \begin{pmatrix} -a_1 & 0 & a_2 \\ -b_1 & 0 & b_2 \\ -c_1 & 0 & c_2 \end{pmatrix} \begin{pmatrix} \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_1 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{b}_1 & \bar{b}_3 \\ \bar{b}_1 & 0 & -\bar{b}_2 \\ -\bar{b}_3 & \bar{b}_2 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{E}}_2 \\ \bar{\mathbf{E}}_3 \\ \bar{\mathbf{E}}_1 \end{pmatrix}, \tag{12}
$$

$$(13)\quad \begin{pmatrix} [\mathbf{E}_1, \bar{\mathbf{E}}_2] \\ [\mathbf{E}_1, \bar{\mathbf{E}}_3] \\ [\mathbf{E}_1, \bar{\mathbf{E}}_1] \end{pmatrix} = \begin{pmatrix} a_3 & -a_2 & 0 \\ b_3 & -b_2 & 0 \\ c_3 & -c_2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_1 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{c}_1 & \bar{c}_3 \\ \bar{c}_1 & 0 & -\bar{c}_2 \\ -\bar{c}_3 & \bar{c}_2 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathbf{E}}_2 \\ \bar{\mathbf{E}}_3 \\ \bar{\mathbf{E}}_1 \end{pmatrix}.$$

Proof. Since the sum of $so(4)[[\lambda]]$ and $\mathcal{G}$ is supposed to be direct, the dimension of the vector space generated by elements (7), (8) and their pairwise commutators must be equal to 12. In other words, for any i and j commutators $[\mathbf{E}_i,\, \bar{\mathbf{E}}_j]$ are some linear combinations of generators (7), (8). For instance,

$$[\mathbf{E}_1,\, \bar{\mathbf{E}}_2] = \sum_{i=1}^{3} m_i \mathbf{E}_i + n_i \bar{\mathbf{E}}_i$$

for some constants m_i and n_i. Substituting the corresponding decompositions from (7), (8) for $\mathbf{E}_1$, $\bar{\mathbf{E}}_2$ into the above identity and equating the coefficients of λ^{-1}, we obtain explicit expressions for m_i and n_i in terms of the coefficients from formula (9). These expressions are given by the first row of matrix formula (13). In this manner, we obtain all identities (11-13). □

It is easy to see that for any factoring subalgebra $\mathcal{G}$ its projections onto the first and second blocks are factoring subalgebras for the algebra $so(3)((\lambda))$. Relations (11-13) describe the "interaction" between these two factoring subalgebras. If all constants a_i, b_i, c_i, $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$ are equal to zero simultaneously, then the algebra $\mathcal{G}$ is the direct sum of two factoring subalgebras for $so(3)((\lambda))$.

Another necessary condition for the sum of $so(4)[[\lambda]]$ and $\mathcal{G}$ to be direct is that the dimension of the vector space spanned over the vectors $\mathbf{E}_i$, $[\mathbf{E}_i,\, \mathbf{E}_j]$, and $[\mathbf{E}_i,\, [\mathbf{E}_j,\, \mathbf{E}_k]]$ be equal to 9. The same condition should be fulfilled for the generators $\bar{\mathbf{E}}_1$, $\bar{\mathbf{E}}_2$, $\bar{\mathbf{E}}_3$. The corresponding commutator relations coincide with the identities for the loop algebra $so(3)((\lambda))$ obtained in [4]. Namely, the following statement is valid.

Proposition 2. *Suppose $\mathcal{G}$ is a factoring subalgebra; then the following relations are satisfied*

$$(14)\quad \begin{pmatrix} [\mathbf{E}_1,\, [\mathbf{E}_2,\, \mathbf{E}_3]] \\ [\mathbf{E}_3,\, [\mathbf{E}_1,\, \mathbf{E}_2]] \\ [\mathbf{E}_2,\, [\mathbf{E}_3,\, \mathbf{E}_1]] \end{pmatrix} = \mathbf{A} \begin{pmatrix} [\mathbf{E}_3,\, \mathbf{E}_1] \\ [\mathbf{E}_1,\, \mathbf{E}_2] \\ [\mathbf{E}_2,\, \mathbf{E}_3] \end{pmatrix} + \mathbf{B} \begin{pmatrix} \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_1 \end{pmatrix}$$

and

$$(15)\qquad \begin{pmatrix} [\mathbf{E}_3,\,[\mathbf{E}_2,\,\mathbf{E}_3]]+[\mathbf{E}_1,\,[\mathbf{E}_1,\,\mathbf{E}_2]] \\ [\mathbf{E}_1,\,[\mathbf{E}_3,\,\mathbf{E}_1]]+[\mathbf{E}_2,\,[\mathbf{E}_2,\,\mathbf{E}_3]] \\ [\mathbf{E}_2,\,[\mathbf{E}_1,\,\mathbf{E}_2]]+[\mathbf{E}_3,\,[\mathbf{E}_3,\,\mathbf{E}_1]] \end{pmatrix} = \mathbf{C}\begin{pmatrix} [\mathbf{E}_3,\,\mathbf{E}_1] \\ [\mathbf{E}_1,\,\mathbf{E}_2] \\ [\mathbf{E}_2,\,\mathbf{E}_3] \end{pmatrix} + \mathbf{D}\begin{pmatrix} \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_1 \end{pmatrix},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *and* $\mathbf{D}$ *are some matrices of the form*

$$(16)\qquad \mathbf{A}=\begin{pmatrix} -u & w & 0 \\ u & 0 & -v \\ 0 & -w & v \end{pmatrix},\qquad \mathbf{B}=\begin{pmatrix} -\alpha & \beta & 0 \\ \alpha & 0 & -\gamma \\ 0 & -\beta & \gamma \end{pmatrix},$$

$$(17)\qquad \mathbf{C}=\begin{pmatrix} x & v & -w \\ -v & y & u \\ w & -u & z \end{pmatrix},\qquad \mathbf{D}=\begin{pmatrix} \varepsilon & \gamma & -\beta \\ -\gamma & \tau & \alpha \\ \beta & -\alpha & \delta \end{pmatrix}$$

and u, v, w, x, y, z, α, β, γ, δ, τ, *and* ε *are some numbers such that the conditions* $\operatorname{tr}\mathbf{C}=\operatorname{tr}\mathbf{D}=0$ *are satisfied. Moreover, there exist the constants* k_1 *and* k_2 *such that*

$$(18)\qquad k_1\mathbf{A}+k_2\mathbf{B}=0,\qquad k_1\mathbf{C}+k_2\mathbf{D}=0.$$

The analogous commutator relations link the generators $\bar{\mathbf{E}}_1$, $\bar{\mathbf{E}}_2$, $\bar{\mathbf{E}}_3$: one should put bars over all symbols in Eq. (14-18).

Now we recall a general construction from the paper [2]. Let

$$(19)\qquad L=\frac{d}{d\eta}+\sum u_i\mathbf{E}_i,\qquad M=\frac{d}{d\xi}+\sum v_i\bar{\mathbf{E}}_i.$$

Then the relation $[L,\,M]=0$ is equivalent to the system

$$(20)\qquad u_\xi=[u,\;v_1\mathbf{c}+v_2\mathbf{a}+v_3\mathbf{b}],\qquad v_\eta=[v,\;u_1\bar{\mathbf{c}}+u_2\bar{\mathbf{a}}+u_3\bar{\mathbf{b}}],$$

where $u=\sum u_i\mathbf{e}_i$, $v=\sum v_i\mathbf{e}_i$. Indeed, the commutator $[L,\,M]$ lies in $\mathcal{G}$, and its asymptotic behaviour is $Z/\lambda+O(1)$, while system (20) is equivalent to the condition that the residue Z be equal to zero. The algebra $\mathcal{G}$ does not contain non-zero Taylor series, therefore the equation $Z=0$ is equivalent to the condition $[L,\,M]=0$.

One easily checks that Eq. (20) can be rewritten in the form (2), where

$$(21)\qquad S=\begin{pmatrix} \frac{1}{2}(a_2-b_3+c_1) & a_3 & -c_3 \\ b_2 & \frac{1}{2}(-a_2+b_3+c_1) & c_2 \\ -b_1 & a_1 & \frac{1}{2}(a_2+b_3-c_1) \end{pmatrix}$$

and $\bar{S}$ is obtained by putting the bar over each symbol in the formula above.

Within this paper, we assume that the matrices S and $\bar{S}$ are diagonal, which implies

$$a_1=\bar{a}_1=a_3=\bar{a}_3=b_1=\bar{b}_1=b_2=\bar{b}_2=c_2=\bar{c}_2=c_3=\bar{c}_3=0.$$

Then the matrices Λ and $\bar\Lambda$ in Eq. (3) are

$$\Lambda = \operatorname{diag}(c_1,\ a_2,\ b_3), \qquad \bar\Lambda = \operatorname{diag}(\bar c_1\ ,\bar a_2,\ \bar b_3). \tag{22}$$

We also assume that none of these matrices is zero, otherwise system (3) becomes trivial.

One can check that in this case the structure of the generators (7),(8) is of the form

$$\mathbf{E}_i = \begin{pmatrix} q_i\,\mathbf{e}_i & 0 \\ 0 & \bar p_i\,\mathbf{e}_i \end{pmatrix}, \qquad \bar{\mathbf{E}}_i = \begin{pmatrix} p_i\,\mathbf{e}_i & 0 \\ 0 & \bar q_i\,\mathbf{e}_i \end{pmatrix}, \tag{23}$$

where q_i and $\bar q_i$ are some scalar Laurent series with the asymtotics $1/\lambda + O(1)$ and p_i, $\bar p_i$ are Taylor series.

Now we describe all factoring subalgebras with the generators given by Eq. (23). Substituting Eq. (23) into commutation relations (11-15), we get an overdetermined system of algebraic equations for q_i, p_i, $\bar q_i$, $\bar p_i$. Quite obviously, the relations

$$u = \bar u = v = \bar v = w = \bar w = \alpha = \bar\alpha = \beta = \bar\beta = \gamma = \bar\gamma = 0$$

are its compatibility conditions.

The system consists of the equations

$$\begin{aligned} c_1q_2 - q_3p_1 + \bar b_3p_2 &= 0, & c_1q_3 - q_2p_1 + \bar a_2p_3 &= 0, \\ a_2q_1 - q_3p_2 + \bar b_3p_1 &= 0, & a_2q_3 - q_1p_2 + \bar c_1p_3 &= 0, \\ b_3q_2 - q_1p_3 + \bar c_1p_2 &= 0, & b_3q_1 - q_2p_3 + \bar a_2p_1 &= 0, \end{aligned} \tag{24}$$

$$\begin{aligned} y\ q_1q_2 - \tau\ q_3 + q_1^2q_3 - q_2^2q_3 &= 0, \\ z\ q_2q_3 - \delta\ q_1 + q_1q_2^2 - q_1q_3^2 &= 0, \\ x\ q_1q_3 - \varepsilon\ q_2 + q_2q_3^2 - q_1^2q_2 &= 0, \end{aligned} \tag{25}$$

$$\begin{aligned} \bar y\ p_1p_2 - \bar\tau\ p_3 + p_1^2p_3 - p_2^2p_3 &= 0, \\ \bar z\ p_2p_3 - \bar\delta\ p_1 + p_1p_2^2 - p_1p_3^2 &= 0, \\ \bar x\ p_1p_3 - \bar\varepsilon\ p_2 + p_2p_3^2 - p_1^2p_2 &= 0. \end{aligned} \tag{26}$$

There is also the system symmetric to (24-26) with bars over all symbols.

¿From Eq. (24) and (25) it follows that

$$p_1 = c_1 + O(\lambda), \qquad p_2 = a_2 + O(\lambda), \qquad p_3 = b_3 + O(\lambda), \tag{27}$$

$$q_2^2 - q_1^2 = \frac{y}{\lambda} + O(1), \qquad q_3^2 - q_2^2 = \frac{z}{\lambda} + O(1), \qquad q_1^2 - q_3^2 = \frac{x}{\lambda} + O(1).$$

Excluding the unknowns q_1, q_2, q_3 from Eq. (24), we obtain the equation

$$a_2\bar a_2(p_1^2 - p_3^2) + b_3\bar b_3(p_2^2 - p_1^2) + c_1\bar c_1(p_3^2 - p_2^2) = 0,$$

whence, by using Eq. (27),(22), we deduce that condition (4) is necessary. Condition (5) is obtained if we start from the system symmetric to Eq. (24). The equation

$$p_1^2(q_2^2-q_3^2)+p_2^2(q_3^2-q_1^2)+p_3^2(q_1^2-q_2^2)=0 \tag{28}$$

also follows from Eq. (24). Also, formulae (28) and (27) imply the equation

$$c_1^2z+a_2^2x+b_3^2y=0.$$

Taking into account the condition $x+y+z=0$, we conclude that either the relation

$$z=\kappa(a_2^2-b_3^2),\qquad x=\kappa(b_3^2-c_1^2),\qquad y=\kappa(c_1^2-a_2^2) \tag{29}$$

holds for some κ, or the relation $c_1^2=a_2^2=b_3^2$ is valid. The second case corresponds to the unit matrix Λ (up to the transformations mentioned in the introduction).

The unknows p_1, p_2, p_3 are also found from system (24): they are

$$\begin{aligned} p_1&=\frac{c_1q_1q_2^2+a_2\bar{b}_3q_2q_3+\bar{c}_1b_3\bar{b}_3q_1}{q_1q_2q_3-\bar{c}_1\bar{a}_2\bar{b}_3},\\ p_2&=\frac{a_2q_2q_3^2+b_3\bar{c}_1q_1q_3+\bar{a}_2c_1\bar{c}_1q_2}{q_1q_2q_3-\bar{c}_1\bar{a}_2\bar{b}_3},\\ p_3&=\frac{b_3q_3q_1^2+c_1\bar{a}_2q_1q_2+\bar{b}_3a_2\bar{a}_2q_3}{q_1q_2q_3-\bar{c}_1\bar{a}_2\bar{b}_3}. \end{aligned} \tag{30}$$

The denominators in these expressions are nonzero due to Eq. (27). One can check that the other three equations in system (24) are equivalent to the relations

$$c_1z=a_2\bar{b}_3-\bar{a}_2b_3,\qquad a_2x=b_3\bar{c}_1-\bar{b}_3c_1,\qquad b_3y=c_1\bar{a}_2-\bar{c}_1a_2, \tag{31}$$

$$c_1\delta=\bar{c}_1(a_2\bar{a}_2-b_3\bar{b}_3),\quad a_2\varepsilon=\bar{a}_2(b_3\bar{b}_3-c_1\bar{c}_1),\quad b_3\tau=\bar{b}_3(c_1\bar{c}_1-a_2\bar{a}_2), \tag{32}$$

since equations (25) hold.

Now we consider system (25). Taking into account the condition $x+y+z=\varepsilon+\delta+\tau=0$, from Eq. (25) we get

$$\delta q_1^2+\varepsilon q_2^2+\tau q_3^2=0,\qquad zq_2^2q_3^2+xq_1^2q_3^2+yq_1^2q_2^2=0.$$

Three cases follow from these relations and the second condition in Eq. (18). Namely, either we obtain the equality $x=y=z=0$, or the equation $\varepsilon=\delta=\tau=0$ holds, or, finally, two series among q_1, q_2, and q_3 coincide. In the last case, we put $q_2=q_1$ without loss of generality; then we also conclude that either $\tau=y=0$ or $q_3=q_1$.

Case 1. Consider the case $x=y=z=0$. In addition, suppose that $q_i\neq q_j$ for any $i\neq j$ (unless we get Case 3). System (25) acquires the form

$$q_2^2-q_3^2=\delta,\qquad q_3^2-q_1^2=\varepsilon,\qquad q_1^2-q_2^2=\tau.$$

¿From Eq. (31) and (32) it follows that the relations

$$\bar c_1 = kc_1, \qquad \bar a_2 = ka_2, \qquad \bar b_3 = kb_3,$$
$$\delta = k^2(a_2^2 - b_3^2), \qquad \varepsilon = k^2(b_3^2 - c_1^2), \qquad \tau = k^2(c_1^2 - a_2^2)$$

hold. Here k is a nonzero parameter which can be eliminated from system (3) by a scaling of the independent variable η. The parameter λ in the factoring subalgebra problem is defined up to the change

$$\lambda \mapsto \lambda + k_2\lambda^2 + k_3\lambda^3 + \cdots , \tag{33}$$

therefore, without loss of generality, we assume that the conditions

$$q_1^2 - k^2c_1^2 = q_2^2 - k^2a_2^2 = q_3^2 - k^2b_3^2 = \frac{1}{\lambda^2}$$

are fulfilled. We finally have

$$q_1 = \frac{\sqrt{1+k^2c_1^2\lambda^2}}{\lambda}, \qquad q_2 = \frac{\sqrt{1+k^2a_2^2\lambda^2}}{\lambda}, \qquad q_3 = \frac{\sqrt{1+k^2b_3^2\lambda^2}}{\lambda},$$
$$p_1 = c_1\sqrt{1+k^2a_2^2\lambda^2}\sqrt{1+k^2b_3^2\lambda^2} + ka_2b_3\lambda\sqrt{1+k^2c_1^2\lambda^2},$$
$$p_2 = a_2\sqrt{1+k^2c_1^2\lambda^2}\sqrt{1+k^2b_3^2\lambda^2} + kc_1b_3\lambda\sqrt{1+k^2a_2^2\lambda^2},$$
$$p_3 = b_3\sqrt{1+k^2c_1^2\lambda^2}\sqrt{1+k^2a_2^2\lambda^2} + kc_1a_2\lambda\sqrt{1+k^2b_3^2\lambda^2}.$$

The formulae for the functions $\bar q_i$, $\bar p_i$ are similar.

Now we consider the upper block in the equation $[L, M] = 0$ for the operators L and M, which are given in Eq. (19), and obtain the Lax operators

$$L = \frac{d}{d\eta} + \sum_{i=1}^{3} u_i q_i \mathbf{e}_i, \qquad M = \frac{d}{d\xi} + \sum_{i=1}^{3} v_i p_i \mathbf{e}_i \tag{34}$$

for Cherednik's model. Here the coefficients belong to $so(3)$. The lower block defines another Lax pair which is completely analogous.

Case 2. Consider the case $\varepsilon = \delta = \tau = 0$; assume that $q_i \neq q_j$ for any $i \neq j$. Taking into account Eq. (29), we rewrite system (25) in the form

$$\frac{q_1q_2}{q_3} - \kappa b_3^2 = \frac{q_1q_3}{q_2} - \kappa a_2^2 = \frac{q_2q_3}{q_1} - \kappa c_1^2 = \frac{1}{\lambda},$$

whence we get

$$q_1 = \frac{\sqrt{1+\kappa a_2^2\lambda}\sqrt{1+\kappa b_3^2\lambda}}{\lambda}, \qquad q_2 = \frac{\sqrt{1+\kappa c_1^2\lambda}\sqrt{1+\kappa b_3^2\lambda}}{\lambda},$$
$$q_3 = \frac{\sqrt{1+\kappa c_1^2\lambda}\sqrt{1+\kappa a_2^2\lambda}}{\lambda},$$
$$p_1 = c_1\lambda q_1, \qquad p_2 = a_2\lambda q_2, \qquad p_3 = b_3\lambda q_3,$$
$$c_1\bar c_1 = a_2\bar a_2 = b_3\bar b_3 = \kappa c_1a_2b_3.$$

One can easily obtain analogous expressions that define the functions $\bar{q}_i$, $\bar{p}_i$.

Case 3a. Consider the case $q_1 = q_2$ and assume $\tau = y = 0$. ¿From Eq. (28) it follows that $p_1^2 = p_2^2$, and Eq. (27) implies the condition $c_1^2 = a_2^2$. Also, from Eq. (24) we easily get $\bar{c}_1^2 = \bar{a}_2^2$. Consider the case $p_2 = p_1$, $c_1 = a_2$, $\bar{c}_1 = \bar{a}_2$. By definition, put $\bar{c}_1 = tc_1$ and $k = \bar{b}_3 + b_3 t$, where k and t are arbitrary constants; further on, we define λ from the equation

$$q_1^2 - \bar{c}_1^2 = \frac{1}{\lambda^2}.$$

Then, the solution is

$$q_1 = q_2 = \frac{\sqrt{1 + t^2 c_1^2 \lambda^2}}{\lambda}, \qquad q_3 = \frac{\sqrt{4 + k^2\lambda^2}}{2\lambda} + \frac{k}{2} - b_3 t,$$
$$p_1 = p_2 = \frac{1}{2} c_1 \,(k\lambda + \sqrt{4 + k^2\lambda^2})\sqrt{1 + t^2 c_1^2 \lambda^2},$$
$$p_3 = b_3 + \frac{1}{2} t c_1^2 \lambda\,(k\lambda + \sqrt{4 + k^2\lambda^2}).$$

The Lax pair (19), (23) assigned to this solution is one of the main results of the present paper. The constants t and k look a bit artificial; they are introduced in order to absorb the degenerate case which is considered below. The case $p_2 = -p_1$ is treated quite analogously and results in the same system (3).

Case 3b. Let the relation $q_1 = q_2 = q_3$ hold. ¿From Eq. (25) we obviously get $x = y = z = \delta = \varepsilon = \tau = 0$. Also, from Eq. (24) we obtain

$$(p_1, p_2, p_3) = t\,(c_1, a_2, b_3), \tag{35}$$

where t is a Taylor series in λ.

The terms a_2, b_3, c_1 do not equal zero simultaneously; consequently, the constant term t_0 of the series t is also nontrivial and the relation

$$(\bar{c}_1, \bar{a}_2, \bar{b}_3) = t_0\,(c_1, a_2, b_3) \tag{36}$$

holds. By using Eq. (35), from Eq. (24) we obtain the equations

$$(c_1^2 - a_2^2)(1 - t) = 0, \qquad (b_3^2 - a_2^2)(1 - t) = 0, \qquad (c_1^2 - b_3^2)(1 - t) = 0.$$

Clearly enough, either all p_i are constants or $a_2^2 = b_3^2 = c_1^2$. In the first case, without loss of generality we assume that $a_2 = c_1 = \bar{a}_2 = \bar{c}_1 = p_1 = p_2 = 0$. The final result is

$$q_1 = q_2 = q_3 = \frac{1}{\lambda}, \quad p_1 = p_2 = 0, \quad p_3 = b_3.$$

We finally note that these formulae are a particular subcase within Case 3a if we put $c_1 = 0$.

In the second case, suppose that $c_1 = a_2 = b_3$. Then Eq. (36) implies $\bar{c}_1 = \bar{a}_2 = \bar{b}_3$, and therefore $p_1 = p_2 = p_3$. The solution

$$q_i = \frac{1}{\lambda} + \bar{c}_1, \qquad p_i = c_1(1 + \bar{c}_1\lambda)$$

is absorbed in Case 2 if $\kappa = 1/\bar{c}_1$ and $c_1 = b_3 = a_2$.

Acknowledgements. The authors thank M. V. Pavlov for stimulating discussions.

References

[1] *I. Z. Golubchik, V. V. Sokolov*, One more instance of the classical Yang–Baxter equation, in: Funkts. analiz i ego pilozh., **34** (2000) 4, 75–78

[2] *I. Z. Golubchik, V. V. Sokolov*, Compatible Lie brackets and integrable equations of the type of the principal chiral field model, in: Funkts. analiz i ego pilozh., **36** (2002) 3, 9-19 (in Russian).

[3] *I. V. Cherednik*, On the integrability of the two–dimensional asymmetric chiral $O(3)$-field and its quantum analog, in: Yadernaya fizika, **33** (1981), 278–282 (in Russian).

[4] *V. V. Sokolov*, On the decompositions of the loop algebra over $so(3)$ to a sum of two subalgebras, in: Doklady RAN(to appear).

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