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TOWARDS CLASSIFICATION OF CONDITIONALLY INTEGRABLE EVOLUTION SYSTEMS IN $(1+1)$ DIMENSIONS

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ABSTRACT. We generalize earlier results of Fokas and Liu and find all locally analytic (1+1)-dimensional evolution equations of order n that admit an N-shock type solution with $N \leq n+1$.

To this end, we develop a refinement of the technique from our earlier work (A. Sergyeyev, *J. Phys. A*: *Math. Gen*, **35** (2002), 7653–7660), where we completely characterized all (1+1)-dimensional evolution systems $\boldsymbol{u}_t = \boldsymbol{F}(x, t, \boldsymbol{u}, \partial\boldsymbol{u}/\partial x, \dots, \partial^n\boldsymbol{u}/\partial x^n)$ that are conditionally invariant under a given generalized (Lie–Bäcklund) vector field $\boldsymbol{Q}(x, t, \boldsymbol{u}, \partial\boldsymbol{u}/\partial x, \dots, \partial^k\boldsymbol{u}/\partial x^k)\partial/\partial\boldsymbol{u}$ under the assumption that the system of ODEs $\boldsymbol{Q} = 0$ is totally nondegenerate. Every such conditionally invariant evolution system admits a reduction to a system of ODEs in t, thus being a nonlinear counterpart to quasi-exactly solvable models in quantum mechanics.

Introduction. In order to be relevant, mathematical models of natural phenomena are often required to have solutions of prescribed form (e.g. travelling wave solutions, kinks, solitons, etc.). As was first shown by Fokas and Liu [4, 5] (see also Zhdanov [15]), a natural way to achieve this for evolution systems in (1+1) dimensions is to require that the system in question admit a generalized conditional symmetry (GCS) such that the solutions invariant under this symmetry have the desired form. It turns out that the set of systems admitting GCS (such systems, or even more broadly, the systems possessing integrable reductions, are often referred to as *conditionally integrable*, see e.g. [9]) is considerably larger than that of systems integrable via the inverse scattering transform or direct linearization, see e.g. [4, 5]. Notice [16]

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that if an (1+1)-dimensional evolution equation admits a reduction to a system of ODEs in the evolution parameter (time t), then it admits a GCS, and vice versa.

This naturally leads to the following problem: how to describe all evolution systems that admit a given GCS? For the case of linear GCS with time-independent coefficients this problem, restated in the terms of the so-called invariant modules, was completely solved in the seminal paper [6] (see also an important earlier work [13]). For the nonlinear GCS, some partial results were obtained in [2], and the complete solution of the problem was obtained in [12] under certain nondegeneracy assumptions, see below for details; see also [6] and [12] for the survey of earlier results in the field and the discussion of the role played by the GCS in the search of exact solutions of nonlinear PDEs.

However, the formulas found in [12] are not very convenient for applications. In the present paper we obtain alternative, easier to use, formulas for the locally analytic (1+1)-dimensional evolution systems that are conditionally invariant under a given *analytic* GCS. This is done in Section 3. In Section 4 we illustrate the application of our results by a number of examples. In particular, we completely characterize all locally analytic (1+1)-dimensional evolution equations of order n that admit an N-shock type solution with $N \leq n+1$.

1. Preliminaries

Consider an evolution system

$$\partial \boldsymbol{u}/\partial t = \boldsymbol{F}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \dots, \boldsymbol{u}_n), \quad n \geq 0, \tag{1}$$

for an s-component vector function $\boldsymbol{u} = (u^1, \dots, u^s)^T$, where $\boldsymbol{u}_l = \partial^l \boldsymbol{u}/\partial x^l$, $l = 0, 1, 2, \dots$, $\boldsymbol{u}_0 \equiv \boldsymbol{u}$, and the superscript '$T$' denotes the matrix transposition.

A smooth function of $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}_2, \dots$ is called *local* (see [7]; cf. also [14] and [8]) if it depends on a finite number of $\boldsymbol{u}_j$. The greatest integer m such that $\partial f/\partial \boldsymbol{u}_m \neq 0$ is called the *order* of a differential function f, and we shall write that as $m = \operatorname{ord} f$. If f depends only on x and t, then we shall assume that $\operatorname{ord} f = 0$. Unless otherwise explicitly stated, all functions considered below will be assumed to be local.

A generalized vector field $\mathcal{Q} = \boldsymbol{Q}\,\partial/\partial \boldsymbol{u}$, where $\boldsymbol{Q}$ is an s-component local vector function is called [4, 5, 15] a *generalized conditional symmetry* (GCS) for (1) if the system $\boldsymbol{Q} = 0$ is compatible with (1).

Eq.(1) is compatible with $\boldsymbol{Q} = 0$ if and only if (cf. e.g. [4, 15])

$$D_t(\boldsymbol{Q})|_{\mathcal{M}} = 0, \tag{2}$$

where $D_t = \partial/\partial t + \sum_{i=0}^{\infty} D^i(\boldsymbol{F})\partial/\partial \boldsymbol{u}_i$ and $D = \partial/\partial x + \sum_{i=0}^{\infty} \boldsymbol{u}_{i+1}\partial/\partial \boldsymbol{u}_i$ are total t- and x-derivatives, and $\mathcal{M}$ is the solution manifold of the system $\boldsymbol{Q} = 0$.

In what follows we shall treat $\boldsymbol{Q}=0$ as a system of ODEs involving an extra parameter t, and assume that this system is totally nondegenerate, i.e., the systems $D^j(\boldsymbol{Q})=0$ are locally solvable and have maximal rank for all $j=0,1,2,\dots$, see Ch. 2 of [8] for more details on this. Then the condition (2) is equivalent [8] to the following: there exist s-component local vector functions $\boldsymbol{\eta}_{\alpha,j}$ and an integer p such that

$$D_t(\boldsymbol{Q})=\sum_{\alpha=1}^{s}\sum_{j=0}^{p}\boldsymbol{\eta}_{\alpha,j}D^j(Q^\alpha). \tag{3}$$

Finding all GCS admitted by a given system (1) is an extremely difficult task, whose complexity is comparable to that of the problem of finding all solutions for the system in question. Surprisingly enough, we succeeded [12] in completely solving the inverse problem of describing all systems (1) that admit a given GCS $\mathcal{Q}=\boldsymbol{Q}\partial/\partial\boldsymbol{u}$, provided the system $\boldsymbol{Q}=0$, considered as a system of ODEs, is totally nondegenerate.

Let us briefly review the results of [12]. Consider a system of ODEs

$$\boldsymbol{Q}(x,t,\boldsymbol{u},\boldsymbol{u}_1,\dots,\boldsymbol{u}_k)=0, \tag{4}$$

involving t as a parameter. Here $\boldsymbol{Q}=(Q^1,\dots,Q^s)^T$ is an s-component local vector function.

Assume that the general solution of (4) in implicit form can be written as

$$\boldsymbol{G}(x,t,\boldsymbol{u},c_1(t),\dots,c_N(t))=0. \tag{5}$$

The number N is sometimes called the total order of (4). It is tacitly assumed here that $\boldsymbol{G}$ depends on N arbitrary functions $c_i(t)$, $i=1,\dots,N$, in an essential way, and that $\det\partial\boldsymbol{G}/\partial\boldsymbol{u}\neq 0$.

Then using the implicit function theorem we can, at least locally, obtain the general solution in the explicit form as $\boldsymbol{u}=\boldsymbol{P}(x,\ t,\ c_1(t),\ \dots,\ c_N(t))$, whence

$$u_j^\alpha=\partial^j P^\alpha(x,t,c_1(t),\dots,c_N(t))/\partial x^j.$$

Using these equations we can express c_i as functions of x, t, u^1, $\dots$, $u^1_{n_1-1}$, $\dots$, u^s, $\dots$, $u^s_{n_s-1}$ for some numbers $n_1,\dots,n_s$:

$$c_i=h_i(x,t,u^1,\dots,u^1_{n_1-1},\dots,u^s,\dots,u^s_{n_s-1}),\quad i=1,\dots,N.$$

Let $\tilde{\boldsymbol{B}}_i=-\left(\partial\boldsymbol{G}/\partial\boldsymbol{u}\right)^{-1}\partial\boldsymbol{G}/\partial c_i$, $\tilde{\boldsymbol{R}}=-\left(\partial\boldsymbol{G}/\partial\boldsymbol{u}\right)^{-1}\partial\boldsymbol{G}/\partial t$, $\boldsymbol{B}_i=\tilde{\boldsymbol{B}}_i(x,\ t,\ h_1,\ \dots,\ h_N)$, and $\boldsymbol{R}(x,\ t,\ \boldsymbol{u},\ \boldsymbol{u}_1,\ \dots,\ \boldsymbol{u}_r)\equiv\tilde{\boldsymbol{R}}(x,\ t,\ h_1,\ \dots,\ h_N)$, i.e., $\boldsymbol{B}_i$ and $\boldsymbol{R}$ are obtained from $\tilde{\boldsymbol{B}}_i$ and $\tilde{\boldsymbol{R}}$ by substituting h_i for c_i. Here it is understood that c_i are *not* differentiated with respect to t while evaluating $\partial\boldsymbol{G}/\partial t$.

Let V be an open domain in the space $\mathcal{V}$ of variables $x,t,\boldsymbol{u},\boldsymbol{u}_1,\dots$, and let W be the set of all points in V satisfying the equations $D^j(\boldsymbol{Q})=0$, $j=0,1,2,\dots$, considered as algebraic equations.

Theorem 1 ([12]). *Suppose that $\boldsymbol{Q}(x,t,\boldsymbol{u},\dots,\boldsymbol{u}_k)=0$, considered as a system of ODEs, is analytic on V, totally nondegenerate on W, and has the same total order N on the whole of W.*

If $\boldsymbol{Q}=0$ is compatible with (1), i.e., $\boldsymbol{Q}\partial/\partial\boldsymbol{u}$ is a generalized conditional symmetry for (1), then $\boldsymbol{F}$ on V can be represented in the form

$$\begin{aligned}\boldsymbol{F}=\boldsymbol{R}+\sum_{i=1}^{N}\zeta_i(t,h_1,\dots,h_N)\boldsymbol{B}_i\\ +\sum_{p=0}^{m}\sum_{\alpha=1}^{s}\boldsymbol{\chi}_{p,\alpha}(t,x,\boldsymbol{u},\dots,\boldsymbol{u}_{j_{p,\alpha}})D^p(Q^\alpha),\end{aligned} \tag{6}$$

where m and $j_{p,\alpha}$ are nonnegative integers, and ζ_i and $\boldsymbol{\chi}_{p,\alpha}$ are smooth functions of their arguments.

By construction [12], the system $\boldsymbol{u}_t=\boldsymbol{F}$ with $\boldsymbol{F}$ given by (6) admits a solution of the same form as the general solution (5) of $\boldsymbol{Q}=0$, that is,

$$\boldsymbol{G}(x,t,\boldsymbol{u},c_1(t),\dots,c_N(t))=0,$$

but now the $c_i(t)$ must satisfy

$$dc_i/dt=\zeta_i(t,c_1,\dots,c_N),\quad i=1,\dots,N, \tag{7}$$

rather than be arbitrary functions of t.

In other words, under the assumptions of Theorem 1 if $\boldsymbol{Q}\partial/\partial\boldsymbol{u}$ is a GCS for $\boldsymbol{u}_t=\boldsymbol{F}$, then the substitution of the general solution (5) of $\boldsymbol{Q}=0$ into $\boldsymbol{u}_t=\boldsymbol{F}$ reduces the latter to the system of ODEs (7).

2. Solving the inverse problem: which systems admit a given GCS?

How to pick among the $\boldsymbol{F}$'s (6) those of order $\leq n$, where n is a given natural number? This can be done using the following result.

Theorem 2. *Under the assumptions of Theorem 1 on $\boldsymbol{Q}$, suppose that*

$$Q^\alpha=u^\alpha_{n_\alpha}-g^\alpha(x,t,\tilde{u}), \tag{8}$$

where g^α are analytic functions of their arguments and

$$\tilde{u}=(u^1,\dots,u^1_{n_1-1},\dots,u^s,\dots,u^s_{n_s-1}).$$

Then the most general locally analytic $\boldsymbol{F}$ of order $n\geq\max(\operatorname{ord}\boldsymbol{R},\max\limits_{i=1,\dots,N}\operatorname{ord}\boldsymbol{B}_i,\max\limits_{j=1,\dots,N}\operatorname{ord}h_j)$ such that (1) is compatible with $\boldsymbol{Q}=0$ can be locally written as

$$\boldsymbol{F}=\boldsymbol{R}+\sum_{i=1}^{N}\zeta_i(t,h_1,\dots,h_N)\boldsymbol{B}_i+\sum_{\alpha=1}^{s}\sum_{m=0}^{n-n_\alpha}D^m(Q^\alpha)\boldsymbol{K}_{\alpha,m}(x,t,\tilde{u},\tilde{Q}), \tag{9}$$

where

$$\tilde{Q}=(Q^1,D(Q^1),\dots,D^{n-n_1}(Q^1),\dots,Q^s,D(Q^s),\dots,D^{n-n_s}(Q^s)),$$

ζ_i and $\boldsymbol{K}_{\alpha,m}$ are arbitrary locally analytic functions of their arguments, $N=\sum_{\alpha=1}^{s} n_\alpha$, and for $n<n_\alpha$ $\boldsymbol{K}_{\alpha,m}\equiv 0$.

Proof. We have $u^\alpha_{n_\alpha}=Q^\alpha+g^\alpha(x,t,\tilde{u})$. Using this equality and proceeding inductively with usage of the formulas found at the previous steps, we obtain $u^\alpha_j=\psi^\alpha_j(x,t,\tilde{u},Q^1,\dots,D^{j-n_1}(Q^1),\dots,Q^s,\dots,D^{j-n_s}(Q^s))$ for $j\geq \max\limits_\alpha n_\alpha$ and similar formulas for $\max\limits_\alpha n_\alpha > j > n_\alpha$.

Plug the above formulas for u^α_j into (6) and expand the resulting expression in Taylor series with respect to $D^j(Q^\alpha)$. As the order of $\boldsymbol{F}$ is n and $\boldsymbol{F}$ is locally analytic, it must be independent of $D^j(Q^\alpha)$ with $j>n-n_\alpha$. Taking this into account, we can rewrite (6) in the form (9), and the result follows. □

It is easily seen that the most general locally analytic $\boldsymbol{F}$ of order $n<\tilde{n}\equiv\max(\operatorname{ord}\boldsymbol{R},\max\limits_{i=1,\dots,N}\operatorname{ord}\boldsymbol{B}_i,\max\limits_{j=1,\dots,N}\operatorname{ord}h_j)$ such that $\boldsymbol{u}_t=\boldsymbol{F}$ admits a given GCS $\boldsymbol{Q}\partial/\partial\boldsymbol{u}$ meeting the requirements of Theorem 2 also has the form (9) with n replaced by $\tilde{n}$ and $\boldsymbol{K}_{\alpha,m}$ and ζ_i subjected to the extra conditions $\partial\boldsymbol{F}/\partial\boldsymbol{u}_j=0$ for $j>n$.

Note that for $\boldsymbol{Q}=\boldsymbol{u}_k-\boldsymbol{g}(x,t,\boldsymbol{u},\dots,\boldsymbol{u}_{k-1})$ we have $N=s\cdot k$, and (9) boils down to

$$\begin{aligned}\boldsymbol{F}&=\boldsymbol{R}+\sum_{i=1}^{N}\zeta_i(t,h_1,\dots,h_N)\boldsymbol{B}_i\\&+\sum_{\alpha=1}^{s}\sum_{m=0}^{n-k}D^m(Q^\alpha)\boldsymbol{K}_{\alpha,m}(x,t,\boldsymbol{u},\boldsymbol{u}_1,\dots,\boldsymbol{u}_{k-1},\boldsymbol{Q},D(\boldsymbol{Q}),\dots,D^{n-k}(\boldsymbol{Q})).\end{aligned}\tag{10}$$

3. Examples

Example 1. Let

$$Q^\alpha=u^\alpha_{n_\alpha}-\sum_{\beta=1}^{s}\sum_{j=0}^{n_\beta-1}g^\alpha_{\beta,j}(x,t)u^\beta_j,\ \alpha=1,\dots,s,\tag{11}$$

and hence $N=\sum_{\alpha=1}^{s} n_\alpha$. Then a general solution of $\boldsymbol{Q}=0$ reads $\boldsymbol{u}=\sum_{i=1}^{N}c_i(t)\cdot\boldsymbol{f}_i(x,\,t)$, where $c_i(t)$ are arbitrary functions of t and $\boldsymbol{f}_i\equiv(f_i^1,\dots,f_i^s)^T$ are linearly independent solutions of $\boldsymbol{Q}=0$.

We have [10, 11] $h_i = Z_i/Z$, where

$$Z = \begin{vmatrix} f_1^1 & \dots & f_i^1 & \dots & f_N^1 \\ \partial f_1^1/\partial x & \dots & \partial f_i^1/\partial x & \dots & \partial f_N^1/\partial x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial^{n_1-1} f_1^1/\partial x^{n_1-1} & \dots & \partial^{n_1-1} f_i^1/\partial x^{n_1-1} & \dots & \partial^{n_1-1} f_N^1/\partial x^{n_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^s & \dots & f_i^s & \dots & f_N^s \\ \partial f_1^s/\partial x & \dots & \partial f_i^s/\partial x & \dots & \partial f_N^s/\partial x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial^{n_s-1} f_1^s/\partial x^{n_s-1} & \dots & \partial^{n_s-1} f_i^s/\partial x^{n_s-1} & \dots & \partial^{n_s-1} f_N^s/\partial x^{n_s-1} \end{vmatrix}$$

and Z_i are obtained from Z by replacing $\partial^j f_i^\alpha/\partial x^j$ by u_j^α. It is easily seen that $\boldsymbol{B}_i = \boldsymbol{f}_i(x,t)$, and $\boldsymbol{R} = \sum_{i=1}^N (Z_i/Z)\partial \boldsymbol{f}_i(x,t)/\partial t$.

If $g_{\beta,j}^\alpha$ are analytic in x and t, then by Theorem 2 the most general locally analytic $\boldsymbol{F}$ of order $n \geq \max_i \operatorname{ord} Z_i/Z$ that admits a GCS $\boldsymbol{Q}\partial/\partial \boldsymbol{u}$ with $\boldsymbol{Q}$ (11) locally has the form

$$\boldsymbol{F} = \sum_{i=1}^N \frac{Z_i}{Z}\frac{\partial \boldsymbol{f}_i(x,t)}{\partial t} + \sum_{i=1}^N \zeta_i(t, Z_1/Z, \dots, Z_N/Z)\boldsymbol{f}_i(x,t) + \sum_{\alpha=1}^s \sum_{m=0}^{n-n_\alpha} D^m(Q^\alpha)\boldsymbol{K}_{\alpha,m}(x,t,\tilde{u},\tilde{Q}), \qquad (12)$$

where ζ_i and $\boldsymbol{K}_{\alpha,m}$ are arbitrary locally analytic functions of their arguments,

$$\tilde{Q} = (Q^1, D(Q^1), \dots, D^{n-n_1}(Q^1), \dots, Q^s, D(Q^s), \dots, D^{n-n_s}(Q^s)),$$
$$\tilde{u} = (u^1, \dots, u^1_{n_1-1}, \dots, u^s, \dots, u^s_{n_s-1}),$$

and $\boldsymbol{K}_{\alpha,m} \equiv 0$ for $n < n_\alpha$.

In the rest of this section we assume that $s = 1$, so for simplicity we shall write $\boldsymbol{u} \equiv u$, $\boldsymbol{Q} \equiv Q$, $\boldsymbol{F} \equiv F$.

Example 2. Let $Q = L(u)$, where $L = \prod_{j=1}^N (D - k_j)$, and k_i, $i = 1, \dots, N$, are distinct ($k_i \neq k_j$ if $i \neq j$) nonzero constants. Then the general solution of $Q = 0$ is $\sum_{i=1}^N c_i(t)\exp(k_i x)$, where $c_i(t)$ are arbitrary functions of t, and using the results of Example 1 we obtain

$$h_i = \frac{\exp(-k_i x)}{\prod\limits_{j=1, j\neq i}^N (k_i - k_j)} L_i(u),$$

where $L_i = \prod_{j=1, j\neq i}^N (D - k_j)$. Thus, the most general locally analytic F of order $n \geq N-1$ such that $u_t = F$ admits a GCS $Q\partial/\partial u$ with

$Q = L(u)$ can be locally written as

$$F = \sum_{i=1}^{N} \zeta_i(t, h_1, \dots, h_N) \exp(k_i x) + \sum_{m=0}^{n-N} D^m(Q) K_m(x, t, u, u_1, \dots, u_{N-1}, Q, D(Q), \dots, D^{n-N}(Q)), \tag{13}$$

where ζ_i and K_m are arbitrary locally analytic functions of their arguments, and $K_m \equiv 0$ if $n < N$.

Example 3. Let $Q = u_2 - f(u,t)$, where f is an arbitrary analytic function of u and t. Set $a(z,t) = \int f(z,t)dz$ and $\psi(y,z,t) = \int (2a(y,t)+z)^{-1/2}dy$. Then the general solution of $Q = 0$ in implicit form reads $\psi(u, c_1(t), t) = x + c_2(t)$, where $c_i(t)$ are arbitrary functions of t, and we have $h_1 = u_1^2 - 2a(u,t)$, $h_2 = \psi(u,z,t)|_{z=h_1} - x$.

Hence by Theorem 2 the most general locally analytic F or order $n \geq 1$ such that $u_t = F$ admits a GCS $Q\partial/\partial u$ with $Q = u_2 - f(u,t)$ locally has the form

$$F = -(2a(u,t)+h_1)^{1/2}\left(\left(\frac{\partial \psi(u,z,t)}{\partial t}\right)\bigg|_{z=h_1} - \zeta_1(t,h_1,h_2)\int^u \frac{dy}{2(2a(y,t)+h_1)^{3/2}} - \zeta_2(t,h_1,h_2)\right) + \sum_{m=0}^{n-2} D^m(Q)K_m(x,t,u,u_1,Q,D(Q),\dots,D^{n-2}(Q)),$$

where ζ_i and K_m are arbitrary locally analytic functions of their arguments; if $n < 2$, then $K_m \equiv 0$ for all m.

Example 4. As a somewhat more elaborated example, consider $Q = u_2 - \varphi(x,t)f(u_1,t)$, where φ and f are arbitrary analytic functions of their arguments, and $f \not\equiv 0$. Let $a(z,t) = \int dz/f(z,t)$, $\tilde{\varphi}(x,t) = \int \varphi(x,t)dx$, and let $b(y,t)$ denote a solution of the equation $a(z,t) = y$ with respect to z, so that $a(b(z,t),t) \equiv z$. Then the general solution of $Q = 0$ reads $u = c_1(t) + \chi(x, c_2(t), t)$, where $\chi(x,z,t) = \int b(\tilde{\varphi}(x,t)+z,t)dx$, $c_i(t)$ are arbitrary functions of t, and we have $h_1 = u - \chi(x,z,t)|_{z=h_2}$, $h_2 = a(u_1,t) - \tilde{\varphi}(x,t)$.

Hence, the most general locally analytic F of order $n \geq 1$ such that $u_t = F$ admits a GCS $Q\partial/\partial u$ with $Q = u_2 - \varphi(x,t)f(u_1,t)$ locally has the form

$$F = \left(\frac{\partial \chi(x,z,t)}{\partial t}\right)\bigg|_{z=h_2} + \zeta_1(t,h_1,h_2) + \zeta_2(t,h_1,h_2)\left(\frac{\partial \chi(x,z,t)}{\partial z}\right)\bigg|_{z=h_2} + \sum_{m=0}^{n-2} D^m(Q)K_m(x,t,u,u_1,Q,D(Q),\dots,D^{n-2}(Q)),$$

where ζ_i and K_m are arbitrary locally analytic functions of their arguments, and if $n < 2$, then $K_m \equiv 0$ for all m.

Example 5. Assume now that $Q = M(1)$, where $M = \prod_{j=1}^{N+1}(D - k_j - u)$. The general solution of $Q = 0$ has the form $-v_x/v$, where $v \equiv \sum_{i=1}^{N+1} b_i(t)\exp(k_i x)$, and $b_i(t)$ are arbitrary functions of t, cf. [4].

Notice that $-v_x/v$ actually involves only N independent arbitrary functions of t. Indeed, renumbering b_i if necessary, we can assume without loss of generality that $b_{N+1}(t) \neq 0$. Then we can rewrite $u = -v_x/v$ in the form

$$u = -D\Bigg(\ln\Bigg(\exp(k_{N+1}x) + \sum_{i=1}^{N} c_i(t)\exp(k_i x)\Bigg)\Bigg), \tag{14}$$

where $c_i(t) = b_i(t)/b_{N+1}(t)$, whence

$$h_i = \frac{\exp((k_{N+1} - k_i)x) M_i(1) \prod\limits_{j=1}^{N}(k_{N+1} - k_j)}{M_{N+1}(1) \prod\limits_{j=1, j\neq i}^{N+1} (k_i - k_j)}, \quad i = 1, \dots, N. \tag{15}$$

Here $M_i = \prod_{j=1, j\neq i}^{N+1}(D - k_j - u)$.

The most general locally analytic F of order $n \geq N - 1$ such that $u_t = F$ admits a GCS $Q\partial/\partial u$ with $Q = M(1)$ (and hence an N-shock type solution (14)) locally has the form

$$\begin{aligned} F = &-\sum_{i=1}^{N} \zeta_i(t, h_1, \dots, h_N)\exp(k_i x)\frac{\left(\sum\limits_{j=1}^{N+1}(k_i - k_j)h_j\exp(k_j x)\right)}{\left(\sum\limits_{q=1}^{N+1} h_q\exp(k_q x)\right)^2} \\ &+ \sum_{m=0}^{n-N} D^m(Q) K_m(x, t, u, u_1, \dots, u_{N-1}, Q, D(Q), \dots, D^{n-N}(Q)), \end{aligned} \tag{16}$$

where ζ_i and K_m are arbitrary locally analytic functions of their arguments, and for $n < N$ we have $K_m \equiv 0$ for all m. This time the h_i are of the form (15), and for the sake of brevity we set $h_{N+1} = 1$. The corresponding N-shock type solution of the evolution equation $u_t = F$ is of the form (14) with $c_i(t)$ that satisfy (7).

Notice that one can readily pick among the F's (16) those independent of x and t. They are of the form

$$\begin{aligned} F = &-\sum_{i=1}^{N} \eta_i\left(\frac{\tilde{h}_1^{k_{N+1}-k_N}}{\tilde{h}_N^{k_{N+1}-k_1}}, \dots, \frac{\tilde{h}_{N-1}^{k_{N+1}-k_N}}{\tilde{h}_N^{k_{N+1}-k_{N-1}}}\right)\frac{\tilde{h}_i\sum\limits_{j=1}^{N+1}(k_i - k_j)\tilde{h}_j}{\left(\sum\limits_{q=1}^{N+1}\tilde{h}_q\right)^2} \\ &+ \sum_{m=0}^{n-N} D^m(Q) K_m(u, u_1, \dots, u_{N-1}, Q, D(Q), \dots, D^{n-N}(Q)), \end{aligned} \tag{17}$$

where $\tilde{h}_i = h_i \exp((k_i - k_{N+1})x)$, and η_i and K_m are arbitrary locally analytic functions of their arguments.

The substitution of (14) into the equation $u_t = F$ with F of the form (17) reduces this equation to the following system of ODEs:

$$\frac{dc_i}{dt} = c_i \eta_i \left(\frac{c_1^{k_{N+1}-k_N}}{c_N^{k_{N+1}-k_1}}, \ldots, \frac{c_{N-1}^{k_{N+1}-k_N}}{c_N^{k_{N+1}-k_{N-1}}} \right), \quad i = 1, \ldots, N.$$

If $\zeta_i = -h_i \sum_{j=0}^{m} \alpha_j (k_i^j - k_{N+1}^j)$, where α_j are arbitrary constants, then (16) represents the most general F of order n such that the equation $u_t = F$ admits an N-shock solution of the form [4]

$$u = -D\left(\ln\left(\sum_{i=1}^{N+1} A_i \exp\left(k_i x - t \sum_{j=0}^{m} \alpha_j k_i^j\right)\right)\right), \tag{18}$$

where A_i are arbitrary constants.

In other words, the class of evolution equations admitting the N-shock solutions of the form (18) contains not only the equation [4]

$$u_t = \sum_{j=0}^{m} \alpha_j D(D-u)^j(u)$$

but infinitely many other equations $u_t = F$ as well. In particular, if the order n of F is greater than $N-2$, then Theorem 2 implies that the corresponding F's have the form

$$F = \sum_{i=1}^{N} \sum_{r=0}^{m} \alpha_r (k_i^r - k_{N+1}^r) \frac{\tilde{h}_i \sum_{j=1}^{N+1} (k_i - k_j) \tilde{h}_j}{\left(\sum_{q=1}^{N+1} \tilde{h}_q\right)^2}$$
$$+ \sum_{m=0}^{n-N} D^m(Q) K_m(x, t, u, u_1, \ldots, u_{N-1}, Q, D(Q), \ldots, D^{n-N}(Q)).$$

4. Conclusions and discussion

In Theorem 2 of the present paper we completely characterized locally analytic evolution systems (1) of a given order $n \geq k-1$ that admit a generalized conditional symmetry $\boldsymbol{Q}(x, t, \boldsymbol{u}, \ldots, \boldsymbol{u}_k)\partial/\partial\boldsymbol{u}$ with a given analytic $\boldsymbol{Q}$ of the form (8). In particular, for the case of linear GCS, i.e., when $\boldsymbol{Q}$ is linear in $\boldsymbol{u}_j$ for all j, the right-hand sides $\boldsymbol{F}$ of such systems (1) are given by (12).

The results of the present paper are somewhat easier to use in applications than the formulas found by us earlier in [12]. For instance, the usage of Theorem 2 enabled us to find the explicit form of *all* locally analytic evolution equations of order $n \geq N-1$ that admit N-shock (and, more broadly, N-shock type) solutions for a given number N, and thus generalize the results of Fokas and Liu [4].

Moreover, Theorem 2 can be employed for the classification of exactly solvable initial value problems along the lines of [3, 17] and for finding evolution equations whose symmetries are compatible with prescribed boundary conditions, and one can find exact solutions for the corresponding boundary value problems, cf. [1] and discussion in [12]. We plan to address these topics elsewhere.

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References

[1] V.E. Adler, B. Gürel, M. Gürses, I. Habibullin, "Boundary conditions for integrable equations," *J. Phys. A: Math. Gen.*, **30**, 3505–3513 (1997).

[2] A. Andreytsev, "Classification of systems of nonlinear evolution equations admitting higher-order conditional symmetries, " in: *Proc. 4th Int. Conf. Symmetry in Nonlinear Mathematical Physics (Kyiv, 2001)*, Part 1, Institute of Mathematics, Kyiv (2002), pp. 72–79.

[3] P. Basarab-Horwath, R.Z. Zhdanov, "Initial-value problems for evolutionary partial differential equations and higher-order conditional symmetries," *J. Math. Phys.*, **42**, 376–389 (2001).

[4] A.S. Fokas, Q.M. Liu, "Generalized conditional symmetries and exact solutions of non-integrable equations," *Theor. Math. Phys.*, **99**, 571–582 (1994).

[5] A.S. Fokas, Q.M. Liu, "Nonlinear interaction of traveling waves of nonintegrable equations," *Phys. Rev. Lett.* **72**, 3293–3296 (1994).

[6] N. Kamran, R. Milson and P. Olver, "Invariant modules and the reduction of nonlinear partial differential equations to dynamical systems, " *Adv. Math.*, **156**, 286–319 (2000).

[7] A. V. Mikhailov, A. B. Shabat, A. B.; V. V. Sokolov, "The symmetry approach to classification of integrable equations, " in: *What is integrability?*, Springer, Berlin (1991), pp.115–184.

[8] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.

[9] J. Rubin and P. Winternitz, "Point symmetries of conditionally integrable nonlinear evolution equations," *J. Math. Phys.* **31**, 2085–2090 (1990).

[10] A.V. Samokhin, "Full Symmetry Algebra for ODEs and Control Systems," *Acta Appl. Math.*; **72**, no. 1-2, 87–99 (2002).

[11] A.V. Samokhin, "Symmetries of linear and linearizable systems of differential equations," *Acta Appl. Math.*, **56**, 253–300 (1999).

[12] A. Sergyeyev, "Constructing conditionally integrable evolution systems in (1+1) dimensions: a generalization of invariant modules approach," *J. Phys. A: Math. Gen.*, **35**, 7653–7660 (2002).

[13] Svirshchevskii S.R., "Lie-Bäcklund symmetries of linear ODEs and invariant linear spaces," in: *Modern Group Analysis*, MPhTI, Moscow (1993), pp.75 83.

[14] *Symmetries and conservation laws for differential equations of mathematical physics*, by A.V. Bocharov et al., I.S. Krasil'shchik and A.M. Vinogradov eds., AMS, Providence, RI (1998).

[15] R.Z. Zhdanov, "Conditional Lie-Bäcklund symmetry and reduction of evolution equations," *J. Phys. A: Math. Gen.*, **28**, 3841–3850 (1995).

[16] R.Z. Zhdanov, “Higher conditional symmetries and reduction of initial value problems for nonlinear evolution equations,” in: *Proc. Int. Conf. Symmetry in nonlinear mathematical physics (Kyiv, 1999)*, Part 1, Institute of Mathematics, Kyiv (2000), pp.255–263.

[17] R.Z. Zhdanov, “Higher conditional symmetry and reduction of initial value problems,” *Nonlinear Dynamics*, **28**, 17–27 (2002).

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