The Diffiety Institute Preprint Series

Preprint DIPS–3/2004 **June 18, 2004**

Classes of the Maxwell spaces that admit subgroups of the Poincaré group

by

M. A. Parinov

Available via INTERNET:
http://diffiety.ac.ru; http://diffiety.org

The Diffiety Institute
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

Classes of the Maxwell spaces that admit subgroups of the Poincaré group

M. A. PARINOV

Abstract. The Maxwell space is the triple $(M,\ g,\ F)$, where M is the four–dimensional Minkowski space or a domain in it, g is a pseudo-Euclidean metric on M, and F is a closed exterior 2-form on M. In this paper, we give the exhaustive description of classes of the Maxwell spaces that admit subgroups of the Poincaré group. Representatives of all classes are constructed.

Contents

Introduction. In classical theory, the electromagnetic field is described by a skew-symmetric tensor F_{ij} on a 4-dimensional real manifold $M \subset \mathbf{R}^4_1$ (a domain in the Minkowski space) satisfying the Maxwell equations [1]

$$\partial_{[i}F_{jk]} = 0, \quad \nabla_k F^{ik} = -\frac{4\pi}{c} J^i \quad (i, j, k = 1, \ldots, 4)$$

(the current J^i must satisfy the continuity equation $\nabla_i J^i = 0$).

We shall say that the *Maxwell space* is a triple (M, g, F), where M is a smooth, real, four-dimentional manifold, $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ is a generalized symplectic structure on M, and $g = g_{ij}dx^i dx^j$ is a pseudo-Euclidean metric on M of the Lorentz signature $(- - -+)$.

2000 *Mathematics Subject Classification.* 83A05, 83C50, 53D99.

Key words and phrases. Minkowski space, Poincaré group, Maxwell equations, symplectic stucture, Maxwell space, classification.

Partially supported by the Ministry of Education of the Russian Federation grant 97-0-1.3-99.

This paper is submitted to the special Issue in Geometry of integrable models in: *Fundamental'naya i Prikladnaya Matematika / Journal of Mathematical Sciences* (2004).

The equation $dF = 0$ means that the form F is closed and is equivalent to the first Maxwell equation. If the second Maxwell equation and the continuity equation for the tensor F_{ij} hold, then the Maxwell space is associated with an electromagnetic field.

Let G_g be the Poincaré group, i. e., the set of the Minkowski space motions (or, equivalently, the group of diffeomorphisms of manifold M that preserve the structure (M, g)). Further, let G_F be the group of symplectomorphisms of the structure (M, F). By G_S we denote the group of diffeomorphisms of the manifold M that preserve both g and F: $G_S = G_g \cap G_F$. Note that G_S is a subgroup of G_g. The Maxwell spaces with non-trivial groups G_S are interesting, for example, in connection with the well-known method for obtaining the first integrals of the Lorentz equations ([2]).

The electromagnetic fields that admit the group G_S were intensively studied in 1960–70s (see [3, 4, 5, 6, 7, 8]). In [3, 4, 5], maximal subgroups of the Poincaré group that preserve the tensor F_{ij} (relativistic symmetry groups) were found for particular fields F_{ij} (homogeneous fields, plane waves, etc.) and the structure of these subgroups was studied. In [6, 7, 8], connected subgroups of the Poincaré group that are invariant transformation groups of electromagnetic fields (or, equivalently, relativistic symmetry groups) were studied. In particular, it was proved that the dimension of such a group is not greater than 6, see [8], and the classification of these groups was obtained ([6, 7]). The problem of classification with respect to conjugation for connected subgroups of the Poincaré group up to the was solved in [9] without any reference to electrodynamics.

In [10, 11], the author formulated the problem of classification with respect to the groups G_S for the Einstein–Maxwell spaces and related this problem with the method of obtaining the first integrals of the Lorentz equations ([2]). An Einstein–Maxwell space is a more generic object than a Maxwell space: in this case, g is a pseudo-Riemannian metric such that the pair (M, g) is an Einstein space ([12]). In case the metric be flat, G_g is the Poincaré group and G_S is its subgroup. Hence we obtain the classification problem for the Maxwell spaces with respect subgroups of the Poincaré group.

1. Formulation of the problem and method of its solution

First, we note that the class of the Maxwell spaces admitting a group G_S can be described as follows. Let $\mathcal{L}_S$ be the Lie algebra of vector fields corresponding to the group G_S. The tensor F_{ij} that defines this class is a solution of the first Maxwell equation

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0 \quad (i,\, j,\, k = 1,\, \dots,\, 4) \tag{1}$$

and is a solution of the invariance conditions with respect to G_S for F_{ij}

$$L_{\xi_\alpha} F_{ij} = 0 \quad (\alpha = 1,\, \dots,\, p = \dim \mathcal{L}_S), \tag{2}$$

where ξ_α are basis vectors in $\mathcal{L}_S$ and L_{ξ_α} is the Lie derivative.

Note that since equations (1) and (2) are linear, the set of solutions of this system is a linear space (class) each of whose elements F_{ij} defines a Maxwell space with a symmetry group that is not less than G_S. But in reality for some of the Maxwell spaces of this class (sometimes, for all of them) their symmetry groups are wider than the group G_S. Therefore, to find the genuine symmetry group we must solve the equation $L_\xi F_{ij} = 0$

with respect to $\xi \in \mathcal{L}_g$[1] for a given F_{ij}. The space of its solutions is the Lie algebra of the symmetry group.

Further, we note that the Poincaré group has infinitely many subgroups; therefore, it is impossible to list all classes of the Maxwell spaces that are invariant under G_S. However, we can consider a finite number of subgroups, typical in some sense, and describe classes of Maxwell spaces for them. The list of representatives of classes of conjugate subgroups of the Poincaré group is a naturally obtained set. Indeed, if subgroup G'_S is conjugate to G_S, then there is a transformation of the coordinates $A \in G_g$, $x^i = A^i_{i'} x^{i'} + a^i$ such that $G'_S = A^{-1} G_S A$. This means that the class of the Maxwell spaces with the symmetry group G_S defined by the tensor F_{ij} transforms into the class with the symmetry group G'_S defined by the tensor $F'_{i'j'} = F_{i'j'} = F_{ij} A^i_{i'} A^j_{j'}$.

We use the classification of subgroups of the Poincaré group up to conjugation presented in [9]. The list of the subalgebras of dimensions 1–6 of the Lie algebra $\mathcal{L}_g$ of the Poincaré group contains 76 items, some of which contain more then one algebra[2]. We denote this algebras by $\mathcal{L}_{p,q}$, where p is the dimension of the algebra and q is its number in the list of subalgebras of dimension p. If necessary, we add the symbol a, b, c, ... to the notation. The problem of description of classes of the Maxwell spaces reduces to solving equations (1)–(2) for each subalgebra $\mathcal{L}_{p,q}$.

Denote by $C_{p,q}$ the class of the Maxwell spaces that corresponds to an algebra $\mathcal{L}_{p,q}$. Obviously, if algebras $\mathcal{L}_{p_1,q_1}$ and $\mathcal{L}_{p_2,q_2}$ are such that $\mathcal{L}_{p_1,q_1} \subset \mathcal{L}_{p_2,q_2}$, $(p_1 < p_2)$, then the corresponding classes are such that $C_{p_2,q_2} \subset C_{p_1,q_1}$. Take the basis of the Lie algebra $\mathcal{L}_g$ in the form

$$\begin{aligned} &e_1 = (1,0,0,0),\ e_2 = (0,1,0,0),\ e_3 = (0,0,1,0),\ e_4 = (0,0,0,1),\\ &e_{12} = (-x^2, x^1, 0, 0),\quad e_{13} = (x^3, 0, -x^1, 0),\quad e_{23} = (0, -x^3, x^2, 0),\\ &e_{14} = (x^4, 0, 0, x^1),\quad e_{24} = (0, x^4, 0, x^2),\quad e_{34} = (0, 0, x^4, x^3), \end{aligned}$$

where $\{x^i\}$ are the Galilean coordinates such that $g_{ij} = \mathrm{diag}(-1, -1, -1, 1)$. In what follows, $L\{\xi_1, \ldots, \xi_p\}$ is the linear combination of vectors $\xi_1, \ldots, \xi_p$.

In [13, 14, 15, 16, 17, 18, 19, 20] classes of the Maxwell spaces were described in some particular cases. In [13], 22 classes of static Maxwell's spaces[3] were described (the smallest among those subalgebras is $\mathcal{L}_S^{\min} = L\{e_4\}$). In [14, 15], 9 classes of the Maxwell spaces admitting hyperbolic helices were described ($\mathcal{L}_S^{\min} = L\{e_{24} + \lambda e_1 + \mu e_3\}$, $\lambda, \mu = \mathrm{const}$). In [16], 15 classes of the Maxwell spaces admitting elliptic helices were described ($\mathcal{L}_S^{\min} = L\{e_{13} + \lambda e_2 + \mu e_4\}$). In [17], 18 classes of the Maxwell spaces admitting parabolic rotation were described ($\mathcal{L}_S^{\min} = L\{e_{12} - e_{14}\}$). In [18], 9 classes of the Maxwell spaces admitting translations along isotropic straight line were described ($\mathcal{L}_S^{\min} = L\{e_2 + e_4\}$). In [19], 10 classes of the Maxwell spaces admitting proportional bi-rotations were described ($\mathcal{L}_S^{\min} = L\{e_{13} + \lambda e_{24}\}$). In [20], 10 classes of the Maxwell spaces admitting translations were described.

[1] $\mathcal{L}_g$ is the Lie algebra of vector fields corresponding to the group G_g; it consists of the vectors $\xi^i = a^i_{.j} x^j + b^i$, where $a^i_{.j} = g^{ik} a_{kj}$ and $a_{kj} = -a_{jk}$, b^i are real numbers.

[2] In fact, the number of these algebras is infinite; for example, one-dimensional subgroups of elliptic helices, parabolic helices, hyperbolic helices, and proportional bi-rotations depend on real parameters, and these subgroups are not conjugate within pairs.

[3] A static Maxwell's space admits the group of translations along the time axis.

In the monograph [21], classes of the Maxwell spaces admitting subgroups of the Poincaré group were described for the whole list of subgroups (subalgebras) in [9]; this classification includes all results mentioned in the previous section. However, in all mentioned publications, there are no examples of the Maxwell spaces admitting the group $G_{p,q}$ exactly; we obtain such examples in this article.

If the evident description of the class $C_{p,q}$ of Maxwell spaces is absent, then we proceed as follows. For the group $G_{p,q}$ (the algebra $\mathcal{L}_{p,q}$) we describe the class $P_{p,q}$ of potentials (covector fields A_i such that $F_{ij} = \partial_i A_j - \partial_j A_i$) is invariant with respect to this group, see [22, 23]; each $A_i \in P_{p,q}$ must satisfy the equations $L_{\xi_\alpha} A_i = 0 \quad (\alpha = 1, \ldots, p)$. Further, we find the subset in $P_{p,q}$ such that each of its elements admits the group $G_{p,q}$ exactly. Then we get the tensor F_{ij} corresponding to the potential A_i and verify the equality $\mathcal{L}_S = \mathcal{L}_{p,q}$; usually, as a result, we receive conditions for the tensor F_{ij} when the corresponding Maxwell space admits the group $G_{p,q}$ exactly.

Remark 1. We denote groups, algebras, and classes of the Maxwell spaces as in [21]. In addition, suppose that components of the tensor F_{ij} always correspond to the Galilean coordinates $\{x^i\}$ even if they are expressed as functions of other coordinates.

In the transversal case the Maxwell space admits the trivial symmetry group $G_S = \{id\}$. We find the example of the Maxwell space with the trivial group G_S.

Example 1. We take the potential $A_i = (0, 0, 0, \Phi)$, where $\Phi = \Phi(x^1, x^2, x^3, x^4)$. Let $\Phi_i = \partial_i \Phi$ and $\Phi_{ij} = \partial_i \partial_j \Phi$. Solving equation $L_\xi A_i = 0$ for $\xi \in \mathcal{L}_g$, we deduce that the group $G_A \cap G_g$ is trivial whenever functions Φ_i, $x^2\Phi_1 - x^1\Phi_2$, $x^3\Phi_1 - x^1\Phi_3$, and $x^3\Phi_2 - x^2\Phi_3$ are linearly independent. (For example, this condition is satisfied for the function $\Phi = x^1x^2x^3x^4$.) Finding the algebra $\mathcal{L}_S$ for the corresponding field F_{ij}

$$F_{12} = F_{13} = F_{23} = 0, \quad F_{14} = \Phi_1, \quad F_{24} = \Phi_2, \quad F_{34} = \Phi_3, \tag{3}$$

we obtain that *the sufficient condition for the group G_S of the Maxwell space defined by* (3) *to be trivial is that functions Φ_1, Φ_2, Φ_{12}, Φ_{13}, Φ_{14}, $x^2\Phi_{11} - x^1\Phi_{12}$, $x^3\Phi_{11} - x^1\Phi_{13}$, $x^3\Phi_{12} - x^2\Phi_{13}$ be linearly independent.* In particular, this condition is provided for the function $\Phi = x^1x^2x^3x^4$.

2. The classes of the Maxwell spaces that admit one-dimensional symmetry groups

In this section, we describe classes of the Maxwell spaces admitting the one-dimensional groups from the list in [9]. Representatives of these classes are presented.

2.1. **Translations.** There are three types of non-conjugate in pairs, one-dimentional subgroups of translations.

2.1.1. *The class $C_{1,1a}$.* The algebra $\mathcal{L}_{1,1a} = L\{e_1\}$ corresponds to the one-dimensional group $G_{1,1a}$ of translations along the space-like vector e_1. *The class $C_{1,1a}$ of the Maxwell spaces is defined by the tensor F_{ij} whose components satisfy the following system*:

$$\begin{aligned} &\partial_2 F_{31} + \partial_3 F_{12} = 0, \quad \partial_2 F_{41} + \partial_4 F_{12} = 0, \quad \partial_3 F_{41} + \partial_4 F_{13} = 0, \\ &\partial_2 F_{34} + \partial_3 F_{42} + \partial_4 F_{23} = 0 \quad \left(F_{ij} = F_{ij}(x^2, x^3, x^4)\right). \end{aligned} \tag{4}$$

Let us note that the class $P_{1,1a}$ of potentials admitting the subgroup $G_{1,1a}$ consists of the following covector fields, $A_i = A_i(x^2, x^3, x^4)$. For $A_i = (0,0,0,\Phi)$, where $\Phi = \Phi(x^2, x^3, x^4)$, we get the following tensor F_{ij}:

$$F_{12} = F_{13} = F_{23} = F_{14} = 0, \quad F_{24} = \partial_2\Phi, \quad F_{34} = \partial_3\Phi. \tag{5}$$

Proposition 1. *Let $\Phi_i = \partial_i\Phi$ and $\Phi_{ij} = \partial_i\partial_j\Phi$. If the partial derivatives Φ_2, Φ_3 are linearly independent and the functions Φ_{23}, Φ_{24}, $x^3\Phi_{22} - x^2\Phi_{23} - \Phi_3$ are also linearly independent, then the Maxwell space defined by the tensor (5) admits the one-dimensional group $G_S = G_{1,1a}$.*

In particular, these conditions are satisfied for the function $\Phi = x^2x^3x^4$.

2.1.2. *The class $C_{1,1b}$.* The algebra $\mathcal{L}_{1,1b} = L\{e_4\}$ corresponds to the one-dimensional group $G_{1,1b}$ of translations along the time-like vector e_4. *The class $C_{1,1b}$ of the Maxwell spaces is defined by the tensor F_{ij} whose components satisfy the following system:*

$$\begin{aligned} &\partial_1F_{24} - \partial_2F_{14} = 0, \quad \partial_1F_{34} - \partial_3F_{14} = 0, \quad \partial_2F_{34} - \partial_3F_{24} = 0, \\ &\partial_1F_{23} + \partial_2F_{31} + \partial_3F_{12} = 0 \quad (F_{ij} = F_{ij}(x^1, x^2, x^3)). \end{aligned} \tag{6}$$

The Maxwell spaces of the class $C_{1,1b}$ are called *static spaces*. In classical vector notations

$$\mathbf{E} = (E_1, E_2, E_3) = (F_{41}, F_{42}, F_{43}), \quad \mathbf{H} = (H_1, H_2, H_3) = (F_{32}, F_{13}, F_{21}), \tag{7}$$

the system (6) has the form $\operatorname{rot}\mathbf{E} = 0$, $\operatorname{div}\mathbf{H} = 0$. Therefore, in particular, *the class $C_{1,1b}$ includes all electrostatic and magnetostatic fields.*

Let us note that the class $P_{1,1b}$ of potentials admitting the subgroup $G_{1,1b}$ consists of the covector fields $A_i = A_i(x^1, x^2, x^3)$. For $A_i = (0,0,0,\Phi)$, $\Phi = \Phi(x^1, x^2, x^3)$, we get the following tensor F_{ij} of electrostatic field:

$$F_{12} = F_{13} = F_{23} = 0, \quad F_{\alpha 4} = \partial_\alpha\Phi \quad (\alpha = 1,2,3). \tag{8}$$

Proposition 2. *Let $\Phi_i = \partial_i\Phi$ and $\Phi_{ij} = \partial_i\partial_j\Phi$. If derivatives Φ_1, Φ_2 are linearly independent and functions Φ_{12}, Φ_{13}, $x^2\Phi_{11} - x^1\Phi_{12} - \Phi_2$, $x^3\Phi_{11} - x^1\Phi_{13} - \Phi_3$, $x^3\Phi_{12} - x^2\Phi_{13}$ are also linearly independent, then the Maxwell space defined by the tensor (8) admits the one-dimensional group $G_S = G_{1,1b}$.*

In particular, these conditions are satisfied for the function $\Phi = x^1x^2x^3$.

2.1.3. *The class $C_{1,1c}$.* The algebra $\mathcal{L}_{1,1c} = L\{e_2 + e_4\}$ corresponds to the one-dimensional group $G_{1,1c}$ of translations along the isotropic vector e_2+e_4. We use the substitution

$$v^1 = x^1, \quad v^2 = x^2 + x^4, \quad v^3 = x^3, \quad v^4 = x^2 - x^4. \tag{9}$$

The class $C_{1,1c}$ of the Maxwell spaces is defined by the tensor $F_{ij} = F_{ij}(v^1, v^3, v^4)$ whose components satisfy the following system:

$$\begin{aligned} &\frac{\partial F_{23}}{\partial v^1} + \frac{\partial F_{12}}{\partial v^3} - \frac{\partial F_{13}}{\partial v^4} = 0, \quad \frac{\partial F_{24}}{\partial v^1} - \frac{\partial F_{12}}{\partial v^4} - \frac{\partial F_{14}}{\partial v^4} = 0, \\ &\frac{\partial F_{34}}{\partial v^1} - \frac{\partial F_{14}}{\partial v^3} - \frac{\partial F_{13}}{\partial v^4} = 0, \quad \frac{\partial F_{34}}{\partial v^4} - \frac{\partial F_{23}}{\partial v^4} - \frac{\partial F_{24}}{\partial v^3} = 0. \end{aligned} \tag{10}$$

Note that the class $P_{1,1c}$ of potentials admitting the subgroup $G_{1,1c}$ consists of the covector fields $A_i = A_i(v^1, v^3, v^4)$. For the potential $A_i = (0, 0, 0, \Phi)$, where $\Phi = \Phi(v^1, v^3, v^4) = \Phi(x^1, x^3, x^2 - x^4)$, we obtain

$$F_{12} = F_{13} = F_{23} = 0, \quad F_{14} = \frac{\partial \Phi}{\partial v^1}, \quad F_{24} = \frac{\partial \Phi}{\partial v^4}, \quad F_{34} = \frac{\partial \Phi}{\partial v^3}. \tag{11}$$

Proposition 3. *Let $\Phi_i = \partial\Phi/\partial v^i$ and $\Phi_{ij} = \partial^2\Phi/\partial v^i \partial v^j$. If partial derivatives Φ_1, Φ_4 are linearly independent and the functions Φ_{13}, Φ_{14}, $x^2\Phi_{11} - x^1\Phi_{14} - \Phi_4$, $x^3\Phi_{11} - x^1\Phi_{13} - \Phi_3$, $x^3\Phi_{14} - x^2\Phi_{13}$ are also linearly independent, then the Maxwell space defined by the tensor* (11) *admits the one-dimensional group $G_S = G_{1,1c}$.*

For example, it is so for the function $\Phi = v^1v^3v^4 = x^1x^3(x^2 - x^4)$.

2.2. **Elliptic helices.** The algebra $\mathcal{L}_{1,2} = L\{e_{13} + \lambda e_2 + \mu e_4\}$ corresponds to the one-dimensional group $G_{1,2}$ of elliptic helices as follows

$$\hat{x}^1 = x^1 \cos a + x^3 \sin a, \quad \hat{x}^2 = \lambda a + x^2,$$
$$\hat{x}^3 = -x^1 \sin a + x^3 \cos a, \quad \hat{x}^4 = \mu a + x^4. \tag{el}$$

If $\lambda = \mu = 0$, then transformations (el) are rotations. If $\lambda \neq 0$, $\mu = 0$, then transformations (el) are helices with the space-like axis. If $\lambda = 0$, $\mu \neq 0$, then transformations (el) are helices with the time-like axis. If $\lambda = \mu \neq 0$, then transformations (el) are helices with the isotropic axis. We use the coordinate system $\{\tilde{x}^i\} = \{r, \tilde{x}^2, \varphi, \tilde{x}^4\}$, connected with $\{x^i\}$ by the formulas

$$x^1 = r \sin\varphi, \quad x^2 = \lambda\varphi + \tilde{x}^2, \quad x^3 = r\cos\varphi, \quad x^4 = \mu\varphi + \tilde{x}^4. \tag{12}$$

The class $C_{1,2}$ of the Maxwell spaces is defined by the tensor

$$\begin{aligned}
&F_{12} = c_1 \cos\varphi + c_2 \sin\varphi, && F_{13} = F_{13}(r, \tilde{x}^2, \tilde{x}^4),\\
&F_{14} = c_3 \cos\varphi + c_4 \sin\varphi, && F_{23} = c_1 \sin\varphi - c_2 \cos\varphi,\\
&F_{24} = F_{24}(r, \tilde{x}^2, \tilde{x}^4), && F_{34} = -c_3 \sin\varphi + c_4 \cos\varphi,
\end{aligned} \tag{13}$$

where $c_i = c_i(r, \tilde{x}^2, \tilde{x}^4)$ are smooth functions which satisfy the following system of equations

$$\begin{aligned}
&\frac{\partial c_1}{\partial r} + \frac{c_1}{r} + \frac{\lambda}{r}\frac{\partial c_2}{\partial \tilde{x}^2} + \frac{\mu}{r}\frac{\partial c_2}{\partial \tilde{x}^4} - \frac{\partial F_{13}}{\partial \tilde{x}^2} = 0, \quad \frac{\partial F_{24}}{\partial r} + \frac{\partial c_2}{\partial \tilde{x}^4} - \frac{\partial c_4}{\partial \tilde{x}^2} = 0,\\
&\frac{\partial c_3}{\partial r} + \frac{c_3}{r} + \frac{\lambda}{r}\frac{\partial c_4}{\partial \tilde{x}^2} + \frac{\mu}{r}\frac{\partial c_4}{\partial \tilde{x}^4} - \frac{\partial F_{13}}{\partial \tilde{x}^4} = 0,\\
&\frac{\lambda}{r}\frac{\partial F_{24}}{\partial \tilde{x}^2} + \frac{\mu}{r}\frac{\partial F_{24}}{\partial \tilde{x}^4} + \frac{\partial c_3}{\partial \tilde{x}^2} - \frac{\partial c_1}{\partial \tilde{x}^4} = 0.
\end{aligned} \tag{14}$$

It is easily proved that the class $P_{1,2}$ of potentials A_i that is invariant with respect to the group $G_{1,2}$ consists of the fields

$$\begin{aligned}
&A_1 = C_1 \cos\varphi + C_2 \sin\varphi, \quad A_2 = A_2(r, \tilde{x}^2, \tilde{x}^4),\\
&A_3 = -C_1 \sin\varphi + C_2 \cos\varphi, \quad A_4 = A_4(r, \tilde{x}^2, \tilde{x}^4),
\end{aligned} \tag{15}$$

where $C_i = C_i(r,\, \tilde{x}^2,\, \tilde{x}^4)$. For the potential $A_i = (0,\, 0,\, 0,\, \Phi)$, $\Phi = \Phi(u^1,\, u^2,\, u^3) = \Phi(r,$ $\tilde{x}^2,\, \tilde{x}^4)$, we obtain

$$
\begin{aligned}
&F_{12} = F_{13} = 0,\;\; F_{23} = 0, \quad F_{24} = \Phi_2, \\
&F_{14} = \Phi_1 \sin\varphi - \frac{\lambda}{r}\Phi_2 \cos\varphi - \frac{\mu}{r}\Phi_3 \cos\varphi, \\
&F_{34} = \Phi_1 \cos\varphi + \frac{\lambda}{r}\Phi_2 \sin\varphi + \frac{\mu}{r}\Phi_3 \sin\varphi,
\end{aligned}
\tag{16}
$$

where $\Phi_k = \partial\Phi/\partial u^k$. Consider four cases, which are different from the geometric point of view.

2.2.1. *The class* $C_{1,2a}$ $(\mu = 0,\;\; \lambda \neq 0)$. The algebra $\mathcal{L}_{1,2a} = L\{e_{13} + \lambda e_2\}$ corresponds to the group $G_{1,2a}$ of elliptic helices with the space-like axis Ox^2. Equations (14) and the tensor (16) obtain the form

$$
\begin{aligned}
&\frac{\partial c_1}{\partial r} + \frac{c_1}{r} + \frac{\lambda}{r}\frac{\partial c_2}{\partial \tilde{x}^2} - \frac{\partial F_{13}}{\partial \tilde{x}^2} = 0, \quad \frac{\partial F_{24}}{\partial r} + \frac{\partial c_2}{\partial \tilde{x}^4} - \frac{\partial c_4}{\partial \tilde{x}^2} = 0, \\
&\frac{\partial c_3}{\partial r} + \frac{c_3}{r} + \frac{\lambda}{r}\frac{\partial c_4}{\partial \tilde{x}^2} - \frac{\partial F_{13}}{\partial \tilde{x}^4} = 0, \quad \frac{\lambda}{r}\frac{\partial F_{24}}{\partial \tilde{x}^2} + \frac{\partial c_3}{\partial \tilde{x}^2} - \frac{\partial c_1}{\partial \tilde{x}^4} = 0
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
&F_{12} = F_{13} = 0, \quad F_{14} = \Phi_1 \sin\varphi - \frac{\lambda}{r}\Phi_2 \cos\varphi, \\
&F_{23} = 0, \quad F_{24} = \Phi_2, \quad F_{34} = \Phi_1 \cos\varphi + \frac{\lambda}{r}\Phi_2 \sin\varphi.
\end{aligned}
\tag{18}
$$

Proposition 4. *Let* $\Phi_i = \partial\Phi/\partial v^i$ *and* $\Phi_{ij} = \partial^2\Phi/\partial v^i \partial v^j$. *If partial derivatives* Φ_{22}, Φ_{23} *are linearly independent and* $\Phi_{21} \neq 0$, *then the Maxwell space defined by the tensor* (18) *admits the one-dimensional group* $G_S = G_{1,2a}$.

2.2.2. *The class* $C_{1,2b}$ $(\lambda = 0,\;\; \mu \neq 0)$. The algebra $\mathcal{L}_{1,2b} = L\{e_{13} + \mu e_4\}$ corresponds to the group $G_{1,2b}$ of elliptic helices with the time-like axis Ox^4. Equations (14) and the tensor (16) obtain the form

$$
\begin{aligned}
&\frac{\partial c_1}{\partial r} + \frac{c_1}{r} + \frac{\mu}{r}\frac{\partial c_2}{\partial \tilde{x}^4} - \frac{\partial F_{13}}{\partial \tilde{x}^2} = 0, \quad \frac{\partial F_{24}}{\partial r} + \frac{\partial c_2}{\partial \tilde{x}^4} - \frac{\partial c_4}{\partial \tilde{x}^2} = 0, \\
&\frac{\partial c_3}{\partial r} + \frac{c_3}{r} + \frac{\mu}{r}\frac{\partial c_4}{\partial \tilde{x}^4} - \frac{\partial F_{13}}{\partial \tilde{x}^4} = 0, \quad \frac{\mu}{r}\frac{\partial F_{24}}{\partial \tilde{x}^4} + \frac{\partial c_3}{\partial \tilde{x}^2} - \frac{\partial c_1}{\partial \tilde{x}^4} = 0
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
&F_{12} = F_{13} = 0, \quad F_{14} = \Phi_1 \sin\varphi - \frac{\mu}{r}\Phi_3 \cos\varphi, \\
&F_{23} = 0, \quad F_{24} = \Phi_2, \quad F_{34} = \Phi_1 \cos\varphi + \frac{\mu}{r}\Phi_3 \sin\varphi.
\end{aligned}
\tag{20}
$$

Proposition 5. *Let* $\Phi_i = \partial\Phi/\partial v^i$ *and* $\Phi_{ij} = \partial^2\Phi/\partial v^i \partial v^j$. *If the partial derivatives* Φ_{22}, Φ_{23} *are linearly independent and* $\Phi_{21} \neq 0$, *then the Maxwell space defined by the tensor* (20) *admits the one-dimensional group* $G_S = G_{1,2b}$.

2.2.3. *The class* $C_{1,2c}$ $(\lambda = \mu \neq 0)$. The algebra $\mathcal{L}_{1,2c} = L\{e_{13} + \lambda(e_2 + e_4)\}$ corresponds to the group $G_{1,2c}$ of elliptic helices with the isotropic axis. Equations (14) and the tensor (16) acquire the form

$$\begin{aligned}
&\frac{\partial c_1}{\partial r} + \frac{c_1}{r} + \frac{\lambda}{r}\left(\frac{\partial c_2}{\partial \tilde{x}^2} + \frac{\partial c_2}{\partial \tilde{x}^4}\right) - \frac{\partial F_{13}}{\partial \tilde{x}^2} = 0, \quad \frac{\partial F_{24}}{\partial r} + \frac{\partial c_2}{\partial \tilde{x}^4} - \frac{\partial c_4}{\partial \tilde{x}^2} = 0,\\
&\frac{\partial c_3}{\partial r} + \frac{c_3}{r} + \frac{\lambda}{r}\left(\frac{\partial c_4}{\partial \tilde{x}^2} + \frac{\partial c_4}{\partial \tilde{x}^4}\right) - \frac{\partial F_{13}}{\partial \tilde{x}^4} = 0,\\
&\frac{\lambda}{r}\left(\frac{\partial F_{24}}{\partial \tilde{x}^2} + \frac{\partial F_{24}}{\partial \tilde{x}^4}\right) + \frac{\partial c_3}{\partial \tilde{x}^2} - \frac{\partial c_1}{\partial \tilde{x}^4} = 0
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&F_{12} = F_{13} = 0, \quad F_{14} = \Phi_1 \sin\varphi - \frac{\lambda}{r}\cos\varphi(\Phi_2 + \Phi_3),\\
&F_{23} = 0, \quad F_{24} = \Phi_2, \quad F_{34} = \Phi_1 \cos\varphi + \frac{\lambda}{r}\sin\varphi(\Phi_2 + \Phi_3).
\end{aligned} \tag{22}$$

Proposition 6. *Let* $\Phi_i = \partial\Phi/\partial v^i$ *and* $\Phi_{ij} = \partial^2\Phi/\partial v^i \partial v^j$. *If* $\Phi_{21} \neq 0$ *and* $\Phi_{22} + \Phi_{23} \neq 0$, *then the Maxwell space defined by the tensor* (22) *admits the one-dimensional group* $G_S = G_{1,2c}$.

2.2.4. *The class* $C_{1,2d}$ $(\lambda = \mu = 0)$. The algebra $\mathcal{L}_{1,2d} = L\{e_{13}\}$ corresponds to the group $G_{1,2d}$ of rotations in the plane Ox^1x^3. Equations (14) and the tensor (16) have the form

$$\begin{aligned}
&\frac{\partial c_1}{\partial r} + \frac{c_1}{r} - \frac{\partial F_{13}}{\partial \tilde{x}^2} = 0, \quad \frac{\partial F_{24}}{\partial r} + \frac{\partial c_2}{\partial \tilde{x}^4} - \frac{\partial c_4}{\partial \tilde{x}^2} = 0,\\
&\frac{\partial c_3}{\partial r} + \frac{c_3}{r} - \frac{\partial F_{13}}{\partial \tilde{x}^4} = 0, \quad \frac{\partial c_3}{\partial \tilde{x}^2} - \frac{\partial c_1}{\partial \tilde{x}^4} = 0
\end{aligned} \tag{23}$$

and

$$F_{12} = F_{13} = F_{23} = 0, \quad F_{14} = \Phi_1 \sin\varphi, \quad F_{24} = \Phi_2, \quad F_{34} = \Phi_1 \cos\varphi. \tag{24}$$

Proposition 7. *Let* $\Phi_i = \partial\Phi/\partial v^i$ *and* $\Phi_{ij} = \partial^2\Phi/\partial v^i \partial v^j$. *If* $\tilde{x}^2\Phi_{21} - r\Phi_{22} + \Phi_1 \neq 0$, $\Phi_{21} \neq 0$, *and partial derivatives* Φ_{22}, Φ_{23} *are linearly independent, then the Maxwell space defined by the tensor* (24) *admits the one-dimensional group* $G_S = G_{1,2d}$.

2.3. **Hyperbolic helices.** The algebra $\mathcal{L}_{1,3} = L\{e_{24} + \lambda e_1\}$ corresponds to the group $G_{1,3}$ of hyperbolic helices with the axis Ox^1:

$$\begin{aligned}
&\hat{x}^1 = x^1 + \lambda a, \quad \hat{x}^2 = x^2 \cosh a + x^4 \sinh a,\\
&\qquad \hat{x}^3 = x^3, \quad \hat{x}^4 = x^2 \sinh a + x^4 \cosh a.
\end{aligned} \tag{hyp}$$

If $\lambda = 0$, then (hyp) are the Lorentz transformations. We use the coordinate system $\{\tilde{x}^i\} = \{\tilde{x}^1, r, \tilde{x}^3, \varphi\}$, related to $\{x^i\}$ by the formulae

$$x^1 = \lambda\varphi + \tilde{x}^1, \quad x^2 = r\cosh\varphi, \quad x^3 = \tilde{x}^3, \quad x^4 = r\sinh\varphi. \tag{25}$$

2.3.1. *The class $C_{1,3}$ $(\lambda \neq 0)$ of the Maxwell spaces is defined by the following tensor F_{ij}*

$$\begin{aligned} &F_{12} = c_1 \cosh\varphi + c_2 \sinh\varphi, \quad F_{13} = F_{13}(\tilde{x}^1, r, \tilde{x}^3),\\ &F_{14} = -c_1 \sinh\varphi - c_2\cosh\varphi, \quad F_{23} = c_3\cosh\varphi + c_4\sinh\varphi, \\ &F_{24} = F_{24}(\tilde{x}^1, r, \tilde{x}^3), \quad F_{34} = c_3\sinh\varphi + c_4\cosh\varphi, \end{aligned} \tag{26}$$

where $c_i = c_i(\tilde{x}^1, r, \tilde{x}^3)$ $(i = 1, \ldots, 4)$ are smooth functions which satisfy the system

$$\begin{aligned} &\frac{\partial c_2}{\partial r} + \frac{c_2}{r} - \frac{\lambda}{r}\frac{\partial c_1}{\partial \tilde{x}^1} + \frac{\partial F_{24}}{\partial \tilde{x}^1} = 0, \quad \frac{\partial c_1}{\partial \tilde{x}^3} - \frac{\partial F_{13}}{\partial r} + \frac{\partial c_3}{\partial \tilde{x}^1} = 0,\\ &\frac{\partial c_4}{\partial r} + \frac{c_4}{r} - \frac{\lambda}{r}\frac{\partial c_3}{\partial \tilde{x}^1} - \frac{\partial F_{24}}{\partial \tilde{x}^3} = 0, \quad \frac{\lambda}{r}\frac{\partial F_{13}}{\partial \tilde{x}^1} - \frac{\partial c_4}{\partial \tilde{x}^1} - \frac{\partial c_2}{\partial \tilde{x}^3} = 0. \end{aligned} \tag{27}$$

Note that the class $P_{1,3}$ of potentials A_i that is invariant with respect to the group $G_{1,3}$ consists of the fields [23]

$$\begin{aligned} &A_1 = A_1(\tilde{x}^1, r, \tilde{x}^3), \quad A_2 = C_1\cosh\varphi + C_2\sinh\varphi,\\ &A_3 = A_3(\tilde{x}^1, r, \tilde{x}^3), \quad A_4 = -C_1\sinh\varphi - C_2\cosh\varphi, \end{aligned} \tag{28}$$

where $C_i = C_i(\tilde{x}^1, r, \tilde{x}^3)$. For the potential $A_i = (-\Phi, 0, 0, 0)$, $\Phi = \Phi(\tilde{x}^1, \tilde{x}^3)$, we obtain

$$F_{12} = \frac{\lambda}{r}\Phi_1\sinh\varphi, \quad F_{13} = \Phi_3, \quad F_{14} = -\frac{\lambda}{r}\Phi_1\cosh\varphi, \quad F_{23} = F_{24} = F_{34} = 0, \tag{29}$$

where $\Phi_i = \partial\Phi/\partial\tilde{x}^i$ $(i = 1, 3)$. Let $\Phi_{ij} = \partial^2\Phi/\partial\tilde{x}^i\partial\tilde{x}^j$.

Proposition 8. *If the partial derivatives Φ_{31}, Φ_{33} are linearly independent, then the Maxwell space defined by the tensor (29) admits the one-dimensional group $G_S = G_{1,3}$.*

2.3.2. *The class $C_{1,3b}$.* For $\lambda = 0$ *the class of the Maxwell spaces that corresponds to the algebra $\mathcal{L}_{1,3b} = L\{e_{24}\}^3$, is defined by the tensor (26), where the functions $c_i = c_i(\tilde{x}^1, r, \tilde{x}^3)$ must satisfy the equations (instead of (27))*

$$\begin{aligned} &\frac{\partial c_2}{\partial r} + \frac{c_2}{r} + \frac{\partial F_{24}}{\partial \tilde{x}^1} = 0, \quad \frac{\partial c_1}{\partial \tilde{x}^3} - \frac{\partial F_{13}}{\partial r} + \frac{\partial c_3}{\partial \tilde{x}^1} = 0,\\ &\frac{\partial c_4}{\partial r} + \frac{c_4}{r} - \frac{\partial F_{24}}{\partial \tilde{x}^3} = 0, \quad \frac{\partial c_4}{\partial \tilde{x}^1} + \frac{\partial c_2}{\partial \tilde{x}^3} = 0; \end{aligned} \tag{30}$$

we use the Lorentz transformation

$$x^1 = \tilde{x}^1, \quad x^2 = r\cosh\varphi, \quad x^3 = \tilde{x}^3, \quad x^4 = r\sinh\varphi. \tag{31}$$

instead of (25).

For the potential $A_i = (\Phi, 0, 0, 0)$, $\Phi = \Phi(\tilde{x}^1, r, \tilde{x}^3)$, we have

$$F_{12} = \Phi_r\cosh\varphi, \quad F_{13} = \Phi_3, \quad F_{14} = -\Phi_r\sinh\varphi, \quad F_{23} = F_{24} = F_{34} = 0 \tag{32}$$

$(\Phi_r = \partial\Phi/\partial r)$. Let $\Phi_{ij} = \partial^2\Phi/\partial\tilde{x}^i\partial\tilde{x}^j$.

[3] We do not present here the class $C_{1,3a}$ that corresponds to the algebra $\mathcal{L}_{1,3a} = L\{e_{24} + \lambda e_3\}$ since the groups $G_{1,3a}$ and $G_{1,3}$ are conjugated; in [14, 15, 21], this class is used for description of other classes.

Proposition 9. *If the partial derivatives* Φ_{31}, Φ_{3r}, Φ_{33} *are linearly independent, then the Maxwell space defined by the tensor* (32) *admits the one-dimensional group* $G_S = G_{1,3}$.

2.4. **Parabolic helices.** Here we describe the class $C_{1,4}$ of the Maxwell spaces that corresponds to the algebra $\mathcal{L}_{1,4} = L\{e_{12} - e_{14} + \lambda e_2 + \mu e_3\}$ ($\lambda, \mu = \text{const}$, $\lambda\mu = 0$). This class is defined by the pair of equations (1) and (2) for the vector $\xi = e_{12} - e_{14} + \lambda e_2 + \mu e_3$; the second equation is the system

$$\begin{aligned} &XF_{12} + F_{24} = 0, \quad XF_{13} + F_{23} + F_{34} = 0, \quad XF_{14} + F_{24} = 0, \\ &XF_{23} - F_{13} = 0, \quad XF_{24} + F_{12} - F_{14} = 0, \quad XF_{34} + F_{13} = 0, \end{aligned} \tag{33}$$

where

$$X = -(x^2 + x^4)\partial_1 + (x^1 + \lambda)\partial_2 + \mu\partial_3 - x^1\partial_4. \tag{34}$$

We consider three cases: a) $\lambda = \mu = 0$; b) $\lambda = 0$, $\mu \neq 0$; c) $\lambda \neq 0$, $\mu = 0$.

2.4.1. *The class* $C_{1,4a}$. The group $G_{1,4a}$ corresponding to the algebra $\mathcal{L}_{1,4a} = L\{e_{12} - e_{14}\}$ consists of parabolic rotations. We have

$$\begin{aligned} &\hat{x}^1 = x^1 - a(x^2 + x^4), && \hat{x}^2 = x^2 + ax^1 - \frac{a^2}{2}(x^2 + x^4), \\ &\hat{x}^3 = x^3, && \hat{x}^4 = x^4 - ax^1 + \frac{a^2}{2}(x^2 + x^4). \end{aligned}$$

Let $\{\tilde{x}^i\}$ be the coordinate system related with $\{x^i\}$ by formulae

$$\tilde{x}^1 = x^2 + x^4, \quad \tilde{x}^2 = -\frac{x^1}{x^2 + x^4}, \quad \tilde{x}^3 = x^3, \quad \tilde{x}^4 = \frac{1}{2}(x^1)^2 + x^2(x^2 + x^4). \tag{35}$$

Then the operator (34) is replaced by partial derivative with respect to $\tilde{x}^2$ and the solution of the system that follows from (33) is defined by the formulae

$$\begin{aligned} &F_{13} = C_1\tilde{x}^2 + C_2, && F_{24} = C_5\tilde{x}^2 + C_6, \\ &F_{23} = \frac{C_1}{2}\left(\tilde{x}^2\right)^2 + C_2\tilde{x}^2 + C_3, && F_{12} = -\frac{C_5}{2}\left(\tilde{x}^2\right)^2 - C_6\tilde{x}^2 + C_7, \\ &F_{34} = -\frac{C_1}{2}\left(\tilde{x}^2\right)^2 - C_2\tilde{x}^2 + C_4, && F_{14} = -\frac{C_5}{2}\left(\tilde{x}^2\right)^2 - C_6\tilde{x}^2 + C_8, \end{aligned} \tag{36}$$

where $C_k = C_k(\tilde{x}^1, \tilde{x}^3, \tilde{x}^4)$ are smooth functions that satisfy the equations

$$C_1 + C_3 + C_4 = 0, \quad C_5 + C_7 - C_8 = 0. \tag{37}$$

Substituting (36) in the Maxwell equations (1), taking into account the independence of functions C_k from $\tilde{x}^2$, grouping the summands with equal powers of the variable $\tilde{x}^2$, and using the linear independence of the functions 1, $\tilde{x}^2$, $\left(\tilde{x}^2\right)^2$, $\left(\tilde{x}^2\right)^3$, $\left(\tilde{x}^2\right)^4$, we finally obtain that the functions C_k satisfy the equations

$$\tilde{x}^1\frac{\partial C_2}{\partial \tilde{x}^4}+\frac{\partial C_5}{\partial \tilde{x}^3}=0, \tag{38a}$$

$$\tilde{x}^1\frac{\partial C_3}{\partial \tilde{x}^4}+\frac{\partial C_1}{\partial \tilde{x}^1}+\frac{(\tilde{x}^1)^2+\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_1}{\partial \tilde{x}^4}+\frac{\partial C_6}{\partial \tilde{x}^3}=0, \tag{38b}$$

$$-\frac{C_2}{\tilde{x}^1}-\frac{\partial C_2}{\partial \tilde{x}^1}-\frac{(\tilde{x}^1)^2+\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_2}{\partial \tilde{x}^4}+\frac{\partial C_7}{\partial \tilde{x}^3}=0, \tag{38c}$$

$$-\frac{C_5}{\tilde{x}^1}+\frac{\partial C_7}{\partial \tilde{x}^1}-\frac{\partial C_8}{\partial \tilde{x}^1}-\frac{(\tilde{x}^1)^2+\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_8}{\partial \tilde{x}^4}+\frac{\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_7}{\partial \tilde{x}^4}=0, \tag{38d}$$

$$-\tilde{x}^1\frac{\partial C_4}{\partial \tilde{x}^4}+\frac{\partial C_6}{\partial \tilde{x}^3}+\frac{\partial C_1}{\partial \tilde{x}^1}+\frac{\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_1}{\partial \tilde{x}^4}=0, \tag{38e}$$

$$\frac{C_2}{\tilde{x}^1}+\frac{\partial C_2}{\partial \tilde{x}^1}+\frac{\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_2}{\partial \tilde{x}^4}-\frac{\partial C_8}{\partial \tilde{x}^3}=0, \tag{38f}$$

$$\frac{\partial C_4}{\partial \tilde{x}^1}+\frac{(\tilde{x}^1)^2+\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_4}{\partial \tilde{x}^4}-\frac{\partial C_6}{\partial \tilde{x}^3}+\frac{\partial C_3}{\partial \tilde{x}^1}+\frac{\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_3}{\partial \tilde{x}^4}=0. \tag{38g}$$

Thus, *the Maxwell space of the class $C_{1,4a}$, corresponding to the algebra $\mathcal{L}_{1,4a}$, is defined by the tensor F_{ij} of the form (36) satisfying to (37) and (38).*

Proposition 10. *The Maxwell space defined by the tensor F_{ij} is of the form*

$$F_{12}=F_{14}=F_{24}=0,\quad F_{13}=x^1\Phi_3,$$
$$F_{23}=\Phi_1+(2x^2+x^4)\Phi_3,\quad F_{34}=-\Phi_1-x^2\Phi_3,$$

where

$$\Phi(t_1,t_2,t_3)=\Phi(\tilde{x}^1,\tilde{x}^3,\tilde{x}^4)=\Phi\left(x^2+x^4,\ x^3,\ \frac{1}{2}(x^1)^2+x^2(x^2+x^4)\right),$$

$\Phi_1=\partial\Phi/\partial t_1$, $\Phi_3=\partial\Phi/\partial t_3$, admits the one-dimensional group $G_S=G_{1,4a}$ whenever the partial derivatives $\partial^2\Phi/\partial t_3\partial t_1$, $\partial^2\Phi/\partial t_3\partial t_2$ $\partial^2\Phi/\partial t_3^2$ are linearly independent.

In particular, this condition is satisfied for the function $\Phi(t_1,t_2,t_3)=(t_1+t_2^2+t_3^2)t_3$.

2.4.2. *The class $C_{1,4b}$.* The group $G_{1,4b}$ that corresponds to the algebra $\mathcal{L}_{1,4b}=L\{e_{12}-e_{14}+\mu e_3\}$ consists of the parabolic helices of the form

$$\hat{x}^1=x^1-a(x^2+x^4),\ \hat{x}^2=x^2+ax^1-\frac{a^2}{2}(x^2+x^4),$$
$$\hat{x}^3=x^3+\mu a,\ \hat{x}^4=x^4-ax^1+\frac{a^2}{2}(x^2+x^4),$$

which turn into parabolic rotations for $\mu=0$. In this case, the change $\{x^i\}\to\{\tilde{x}^i\}$ of the form

$$\tilde{x}^1=x^2+x^4,\ \tilde{x}^2=-\frac{x^1}{x^2+x^4},\ \tilde{x}^3=x^3+\frac{\mu x^1}{x^2+x^4},\ \tilde{x}^4=\frac{1}{2}(x^1)^2+x^2(x^2+x^4), \tag{39}$$

coinciding for $\mu=0$ with (35), replaces the operator (34) ($\lambda=0$) by the partial derivative with respect to $\tilde{x}^2$. Repeating the same operations as for the class $C_{1,4a}$, we obtain the following result. *The Maxwell space of the class $C_{1,4b}$ is defined by the* *tensor F_{ij} of the form (36), where the change (35) is replaced by (39) and the functions $C_k = C_k(\tilde{x}^1, \tilde{x}^3, \tilde{x}^4)$ satisfy the system of equations (37), (38b), (38e), (38g), and*

$$\frac{\mu}{\tilde{x}^1}\frac{\partial C_1}{\partial \tilde{x}^3} + \tilde{x}^1 \frac{\partial C_2}{\partial \tilde{x}^4} + \frac{\partial C_5}{\partial \tilde{x}^3} = 0, \tag{40a}$$

$$\frac{\mu}{\tilde{x}^1}\frac{\partial C_3}{\partial \tilde{x}^3} - \frac{C_2}{\tilde{x}^1} - \frac{\partial C_2}{\partial \tilde{x}^1} - \frac{(\tilde{x}^1)^2 + \tilde{x}^4}{\tilde{x}^1}\frac{\partial C_2}{\partial \tilde{x}^4} + \frac{\partial C_7}{\partial \tilde{x}^3} = 0, \tag{40b}$$

$$\frac{\mu}{\tilde{x}^1}\frac{\partial C_6}{\partial \tilde{x}^3} - \frac{C_5}{\tilde{x}^1} + \frac{\partial C_7}{\partial \tilde{x}^1} - \frac{\partial C_8}{\partial \tilde{x}^1} - \frac{(\tilde{x}^1)^2 + \tilde{x}^4}{\tilde{x}^1}\frac{\partial C_8}{\partial \tilde{x}^4} + \frac{\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_7}{\partial \tilde{x}^4} = 0, \tag{40c}$$

$$\frac{\mu}{\tilde{x}^1}\frac{\partial C_4}{\partial \tilde{x}^3} + \frac{C_2}{\tilde{x}^1} + \frac{\partial C_2}{\partial \tilde{x}^1} + \frac{\tilde{x}^4}{\tilde{x}^1}\frac{\partial C_2}{\partial \tilde{x}^4} - \frac{\partial C_8}{\partial \tilde{x}^3} = 0. \tag{40d}$$

Note that for $\mu = 0$ the last four equations coincide with (38a), (38c), (38d), and (38f), respectively.

Proposition 11. *The Maxwell space defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = \frac{\mu}{x^2 + x^4}\Phi_2 + x^1\Phi_3,$$

$$F_{23} = \Phi_1 - \frac{\mu x^1}{(x^2 + x^4)^2}\Phi_2 + (2x^2 + x^4)\Phi_3,$$

$$F_{34} = -\Phi_1 + \frac{\mu x^1}{(x^2 + x^4)^2}\Phi_2 - x^2\Phi_3,$$

where

$$\Phi(t_1, t_2, t_3) = \Phi\left(x^2 + x^4,\ x^3 + \frac{\mu x^1}{x^2 + x^4}\Phi_2,\ \frac{1}{2}(x^1)^2 + x^2(x^2 + x^4)\right),$$

$\Phi_i = \partial\Phi/\partial t_i$, admits the one-dimensional group $G_S = G_{1,4b}$ whenever the partial derivatives $\partial^2\Phi/\partial t_3\partial t_1$, $\partial^2\Phi/\partial t_3\partial t_2$ and $\partial^2\Phi/\partial t_3^2$ are linearly independent.

In particular, this condition holds for the function $\Phi(t_1, t_2, t_3) = (t_1 + t_2^2 + t_3^2)t_3$.

2.4.3. *The class $C_{1,4c}$.* The group $G_{1,4c}$ that corresponds to the algebra $\mathcal{L}_{1,4c} = L\{e_{12} - e_{14} + \lambda e_2\}$ consists of the parabolic helices of the form

$$\hat{x}^1 = x^1 - a(x^2 + x^4) - \lambda\frac{a^2}{2}, \quad \hat{x}^2 = x^2 + a(x^1 + \lambda) - \frac{a^2}{2}(x^2 + x^4) - \lambda\frac{a^3}{6},$$
$$\hat{x}^3 = x^3, \quad \hat{x}^4 = x^4 - ax^1 + \frac{a^2}{2}(x^2 + x^4) + \lambda\frac{a^3}{6},$$

which turn into parabolic rotations for $\lambda = 0$. The change $\{x^i\} \to \{\tilde{x}^i\}$ of the form

$$\tilde{x}^1 = 2\lambda x^1 + \left(x^2 + x^4\right)^2, \quad \tilde{x}^2 = \frac{x^2 + x^4}{\lambda},$$
$$\tilde{x}^3 = x^3, \quad \tilde{x}^4 = \lambda x^4 + x^1(x^2 + x^4) + \frac{1}{3\lambda}\left(x^2 + x^4\right)^3, \tag{41}$$

replaces the operator (34) ($\mu = 0$) by the partial derivative with respect to $\tilde{x}^2$. Repeating the same operations as for the class $C_{1,4a}$, we obtain the following result. *The Maxwell space of the class $C_{1,4c}$ is defined by the tensor F_{ij} of the form (36), where* *the change* (35) *is replaced by* (41) *and the functions* $C_k = C_k(\tilde{x}^1, \tilde{x}^3, \tilde{x}^4)$ *satisfy both Eq.* (37) *and the system*

$$\frac{C_1}{\lambda} + \frac{\tilde{x}^1}{2\lambda}\frac{\partial C_2}{\partial \tilde{x}^4} - 2\lambda\frac{\partial C_3}{\partial \tilde{x}^1} - \frac{\partial C_7}{\partial \tilde{x}^3} = 0, \tag{42a}$$

$$\frac{\tilde{x}^1}{2\lambda}\frac{\partial C_1}{\partial \tilde{x}^4} - \lambda\frac{\partial C_3}{\partial \tilde{x}^4} + \frac{\partial C_6}{\partial \tilde{x}^3} = 0, \tag{42b}$$

$$2\lambda\frac{\partial C_1}{\partial \tilde{x}^1} - \lambda\frac{\partial C_2}{\partial \tilde{x}^4} + \frac{\partial C_5}{\partial \tilde{x}^3} = 0, \tag{42c}$$

$$\frac{\tilde{x}^1}{2\lambda}\frac{\partial C_5}{\partial \tilde{x}^4} - 2\lambda\frac{\partial C_6}{\partial \tilde{x}^1} - \lambda\frac{\partial C_7}{\partial \tilde{x}^4} = 0. \tag{42d}$$

Proposition 12. *The Maxwell space defined by the tensor* F_{ij} *such that*

$$F_{12} = F_{14} = F_{24} = 0, \ F_{13} = 2\lambda\Phi_1 + (x^2 + x^4)\Phi_3,$$

$$F_{23} = 2(x^2 + x^4)\Phi_1 + \left(x^1 + \frac{(x^2+x^4)^2}{\lambda}\right)\Phi_3,$$

$$F_{34} = -2(x^2 + x^4)\Phi_1 - \left(\lambda + x^1 + \frac{(x^2+x^4)^2}{\lambda}\right)\Phi_3,$$

where

$$\Phi(t_1, t_2, t_3) = \Phi\left(2\lambda x^1 + \left(x^2+x^4\right)^2,\ x^3,\ \lambda x^4 + x^1(x^2+x^4) + \frac{1}{3\lambda}\left(x^2+x^4\right)^3\right),$$

admits the one-dimensional group $G_S = G_{1,4c}$ *whenever the partial derivatives* $\Phi_i = \partial\Phi/\partial t_i$, Φ_{11}, Φ_{12}, *and* Φ_{13} *are linearly independent* ($\Phi_{ij} = \partial^2\Phi/\partial t_i \partial t_j$).

In particular, this condition holds for the function $\Phi(t_1, t_2, t_3) = t_1(t_1^2 + t_2^2 + t_3^2)$.

2.5. **Proportional bi-rotations.** The algebra $\mathcal{L}_{1,5} = L\{e_{13} + \lambda e_{24}\}$ ($\lambda \neq 0$) corresponds to the group $G_{1,5}$ of proportional bi-rotations of the form

$$\hat{x}^1 = x^3 \sin a + x^1 \cos a, \quad \hat{x}^2 = x^2 \cosh \lambda a + x^4 \sinh \lambda a,$$
$$\hat{x}^3 = x^3 \cos a - x^1 \sin a, \quad \hat{x}^4 = x^2 \sinh \lambda a + x^4 \cosh \lambda a.$$

Here we use the coordinate system $\{r,\ \rho,\ \theta,\ \varphi\}$ related with $\{x^i\}$ by the formulae

$$x^1 = r\cos(\theta - \varphi), \quad x^2 = \rho\cosh(\lambda\varphi), \quad x^3 = r\sin(\theta-\varphi), \quad x^4 = \rho\sinh(\lambda\varphi). \tag{43}$$

The class $C_{1,5}$ *of the Maxwell spaces is defined by the tensor* F_{ij} *of the following form*

$$\begin{aligned} F_{12} &= (-c_1 \cosh\lambda\varphi - c_2 \sinh\lambda\varphi)\sin\varphi + (c_3\cosh\lambda\varphi + c_4\sinh\lambda\varphi)\cos\varphi,\\ F_{14} &= (c_2 \cosh\lambda\varphi + c_1 \sinh\lambda\varphi)\sin\varphi - (c_4\cosh\lambda\varphi + c_3\sinh\lambda\varphi)\cos\varphi,\\ F_{23} &= (c_1 \cosh\lambda\varphi + c_2 \sinh\lambda\varphi)\cos\varphi + (c_3\cosh\lambda\varphi + c_4\sinh\lambda\varphi)\sin\varphi,\\ F_{34} &= (c_2 \cosh\lambda\varphi + c_1 \sinh\lambda\varphi)\cos\varphi + (c_4\cosh\lambda\varphi + c_3\sinh\lambda\varphi)\sin\varphi, \end{aligned} \tag{44}$$

$$F_{13} = F_{13}(\rho, r, \theta), \quad F_{24} = F_{24}(\rho, r, \theta), \tag{45}$$

where the functions $c_i = c_i(\rho, r, \theta)$, $F_{13}(\rho, r, \theta)$, and $F_{24}(\rho, r, \theta)$ satisfy the following system:

$$\begin{aligned}
&\frac{\sin\theta}{r}\frac{\partial F_{24}}{\partial\theta} - \cos\theta\frac{\partial F_{24}}{\partial r} + \frac{c_1}{\lambda\rho} - \frac{c_4}{\rho} - \frac{\partial c_4}{\partial\rho} - \frac{1}{\lambda\rho}\frac{\partial c_3}{\partial\theta} = 0,\\
&\frac{\cos\theta}{r}\frac{\partial F_{24}}{\partial\theta} + \sin\theta\frac{\partial F_{24}}{\partial r} - \frac{c_3}{\lambda\rho} - \frac{c_2}{\rho} - \frac{\partial c_2}{\partial\rho} - \frac{1}{\lambda\rho}\frac{\partial c_1}{\partial\theta} = 0,\\
&\cos\theta\frac{\partial c_1}{\partial r} + \sin\theta\frac{\partial c_3}{\partial r} - \frac{\sin\theta}{r}\frac{\partial c_1}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial c_3}{\partial\theta} - \frac{\partial F_{13}}{\partial\rho} = 0,\\
&\cos\theta\frac{\partial c_2}{\partial r} + \sin\theta\frac{\partial c_4}{\partial r} - \frac{\sin\theta}{r}\frac{\partial c_2}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial c_4}{\partial\theta} + \frac{1}{\lambda\rho}\frac{\partial F_{13}}{\partial\theta} = 0.
\end{aligned} \tag{46}$$

A part of the class $P_{1,5}$ is defined by the formula $A_i = (0, \Phi\cosh(\lambda\varphi), 0, -\Phi\sinh(\lambda\varphi))$, where $\Phi = \Phi(r, \rho, \theta)$ is an arbitrary function. This potential generates the tensor F_{ij} such that

$$\begin{aligned}
&F_{12} = \left(\Phi'_r\cos(\theta-\varphi) - \frac{1}{r}\Phi'_\theta\sin(\theta-\varphi)\right)\cosh\lambda\varphi, \quad F_{13} = 0,\\
&F_{14} = \left(-\Phi'_r\cos(\theta-\varphi) + \frac{1}{r}\Phi'_\theta\sin(\theta-\varphi)\right)\sinh\lambda\varphi,\\
&F_{23} = \left(-\Phi'_r\sin(\theta-\varphi) - \frac{1}{r}\Phi'_\theta\cos(\theta-\varphi)\right)\cosh\lambda\varphi, \quad F_{24} = \frac{1}{\lambda\rho}\Phi'_\theta,\\
&F_{34} = \left(-\Phi'_r\sin(\theta-\varphi) - \frac{1}{r}\Phi'_\theta\cos(\theta-\varphi)\right)\sinh\lambda\varphi.
\end{aligned} \tag{47}$$

Proposition 13. *If the functions Φ'_r, Φ''_{rr}, $\Phi''_{r\rho}$, and $\Phi''_{\theta\theta}$ are linearly independent, then the tensor (47) defines the Maxwell space that admits the group $G_S = G_{1,5}$.*

3. The Maxwell spaces that admit translation groups

In this section, we describe classes of the Maxwell spaces that are invariant with respect to the translation groups whose dimensions are from 2 to 4.

3.1. **Two-dimensional subgroups.**

3.1.1. *The class $C_{2,1a}$.* The algebra $\mathcal{L}_{2,1a} = L\{e_1,\ e_2\}$ corresponds to the group $G_{2,1a}$ of translations along the vectors of the Euclidean plane Ox^1x^2. We have $\mathcal{L}_{1,1a} \subset \mathcal{L}_{2,1a}$, therefore the class $C_{2,1a}$ is a subclass of the class $C_{1,1a}$. *The Maxwell space of the class $C_{2,1a}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = \text{const}, \quad F_{13} = \frac{\partial\Phi}{\partial x^3}, \quad F_{14} = \frac{\partial\Phi}{\partial x^4}, \quad F_{23} = \frac{\partial\Psi}{\partial x^3}, \quad F_{24} = \frac{\partial\Psi}{\partial x^4}, \quad F_{34} = \Theta, \tag{48}$$

where $\Phi = \Phi(x^3, x^4)$, $\Psi = \Psi(x^3, x^4)$, $\Theta = \Theta(x^3, x^4)$ are arbitrary smooth functions.

Proposition 14. *The Maxwell space defined by the tensor (48) admits the two-dimensional group $G_S = G_{2,1a}$ whenever the partial derivatives $\partial_3\Phi$, $\partial_4\Phi$, $\partial_3\Psi$, $\partial_4\Psi$ are linearly independent and the functions $\partial_3\Theta$, $\partial_4\Theta$, $x^3\partial_4\Theta + x^4\partial_3\Theta$ are also linearly independent.*

3.1.2. *The class $C_{2,1b}$.* The algebra $\mathcal{L}_{2,1b} = L\{e_2,\ e_4\}$ corresponds to the group $G_{2,1b}$ of translations along the vectors of the pseudo-Euclidean plane Ox^2x^4. Since $\mathcal{L}_{1,1b} \subset \mathcal{L}_{2,1b}$, the class $C_{2,1b}$ is a subclass of the class $C_{1,1b}$. *The Maxwell space of the class $C_{2,1b}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} F_{12} &= \frac{\partial \Phi}{\partial x^1}, \quad F_{13} = \Theta, F_{14} = \frac{\partial \Psi}{\partial x^1}, \\ F_{23} &= -\frac{\partial \Phi}{\partial x^3}, \quad F_{24} = \text{const}, \quad F_{34} = \frac{\partial \Psi}{\partial x^3}, \end{aligned} \tag{49}$$

where $\Phi = \Phi(x^1, x^3)$, $\Psi = \Psi(x^1, x^3)$, $\Theta = \Theta(x^1, x^3)$ are arbitrary smooth functions.

Proposition 15. *The Maxwell space defined by the tensor (49) admits the two-dimensional group $G_S = G_{2,1b}$ whenever the partial derivatives $\partial_1\Phi$, $\partial_3\Phi$, $\partial_1\Psi$, $\partial_3\Psi$ are linearly independent and the functions $\partial_1\Theta$, $\partial_3\Theta$ $x^3\partial_1\Theta - x^1\partial_3\Theta$ are also linearly independent.*

3.1.3. *The class $C_{2,1c}$.* The algebra $\mathcal{L}_{2,1c} = L\{e_1,\ e_2 + e_4\}$ corresponds to the group $G_{2,1c}$ of translations along the vectors of an isotropic plane. As $\mathcal{L}_{1,1} \subset \mathcal{L}_{2,1}$, the class $C_{2,1c}$ is a subclass of the class $C_{1,1c}$. *The Maxwell space of the class $C_{2,1c}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} F_{12} &= -\frac{\partial \Phi}{\partial v^4}, \quad F_{13} = -\frac{\partial \Phi}{\partial v^3}, \quad F_{14} = \frac{\partial \Phi}{\partial v^4}, \\ F_{23} &= \Theta, \quad F_{24} = \frac{\partial \Psi}{\partial v^4}, \quad F_{34} = \Theta + \frac{\partial \Psi}{\partial v^3}, \end{aligned} \tag{50}$$

where $C = \text{const}$ and $\Phi = \Phi(v^3, v^4)$, $\Psi = \Psi(v^3, v^4)$, and $\Theta = \Theta(v^3, v^4)$ are arbitrary smooth functions ($v^3 = x^3,\ v^4 = x^2 - x^4$).

Set $C = 0$ and $\Theta = 0$ in (50):

$$F_{12} = -\frac{\partial \Phi}{\partial v^4},\ F_{13} = -\frac{\partial \Phi}{\partial v^3},\ F_{14} = \frac{\partial \Phi}{\partial v^4},\ F_{23} = 0,\ F_{24} = \frac{\partial \Psi}{\partial v^4},\ F_{34} = \frac{\partial \Psi}{\partial v^3}. \tag{51}$$

Proposition 16. *The Maxwell space defined by the tensor (50) admits the two-dimensional group $G_S = G_{2,1c}$ whenever the partial derivatives $\partial\Phi/\partial v^3$, $\partial\Phi/\partial v^4$, $\partial\Psi/\partial v^3$, $\partial\Psi/\partial v^4$ are linearly independent and the functions $\Phi_{34} = \partial^2\Phi/\partial v^3\partial v^4$, $\Phi_{44} = \partial^2\Phi/\partial v^4\partial v^4$, $x^3\Phi_{44} - x^2\Phi_{34}$ are also linearly independent.*

3.2. **Three-dimensional subgroups.**

3.2.1. *The class $C_{3,1a}$.* The algebra $\mathcal{L}_{3,1a} = L\{e_1, e_2, e_3\}$ corresponds to the group $G_{3,1a}$ of translations along the vectors of three-dimensional Euclidean space $Ox^1x^2x^3$. The class $C_{3,1a}$ is a subclass of $C_{2,1a}$. *The Maxwell space of the class $C_{3,1a}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} &F_{12} = C_3, \quad F_{23} = C_2, \quad F_{13} = C_1, \\ &F_{14} = \varphi'(x^4), \quad F_{24} = \psi'(x^4), \quad F_{34} = \chi(x^4). \end{aligned} \tag{52}$$

where $C_k = \text{const}$ and $\varphi(x^4)$, $\psi(x^4)$, $\chi(x^4)$ are arbitrary smooth functions (the prime denotes differentiation). In particular, the homogeneous magnetic field crossed with the electric field that depends only on time belongs to the class $C_{3,1a}$.

Proposition 17. *If the functions $\varphi(x^4)$, $\varphi'(x^4)$, and $\psi(x^4)$ are linearly independent and $\chi(x^4) \neq 0$, then the Maxwell space defined by the tensor (52) admits the three-dimensional group $G_S = G_{3,1a}$.*

3.2.2. *The class $C_{3,1b}$.* The algebra $\mathcal{L}_{3,1b} = L\{e_1, e_2, e_4\}$ corresponds to the group $G_{3,1b}$ of translations along the vectors of three-dimensional pseudo-Euclidean space $Ox^1x^2x^4$. The class $C_{3,1b}$ is a subclass of $C_{2,1b}$. *The Maxwell space of the class $C_{3,1b}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = C_1, \quad F_{13} = \varphi(x^3), \quad F_{14} = C_2, \qquad F_{23} = \psi(x^3), \quad F_{24} = C_3, \quad F_{34} = \chi(x^3), \quad (53)$$

where $C_k = \mathrm{const}$ and $\varphi(x^3)$, $\psi(x^3)$, and $\chi(x^3)$ are arbitrary smooth functions.

Proposition 18. *If the functions $\varphi(x^3)$, $\varphi'(x^3)$, $\psi(x^3)$, and $\chi(x^3)$ are linearly independent, then the Maxwell space defined by the tensor (53) admits the three-dimensional group $G_S = G_{3,1b}$.*

For example, this condition holds for the functions $\varphi = x^3$, $\psi = \sin x^3$, $\chi = \cos x^3$.

3.2.3. *The class $C_{3,1c}$.* The algebra $\mathcal{L}_{3,1c} = L\{e_1, e_3, e_2 + e_4\}$ corresponds to the group $G_{3,1c}$ of translations along the vectors of a three-dimensional isotropic space. The class $C_{3,1c}$ is a subclass of $C_{2,1c}$. *The Maxwell space of the class $C_{3,1c}$ is defined by the tensor F_{ij} of the form*

$$F_{ij} = F_{ij}(v^4) \ \ (ij = 12,\ 23,\ 24), \qquad F_{13} = C_1, \quad F_{14} = C_2 - F_{12}, \quad F_{34} = C_3 + F_{23}, \quad (54)$$

where $C_k = \mathrm{const}$ and $F_{ij} = F_{ij}(v^4) = F_{ij}(x^2 - x^4)$ are arbitrary smooth functions.

Proposition 19. *If the functions $F_{ij} = F_{ij}(v^4)$ $(ij = 12,\ 23,\ 24)$ are linearly independent and $F_{24} \neq \mathrm{const}$, then the Maxwell space defined by the tensor (54) admits the three-dimensional group $G_S = G_{3,1c}$.*

3.3. **Four-dimensional subgroup.** The algebra $\mathcal{L}_{4,1} = L\{e_1,\ e_2,\ e_3,\ e_4\}$ corresponds to the group $G_{4,1}$ of translations of the Minkowski space $\mathbf{R}^4_1$. *The Maxwell space of the class $C_{4,1}$ is defined by the constant tensor F_{ij}.* Therefore, $C_{4,1}$ is the class *of homogeneous Maxwell spaces.*

Remark 2. It is easily proved that the group G_S is six-dimensional for each homogeneous Maxwell's space [20]. Hence, *there are no Maxwell's spaces with the symmetry group $G_S = G_{4,1}$.*

4. The classes of the static Maxwell spaces

In this section, we describe classes of the static Maxwell spaces which were not discussed in sections 2 and 3. See details in [21].

4.1. **Two-dimensional subgroup.** The algebra $\mathcal{L}_{2,3} = L\{e_{13} + \lambda e_2, e_4\}$ corresponds to the group $G_{2,3}$ generated by elliptic helices with a space-like axis and by translations along a time-like straight line. *The Maxwell space of the class $C_{2,3}$ is defined by the tensor F_{ij} of the form*

$$
\begin{aligned}
&F_{12} = c_1 \cos\varphi + c_2 \sin\varphi, \quad F_{13} = F_{13}(r, \tilde{x}^2), \quad F_{14} = c_3 \cos\varphi + c_4 \sin\varphi, \\
&F_{23} = c_1 \sin\varphi - c_2 \cos\varphi, \quad F_{24} = F_{24}(r, \tilde{x}^2), \quad F_{34} = -c_3 \sin\varphi + c_4 \cos\varphi,
\end{aligned} \tag{55}
$$

where the functions $F_{13}(r, \tilde{x}^2)$, $F_{24}(r, \tilde{x}^2)$ and $c_i = c_i(r, \tilde{x}^2)$ satisfy the equations

$$
\begin{aligned}
&\frac{\partial c_1}{\partial r} + \frac{c_1}{r} + \frac{\lambda}{r}\frac{\partial c_2}{\partial \tilde{x}^2} - \frac{\partial F_{13}}{\partial \tilde{x}^2} = 0, \quad \frac{\partial F_{24}}{\partial r} - \frac{\partial c_4}{\partial \tilde{x}^2} = 0, \\
&\frac{\partial c_3}{\partial r} + \frac{c_3}{r} + \frac{\lambda}{r}\frac{\partial c_4}{\partial \tilde{x}^2} = 0, \qquad \frac{\lambda}{r}\frac{\partial F_{24}}{\partial \tilde{x}^2} + \frac{\partial c_3}{\partial \tilde{x}^2} = 0;
\end{aligned} \tag{56}
$$

the coordinates $\{x^i\}$ and $\{r, \tilde{x}^2, \varphi, \tilde{x}^4\}$ are related by (12) for $\mu = 0$:

$$
x^1 = r \sin\varphi, \quad x^2 = \lambda\varphi + \tilde{x}^2, \quad x^3 = r\cos\varphi, \quad x^4 = \tilde{x}^4. \tag{57}
$$

The class of the potentials $P_{2,3}$ invariant with respect to the group $G_{2,3}$ consists of the fields

$$
\begin{aligned}
&A_1 = b_1 \cos\varphi + b_2 \sin\varphi, \quad A_2 = A_2(r, \tilde{x}^2), \\
&A_3 = -b_1 \sin\varphi + b_2 \cos\varphi, \quad A_4 = A_4(r, \tilde{x}^2),
\end{aligned} \tag{58}
$$

where $b_k = b_k(r, \tilde{x}^2)$, $A_2(r, \tilde{x}^2)$, and $A_4(r, \tilde{x}^2)$ are arbitrary smooth functions. Setting $A_1 = A_2 = A_3 = 0$, $A_4 = \Phi(r, \tilde{x}^2) \equiv \Phi(t_1, t_2)$, we obtain the following set of electrostatic fields

$$
\begin{aligned}
&F_{12} = F_{13} = F_{23} = 0, \quad F_{14} = \Phi_1 \sin\varphi - \frac{\lambda\Phi_2}{r}\cos\varphi, \\
&F_{24} = \Phi_2, \quad F_{34} = \Phi_1 \cos\varphi + \frac{\lambda\Phi_2}{r}\sin\varphi,
\end{aligned} \tag{59}
$$

where $\Phi_\alpha = \partial\Phi/\partial t_\alpha$. Let $\Phi_{\alpha\beta} = \partial^2\Phi/\partial t_\alpha \partial t_\beta$.

Proposition 20. *If the partial derivatives Φ_1, Φ_2, Φ_{11}, Φ_{12} and Φ_{22} are linearly independent, then the Maxwell space defined by the tensor (59) admits the two-dimensional group $G_S = G_{2,3}$.*

In particular, this condition holds for the function $\Phi(t_1, t_2) = t_1^3 + t_2^3 + t_1^2 + t_1 t_2 + t_2^2$.

4.2. **Three-dimensional subgroup.** The algebra $\mathcal{L}_{3,6} = L\{e_{24} + \lambda e_3, e_2, e_4\}$ corresponds to the group $G_{3,6}$ generated by hyperbolic helices and translations along the vectors of the pseudo-Euclidean plane. *The Maxwell space of the class $C_{3,6}$ is defined by the tensor F_{ij} as follows:*

a) *for* $\lambda \neq 0$ *(class* $C_{3,6a}$*)*

$$\begin{aligned} &F_{12} = -\lambda c_1'(x^1)\cosh\frac{x^3}{\lambda} - \lambda c_2'(x^1)\sinh\frac{x^3}{\lambda}, \quad F_{13} = F_{13}(x^1),\\ &F_{14} = \lambda c_1'(x^1)\sinh\frac{x^3}{\lambda} + \lambda c_2'(x^1)\cosh\frac{x^3}{\lambda}, \quad F_{24} = \text{const},\\ &F_{23} = c_1(x^1)\sinh\frac{x^3}{\lambda} + c_2(x^1)\cosh\frac{x^3}{\lambda},\\ &F_{34} = c_1(x^1)\cosh\frac{x^3}{\lambda} + c_2(x^1)\sinh\frac{x^3}{\lambda}, \end{aligned} \tag{60}$$

where $c_1(x^1)$ *and* $c_2(x^1)$ *are arbitrary functions (prime denotes differentiation)*;

b) *For* $\lambda = 0$ *(class* $C_{3,6b}$*)*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0, \quad F_{13} = F_{13}(x^1, x^3), \quad F_{24} = \text{const}. \tag{61}$$

Putting $c_2 = F_{13} = F_{24} = 0$ in (60), we get

$$\begin{aligned} &F_{12} = -\lambda c_1'(x^1)\cosh\frac{x^3}{\lambda}, \quad F_{13} = 0, \quad F_{14} = \lambda c_1'(x^1)\sinh\frac{x^3}{\lambda},\\ &F_{23} = c_1(x^1)\sinh\frac{x^3}{\lambda}, \qquad F_{24} = 0, \quad F_{34} = c_1(x^1)\cosh\frac{x^3}{\lambda}. \end{aligned} \tag{62}$$

Proposition 21. *If the functions* $c_1(x^1)$ *and* $c_1'(x^1)$ *are linearly independent, then the Maxwell space defined by the tensor* (62) *admits the three-dimensional group* $G_S = G_{3,6a}$.

Proposition 22. *If the functions* $\partial_1 F_{13}$, $\partial_3 F_{13}$, *and* $x^3\partial_1 F_{13} - x^1\partial_3 F_{13}$ *are linearly independent, then the Maxwell space defined by the tensor* (61) *admits the three-dimensional group* $G_S = G_{3,6b}$.

4.3. **Four-dimensional subgroups.**

4.3.1. *The class* $C_{4,3}$. The algebra $\mathcal{L}_{4,3} = L\{e_{13} + \lambda e_2, e_1, e_3, e_4\}$ corresponds to the group $G_{4,3}$ generated by elliptic helices with a space-like axis and translations along the vectors of the pseudo-Euclidean hyperplane. *The Maxwell space of the class* $C_{4,3}$ *is defined by the tensor* F_{ij} *as follows:*

a) *for* $\lambda \neq 0$ *(class* $C_{4,3a}$*)*

$$\begin{aligned} &F_{12} = a_1\sin\frac{x^2}{\lambda} - a_2\cos\frac{x^2}{\lambda}, \quad F_{32} = a_1\cos\frac{x^2}{\lambda} + a_2\sin\frac{x^2}{\lambda},\\ &F_{14} = F_{34} = 0, \quad F_{13} = a_3, \quad F_{24} = a_4 \quad (a_i = \text{const}); \end{aligned} \tag{63}$$

b) *for* $\lambda = 0$ *(class* $C_{4,3b}$*)*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0, \quad F_{13} = \Phi(x^2), \quad F_{24} = \Psi(x^2), \tag{64}$$

where $\Phi(x^2)$ *И* $\Psi(x^2)$ *are arbitrary functions.*

Proposition 23. *If* $a_1 \neq 0$ *or* $a_2 \neq 0$, *then the Maxwell space defined by the tensor* (63) *admits the four-dimensional group* $G_S = G_{4,3a}$.

Proposition 24. *If the functions* $\Phi(x^2)$ *and* $\Psi(x^2)$ *are linearly independent, then the Maxwell space defined by the tensor* (64) *admits the four-dimensional group* $G_S = G_{4,3b}$.

4.3.2. *The class $C_{4,6}$.* The algebra $\mathcal{L}_{4,6} = L\{e_{24} + \lambda e_3,\ e_1,\ e_2,\ e_4\}$ corresponds to the group $G_{4,6}$ generated by hyperbolic helices and translations along the vectors of the pseudo-Euclidean hyperplane. *The Maxwell space of the class $C_{4,6}$ is defined by the tensor F_{ij} of the form:*

a) *for $\lambda \neq 0$ (class $C_{4,6a}$)*

$$\begin{aligned} &F_{23} = a_1 \cosh \frac{x^3}{\lambda} + a_2 \sinh \frac{x^3}{\lambda}, \quad F_{34} = a_1 \sinh \frac{x^3}{\lambda} + a_2 \cosh \frac{x^3}{\lambda}, \\ &F_{12} = F_{14} = 0, \quad F_{24} = a_3, \quad F_{13} = a_4 \quad (a_i = \text{const}); \end{aligned} \tag{65}$$

b) *for $\lambda = 0$ (class $C_{4,6b}$)*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0, \quad F_{13} = F_{13}(x^3), \quad F_{24} = \text{const}. \tag{66}$$

Proposition 25. *If $a_1 \neq 0$ or $a_2 \neq 0$, then the Maxwell space defined by the tensor* (65) *admits the four-dimensional group $G_S = G_{4,6a}$.*

Proposition 26. *If $F_{13}(x^3) \neq \text{const}$, then the Maxwell space defined by the tensor* (66) *admits the four-dimensional group $G_S = G_{4,6b}$.*

4.3.3. *The class $C_{4,8}$.* The algebra $\mathcal{L}_{4,8} = L\{e_{12} - e_{14} + \lambda e_3,\ e_1,\ e_2,\ e_4\}$ corresponds to the group $G_{4,8}$ generated by parabolic helices and translations along the vectors of the pseudo-Euclidean hyperplane. *The Maxwell space of the class $C_{4,8}$ is defined by the tensor F_{ij} of the form:*

a) *for $\lambda = 0$ (class $C_{4,8a}$)*

$$F_{12} = F_{14} = \text{const}, \quad F_{13} = F_{24} = 0, \quad F_{23} = -F_{34}(x^3); \tag{67}$$

b) *for $\lambda \neq 0$ (class $C_{4,8b}$)*

$$\begin{aligned} &F_{12} = F_{14} = C_1, \quad F_{24} = 0, \quad F_{13} = \frac{C_2}{\lambda} x^3 + C_3, \\ &F_{23} = \frac{C_2}{2\lambda^2}(x^3)^2 + \frac{C_3}{\lambda} x^3 + C_4, \quad F_{34} = -F_{23} - C_2, \end{aligned} \tag{68}$$

where $C_1, \ldots, C_4$ are arbitrary constants.

Proposition 27. *If $F_{23} \neq \text{const}$, then the Maxwell space defined by the tensor* (67) *admits the four-dimensional group $G_S = G_{4,8a}$.*

Proposition 28. *If $C_1 \neq 0$ and $C_2 \neq 0$, then the Maxwell space defined by the tensor* (68) *admits the four-dimensional group $G_S = G_{4,8b}$.*

4.3.4. *The class $C_{4,18}$.* The algebra $\mathcal{L}_{4,18} = L\{e_{12}, e_{13}, e_{23}, e_4\}$ corresponds to the group $G_{4,18}$ generated by all rotations around the origin O in Euclidean space $Ox^1x^2x^3$ and translations along the time axis Ox^4. *The Maxwell space of the class $C_{4,18}$ is defined by the tensor F_{ij} of the form:*

$$\begin{aligned} &F_{12} = -Ax^3/r^3, \quad F_{13} = Ax^2/r^3, \quad F_{23} = -Ax^1/r^3, \\ &F_{14} = x^1 C(r), \quad F_{24} = x^2 C(r), \quad F_{34} = x^3 C(r), \end{aligned} \tag{69}$$

where $A = \text{const}$, $C(r)$ an arbitrary function and $\{r,\ \varphi,\ \theta,\ x^4\}$ are "spherical" coordinates

$$x^1 = r \cos\varphi \cos\theta, \quad x^2 = r \sin\varphi \cos\theta, \quad x^3 = r \sin\theta. \tag{70}$$

Example 2. If $A = 0$ and $C(r) = K/r^3$ ($K = \mathrm{const}$), then (69) is the Coulomb field. It admits the four-dimensional group $G_S = G_{4,18}$.

4.4. **Five- and six-dimensional subgroups.**

4.4.1. *For each algebra* $\mathcal{L}_{5,1} = L\{e_{24}, e_1, e_2, e_3, e_4\}$, $\mathcal{L}_{5,2} = L\{e_{13} + \lambda e_{24}, e_1, e_2, e_3, e_4\}$, *and* $\mathcal{L}_{6,2} = L\{e_{13}, e_{24}, e_1, e_2, e_3, e_4\}$, *there is the class* $C_{6,2}$ *of the homogeneous Maxwell spaces defined by the tensor* F_{ij} *of the form*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0, \quad F_{13} = C_1, \quad F_{24} = C_2, \tag{71}$$

where C_1 *and* C_2 *are arbitrary constants.*

Proposition 29. *If* $C_1 \neq 0$ *or* $C_2 \neq 0$, *then the Maxwell space defined by the tensor* (71) *admits the six-dimensional group* $G_S = G_{6,2}$ *which corresponds to the algebra* $\mathcal{L}_{6,2}$. *There are no Maxwell's spaces with the symmetry groups* $G_S = G_{5,1}$ *and* $G_S = G_{5,2}$.

4.4.2. *For each algebra*

$$\mathcal{L}_{5,3} = L\{e_{12} - e_{14}, e_1, e_2, e_3, e_4\} \text{ and } \mathcal{L}_{6,3} = L\{e_{12} - e_{14}, e_{23} + e_{34}, e_1, e_2, e_3, e_4\}$$

there is the class $C_{6,3}$ *of the homogeneous Maxwell spaces defined by the tensor* F_{ij} *of the form*

$$F_{12} = F_{14} = C_1, \quad F_{23} = -F_{34} = C_2, \quad F_{13} = F_{24} = 0, \tag{72}$$

where C_1 *and* C_2 *are arbitrary constants.*

Proposition 30. *If* $C_1 \neq 0$ *or* $C_2 \neq 0$, *then the Maxwell space defined by the tensor* (72) *admits the six-dimensional group* $G_S = G_{6,3}$ *which corresponds to the algebra* $\mathcal{L}_{6,3}$. *There are no Maxwell's spaces with symmetry group* $G_S = G_{5,3}$.

4.4.3. The algebra $\mathcal{L}_{5,6} = L\{e_{12} - e_{14}, e_{24} + \lambda e_3, e_1, e_2, e_4\}$ corresponds to the group $G_{5,6}$ generated by parabolic rotations, hyperbolic helices, and translations along the vectors of the pseudo-Euclidean hyperplane. Since $\mathcal{L}_{4,6} \subset \mathcal{L}_{5,6}$, then $C_{5,6} \subset C_{4,6}$. *The Maxwell space of the class* $C_{5,6}$ *is defined by the tensor* F_{ij} *of the form:*

$$F_{12} = F_{13} = F_{14} = F_{24} = 0, \quad F_{23} = -F_{34} = K\, e^{-x^3/\lambda} \quad (K = \mathrm{const}). \tag{73}$$

For $\lambda = 0$ *this class is empty.*

Proposition 31. *For* $K \neq 0$ *the Maxwell space defined by the tensor* (73), *admits the five-dimensional group* $G_S = G_{5,6}$.

4.4.4. *For the algebras*

$$\mathcal{L}_{6,4} = L\{e_{12} - e_{14}, e_{24}, e_1, e_2, e_3, e_4\} \text{ and } \mathcal{L}_{6,9} = L\{e_{12}, e_{14}, e_{24}, e_1, e_2, e_4\},$$

the corresponding classes of Maxwell spaces are empty.

5. The Maxwell spaces that admit elliptic helices

In this section, we describe classes of the Maxwell spaces that admit elliptic helices and were not included in the previous sections. See details in [21].

5.1. **Two-dimensional subgroups.**

5.1.1. *The class $C_{2,2}$.* The algebra $\mathcal{L}_{2,2} = L\{e_{13}+\mu e_4,\ e_2\}$ corresponds to the group $G_{2,2}$ generated by elliptic helices with a time-like axis and by translations along a space-like straight line. Since $\mathcal{L}_{1,2b} \subset \mathcal{L}_{2,2}$, then $C_{2,2} \subset C_{1,2b}$. *The Maxwell space of the class $C_{2,2}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} &F_{12} = c_1 \cos\varphi + c_2 \sin\varphi, \quad F_{13} = F_{13}(r, \tilde{x}^4), \quad F_{14} = c_3 \cos\varphi + c_4 \sin\varphi, \\ &F_{23} = c_1 \sin\varphi - c_2 \cos\varphi, \quad F_{24} = F_{24}(r, \tilde{x}^4), \quad F_{34} = -c_3 \sin\varphi + c_4 \cos\varphi, \end{aligned} \tag{74}$$

where $c_i = c_i(r, \tilde{x}^4)$ are the functions that satisfy to the following system of equations

$$\begin{aligned} &\frac{\partial c_1}{\partial r} + \frac{c_1}{r} + \frac{\mu}{r}\frac{\partial c_2}{\partial \tilde{x}^4} = 0, && \frac{\partial F_{24}}{\partial r} + \frac{\partial c_2}{\partial \tilde{x}^4} = 0, \\ &\frac{\partial c_3}{\partial r} + \frac{c_3}{r} + \frac{\mu}{r}\frac{\partial c_4}{\partial \tilde{x}^4} - \frac{\partial F_{13}}{\partial \tilde{x}^4} = 0, && \frac{\mu}{r}\frac{\partial F_{24}}{\partial \tilde{x}^4} - \frac{\partial c_1}{\partial \tilde{x}^4} = 0; \end{aligned} \tag{75}$$

the relation of the coordinates $\{x^i\}$ and $\{\tilde{x}^i\} = \{r, \tilde{x}^2, \varphi, \tilde{x}^4\}$ is defined by (12) for $\lambda = 0$:

$$x^1 = r\sin\varphi, \quad x^2 = \tilde{x}^2, \quad x^3 = r\cos\varphi, \quad x^4 = \mu\varphi + \tilde{x}^4. \tag{76}$$

For the potential $A_i = (0, 0, 0, \Phi)$, $\Phi = \Phi(r, \tilde{x}^4)$, we get instead of (20):

$$\begin{aligned} &F_{14} = \Phi_r \sin\varphi - \frac{\mu}{r}\Phi_4 \cos\varphi, \quad F_{12} = F_{13} = F_{23} = F_{24} = 0, \\ &F_{34} = \Phi_r \cos\varphi + \frac{\mu}{r}\Phi_4 \sin\varphi \quad \left(\Phi_r = \frac{\partial \Phi}{\partial r}, \quad \Phi_4 = \frac{\partial \Phi}{\partial \tilde{x}^4}\right). \end{aligned} \tag{77}$$

Proposition 32. *If $\partial^2\Phi/\partial r\partial\tilde{x}^4 \neq 0$, then the Maxwell space defined by the tensor (77) admits the two-dimensional group $G_S = G_{2,2}$.*

5.1.2. *The class $C_{2,4}$.* The algebra $\mathcal{L}_{2,4} = L\{e_{13} + \lambda e_2,\ e_2 + e_4\}$ corresponds to the group $G_{2,4}$ generated by elliptic helices with a space-like axis and by translations along an isotropic straight line. The class $C_{2,4}$ is an intersection of the classes $C_{1,2a}$ and $C_{1,1c}$. *The Maxwell space of the class $C_{2,4}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} &F_{12} = c_1 \cos\varphi + c_2 \sin\varphi, \quad F_{13} = F_{13}(r, \tilde{x}^2 - \tilde{x}^4), \\ &F_{14} = c_3 \cos\varphi + c_4 \sin\varphi, \quad F_{23} = c_1 \sin\varphi - c_2 \cos\varphi, \\ &F_{24} = F_{24}(r, \tilde{x}^2 - \tilde{x}^4), \quad F_{34} = -c_3 \sin\varphi + c_4 \cos\varphi, \end{aligned} \tag{78}$$

where the functions $c_i = c_i(r, \tilde{x}^2 - \tilde{x}^4)$, $i = 1, \ldots, 4$, satisfy to (17) (the change of coordinates is defined by (12) for $\mu = 0$):

$$x^1 = r\sin\varphi, \quad x^2 = \lambda\varphi + \tilde{x}^2, \quad x^3 = r\cos\varphi, \quad x^4 = \tilde{x}^4. \tag{79}$$

For the potential $A_i = (0, 0, 0, \Phi)$, $\Phi = \Phi(r, \tilde{x}^2 - \tilde{x}^4) = \Phi(r, u)$, we get instead of (18):

$$\begin{aligned} &F_{12} = F_{13} = 0, \quad F_{14} = \Phi_r \sin\varphi - \frac{\lambda}{r}\Phi_u \cos\varphi, \\ &F_{23} = 0, \quad F_{24} = \Phi_u, \quad F_{34} = \Phi_r \cos\varphi + \frac{\lambda}{r}\Phi_u \sin\varphi. \end{aligned} \tag{80}$$

Proposition 33. *If $\partial^2\Phi/\partial r\partial u \neq 0$, then the Maxwell space defined by the tensor (80) admits the two-dimensional group $G_S = G_{2,4}$.*

5.2. **Three-dimensional subgroups.**

5.2.1. *The class $C_{3,2}$.* The algebra $\mathcal{L}_{3,2} = L\{e_{13}+\lambda e_2,\ e_1,\ e_3\}$ ($\lambda \neq 0$) corresponds to the group $G_{3,2}$ generated by elliptic helices with a space-like axis and by translations along the vectors of the two-dimensional Euclidian plane. Since $\mathcal{L}_{1,2a} \subset \mathcal{L}_{3,2}$ and $\mathcal{L}_{1,1a} \subset \mathcal{L}_{3,2}$, then $C_{3,2} \subset C_{1,2a}$ and $C_{3,2} \subset C_{1,1a}$. *The Maxwell space of the class $C_{3,2}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned}
&F_{12} = a_1(x^4)\sin\frac{x^2}{\lambda} - a_2(x^4)\cos\frac{x^2}{\lambda}, \quad F_{13} = \text{const},\\
&F_{23} = -a_1(x^4)\cos\frac{x^2}{\lambda} - a_2(x^4)\sin\frac{x^2}{\lambda}, \quad F_{24} = F_{24}(x^4),\\
&F_{14} = -\lambda a_1'(x^4)\cos\frac{x^2}{\lambda} - \lambda a_2'(x^4)\sin\frac{x^2}{\lambda},\\
&F_{34} = \lambda a_1'(x^4)\sin\frac{x^2}{\lambda} - \lambda a_2'(x^4)\cos\frac{x^2}{\lambda},
\end{aligned} \tag{81}$$

where $a_k = a_k(x^4)$ are arbitrary smooth functions.

Proposition 34. *The Maxwell space defined by the tensor (81) admits the three-dimensional group $G_S = G_{3,2}$ whenever one of the following conditions is satisfied:*
1) *the functions $a_1(x^4)$ and $a_1'(x^4)$ are linearly independent,*
2) *the functions $a_2(x^4)$ and $a_2'(x^4)$ are linearly independent.*

5.2.2. *The class $C_{3,3}$.* The algebra $\mathcal{L}_{3,3} = L\{e_{13}+\mu e_4,\ e_1,\ e_3\}$ ($\mu \neq 0$) corresponds to the group $G_{3,3}$ generated by elliptic helices with a time-like axis and by translations along the vectors of the two-dimensional Euclidian plane. Since $\mathcal{L}_{1,2b} \subset \mathcal{L}_{3,3}$ and $\mathcal{L}_{1,1a} \subset \mathcal{L}_{3,3}$, then $C_{3,3} \subset C_{1,2b}$ and $C_{3,3} \subset C_{1,1a}$. *The Maxwell space of the class $C_{3,3}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned}
&F_{12} = \mu a_1'(x^2)\sin\frac{x^4}{\mu} - \mu a_2'(x^2)\cos\frac{x^4}{\mu}, \quad F_{13} = \text{const},\\
&F_{23} = -\mu a_1'(x^2)\cos\frac{x^4}{\mu} - \mu a_2'(x^2)\sin\frac{x^4}{\mu}, \quad F_{24} = F_{24}(x^2),\\
&F_{14} = a_1(x^2)\cos\frac{x^4}{\mu} + a_2(x^2)\sin\frac{x^4}{\mu},\\
&F_{34} = -a_1(x^2)\sin\frac{x^4}{\mu} + a_2(x^2)\cos\frac{x^4}{\mu},
\end{aligned} \tag{82}$$

where $a_1(x^2)$ and $a_2(x^2)$ are arbitrary smooth functions.

Proposition 35. *The Maxwell space defined by the tensor (82) admits the three-dimensional group $G_S = G_{3,3}$, whenever one of the following conditions is satisfied:*
1) *the functions $a_1(x^2)$ and $a_1'(x^2)$ are linearly independent,*
2) *the functions $a_2(x^2)$ and $a_2'(x^2)$ are linearly independent.*

5.2.3. *The class $C_{3,4a}$.* The algebra $\mathcal{L}_{3,4a} = L\{e_{13} + \lambda(e_2 + e_4),\, e_1,\, e_3\}$ $(\lambda \neq 0)$ corresponds to the group $G_{3,4a}$ generated by elliptic helices with an isotropic axis and by translations along the vectors of the two-dimensional Euclidian plane. Since $\mathcal{L}_{1,2c} \subset \mathcal{L}_{3,4a}$ and $\mathcal{L}_{1,1a} \subset \mathcal{L}_{3,4a}$, then $C_{3,4a} \subset C_{1,2c}$ and $C_{3,4a} \subset C_{1,1a}$. *The Maxwell space of the class $C_{3,4a}$ is defined by the tensor F_{ij} of the form*

$$
\begin{aligned}
&F_{12} = F_{14} = a_1 \sin\frac{x^2+x^4}{2\lambda} - a_2\cos\frac{x^2+x^4}{2\lambda}, \quad F_{13} = \mathrm{const},\\
&F_{34} = -F_{23} = a_1\cos\frac{x^2+x^4}{2\lambda} + a_2\sin\frac{x^2+x^4}{2\lambda}, \quad F_{24} = \Phi(x^2 - x^4),
\end{aligned}
\tag{83}
$$

where $a_1, a_2 = \mathrm{const}$ and $\Phi(u)$ is an arbitrary function of one variable.

Proposition 36. *The Maxwell space defined by the tensor (83) admits the three-dimensional group $G_S = G_{3,4a}$, whenever one of the following conditions is satisfied:*
1) $a_1 \neq 0$ and $\Phi(u) \neq \mathrm{const}$,
2) $a_2 \neq 0$ and $\Phi(u) \neq \mathrm{const}$.

5.2.4. *The class $C_{3,4b}$.* The algebra $\mathcal{L}_{3,4b} = L\{e_{13},\, e_1,\, e_3\}$ corresponds to the group $G_{3,4b}$ of motions of the two-dimensional Euclidian plane Ox^1x^3. Since $\mathcal{L}_{1,2d} \subset \mathcal{L}_{3,4b}$ and $\mathcal{L}_{1,1a} \subset \mathcal{L}_{3,4b}$, then $C_{3,4b} \subset C_{1,2d}$ and $C_{3,4b} \subset C_{1,1a}$. *The Maxwell space of the class $C_{3,4b}$ is defined by the tensor F_{ij} of the form*

$$
F_{12} = F_{14} = F_{23} = F_{34} = 0, \quad F_{13} = \Phi(x^2, x^4), \quad F_{24} = \Psi(x^2, x^4), \tag{84}
$$

where $\Phi(x^2,\, x^4)$ and $\Psi(x^2,\, x^4)$ are arbitrary smooth functions.

Proposition 37. *If the functions $\Phi(x^2, x^4)$, $\Psi(x^2, x^4)$ are linearly independent and the partial derivatives $\partial\Phi/\partial x^2$, $\partial\Phi/\partial x^4$ (or $\partial\Psi/\partial x^2$, and $\partial\Psi/\partial x^4$) are also linearly independent, then the Maxwell space defined by the tensor (84) admits the three-dimensional group $G_S = G_{3,4b}$.*

5.3. **Four-dimensional subgroups.**

5.3.1. *The class $C_{4,2a}$.* The algebra $\mathcal{L}_{4,2a} = L\{e_{13} + \mu e_4,\, e_1,\, e_2,\, e_3\}$ $(\mu \neq 0)$ corresponds to the group $G_{4,2a}$ generated by elliptic helices with a time-like axis and by translations along the vectors of the Euclidian hyperplane. Since $\mathcal{L}_{3,3a} \subset \mathcal{L}_{4,2a}$, then $C_{4,2a} \subset C_{3,3a}$. *The Maxwell space of the class $C_{4,2a}$ is defined by the tensor F_{ij} of the form*

$$
\begin{aligned}
&F_{14} = c_1\cos\frac{x^4}{\mu} + c_2\sin\frac{x^4}{\mu}, \quad F_{34} = -c_1\sin\frac{x^4}{\mu} + c_2\cos\frac{x^4}{\mu},\\
&F_{12} = F_{23} = 0, \quad F_{13} = c_3, \quad F_{24} = c_4 \quad (c_k = \mathrm{const}).
\end{aligned}
\tag{85}
$$

Proposition 38. *If $c_1 \neq 0$ or $c_2 \neq 0$, then the Maxwell space defined by the tensor (85) admits the four-dimensional group $G_S = G_{4,2a}$.*

5.3.2. *The class $C_{4,2b}$.* The algebra $\mathcal{L}_{4,2b} = L\{e_{13},\, e_1,\, e_2,\, e_3\}$ corresponds to the group $G_{4,2b}$ generated by rotations in the plane Ox^1x^3 and by translations along the vectors of the Euclidian hyperplane. Since $\mathcal{L}_{3,4b} \subset \mathcal{L}_{4,2b}$, then $C_{4,2b} \subset C_{3,4b}$. *The Maxwell space* *of the class $C_{4,2b}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0, \qquad F_{13} = C = \text{const}, \quad F_{24} = \Phi(x^4), \quad (86)$$

where $\Phi(x^4)$ is an arbitrary smooth function.

Proposition 39. *If $\neq 0$ and $\Phi(x^4) \neq \text{const}$, then the Maxwell space defined by the tensor (86) admits the four-dimensional group $G_S = G_{4,2b}$.*

5.3.3. *The class $C_{4,4a}$.* The algebra $\mathcal{L}_{4,4a} = L\{e_{13} + \lambda e_2,\ e_1, e_3, e_2 + e_4\}$ $(\lambda \neq 0)$ corresponds to the group $G_{4,4a}$ generated by elliptic helices with a space-like axis and by translations along the vectors of the isotropic hyperplane. Since $\mathcal{L}_{3,2} \subset \mathcal{L}_{4,4a}$, then $C_{4,4a} \subset C_{3,2}$. *The Maxwell space of the class $C_{4,2a}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} F_{12} = -F_{14} = b_1 \cos \frac{x^2 - x^4}{\lambda} - b_2 \sin \frac{x^2 - x^4}{\lambda}, \quad F_{13} = b_3, \\ F_{23} = F_{34} = b_1 \sin \frac{x^2 - x^4}{\lambda} + b_2 \cos \frac{x^2 - x^4}{\lambda}, \quad F_{24} = b_4 \quad (b_i = \text{const}). \end{aligned} \quad (87)$$

Proposition 40. *The Maxwell space defined by the tensor (87) admits the four-dimensional group $G_S = G_{4,4a}$, whenever the following conditions are satisfied*

1) $b_1 \neq 0$ or $b_2 \neq 0$,
2) $b_3 \neq 0$ or $b_4 \neq 0$.

5.3.4. *The class $C_{4,4b}$.* The algebra $\mathcal{L}_{4,4b} = L\{e_{13},\ e_1, e_3, e_2 + e_4\}$ corresponds to the group $G_{4,4b}$ generated by rotations in the plane Ox^1x^3 and by translations along the vectors of the isotropic hyperplane. Since $\mathcal{L}_{3,4b} \subset \mathcal{L}_{4,4b}$, then $C_{4,4b} \subset C_{3,4b}$. *The Maxwell space of the class $C_{4,2b}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0,\ F_{13} = \Phi(x^2 - x^4),\ F_{24} = \Psi(x^2 - x^4), \quad (88)$$

where $\Phi(u)$ and $\Psi(u)$ are arbitrary functions of one variable.

Proposition 41. *The Maxwell space defined by the tensor (88) admits the four-dimensional group $G_S = G_{4,4b}$, whenever two following conditions are satisfied*

1) the functions $\Phi(u)$ and $\Psi(u)$ are linearly independent,
2) $\Phi(u) \neq \text{const}$ or $\Psi(u) \neq \text{const}$.

5.4. **Five- and six-dimensional subgroups.**

5.4.1. *For the algebra $\mathcal{L}_{5,4} = L\{e_{13}, e_{24}, e_1, e_3, e_2 + e_4\}$ we have the class $C_{6,2}$ of homogeneous Maxwell spaces defined by the tensor F_{ij} of the form (71).*

Proposition 42. *There are no Maxwell's spaces with the symmetry group $G_S = G_{5,4}$.*

5.4.2. *The classes of the Maxwell spaces that correspond to the algebras*

$$\mathcal{L}_{6,5} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda e_2,\ e_1,\ e_3,\ e_2 - e_4\}$$

and

$$\mathcal{L}_{6,8} = L\{e_{12},\ e_{13},\ e_{23},\ e_1,\ e_2,\ e_3\},$$

are empty.

6. The Maxwell spaces that admit hyperbolic helices

In this section, we describe classes of the Maxwell spaces that admit hyperbolic helices and were not included in the previous sections. See details in [21].

6.1. **Two-dimensional subgroups.**

6.1.1. *The class $C_{2,5}$.* The algebra $\mathcal{L}_{2,5} = L\{e_{24} + \lambda e_3,\, e_1\}$ corresponds to the group $G_{2,5}$ generated by hyperbolic helices and by translations along the space-like straight line Ox^1. *The Maxwell space of the class $C_{2,5}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} &F_{12} = c_1\cosh\varphi + c_2\sinh\varphi, \quad F_{13} = F_{13}(r,\tilde{x}^3),\\ &F_{14} = -c_1\sinh\varphi - c_2\cosh\varphi, \quad F_{23} = c_3\cosh\varphi + c_4\sinh\varphi,\\ &F_{24} = F_{24}(r,\tilde{x}^3), \quad F_{34} = c_3\sinh\varphi + c_4\cosh\varphi, \end{aligned} \tag{89}$$

where the functions $c_i = c_i(r,\tilde{x}^3)$ satisfy the equations

$$\begin{aligned} &\frac{\partial c_2}{\partial r} + \frac{c_2}{r} - \frac{\lambda}{r}\frac{\partial c_1}{\partial \tilde{x}^3} = 0, && \frac{\partial c_1}{\partial \tilde{x}^3} - \frac{\partial F_{13}}{\partial r} = 0,\\ &\frac{\partial c_4}{\partial r} + \frac{c_4}{r} - \frac{\lambda}{r}\frac{\partial c_3}{\partial \tilde{x}^3} - \frac{\partial F_{24}}{\partial \tilde{x}^3} = 0, && \frac{\lambda}{r}\frac{\partial F_{13}}{\partial \tilde{x}^3} - \frac{\partial c_2}{\partial \tilde{x}^3} = 0, \end{aligned} \tag{90}$$

and the transformation of coordinates is defined by formulae

$$x^1 = \tilde{x}^1, \quad x^2 = r\cosh\varphi, \quad x^3 = \lambda\varphi + \tilde{x}^3, \quad x^4 = r\sinh\varphi. \tag{91}$$

The class $P_{2,5}$ of the potentials A_i that are invariant with respect to the group $G_{2,5}$ consists of the following fields [23]

$$\begin{aligned} &A_1 = A_1(r,\tilde{x}^3), \quad A_2 = C_1\cosh\varphi + C_2\sinh\varphi,\\ &A_3 = A_3(r,\tilde{x}^3), \quad A_4 = -C_1\sinh\varphi - C_2\cosh\varphi, \end{aligned} \tag{92}$$

where $C_i = C_i(r,\tilde{x}^3)$. For the field $A_i = (-\Phi, 0, 0, 0)$, $\Phi = \Phi(\tilde{x}^3)$, we obtain

$$F_{12} = \frac{\lambda}{r}\Phi'\sinh\varphi, \quad F_{13} = \Phi', \quad F_{14} = -\frac{\lambda}{r}\Phi'\cosh\varphi, \quad F_{23} = F_{24} = F_{34} = 0, \tag{93}$$

where $\Phi' = d\Phi/d\tilde{x}^3$.

Proposition 43. *If $\Phi''(\tilde{x}^3) \neq 0$, then the Maxwell space defined by the tensor (93) admits the two-dimensional group $G_S = G_{2,5}$.*

6.1.2. *The class $C_{2,6}$.* The algebra $\mathcal{L}_{2,6} = L\{e_{24} + \lambda e_3,\, e_2 - e_4\}$ corresponds to the group $G_{2,6}$ generated by hyperbolic helices and by translations along an isotropic straight line. *The Maxwell space of the class $C_{2,6}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} &F_{12} = c_1\cosh\varphi + c_2\sinh\varphi, \quad F_{13} = F_{13}(\tilde{x}^1, r,\tilde{x}^3),\\ &F_{14} = -c_1\sinh\varphi - c_2\cosh\varphi, \quad F_{23} = c_3\cosh\varphi + c_4\sinh\varphi,\\ &F_{24} = F_{24}(\tilde{x}^1, r,\tilde{x}^3), \quad F_{34} = c_3\sinh\varphi + c_4\cosh\varphi, \end{aligned} \tag{94}$$

where the functions $F_{13}(\tilde{x}^1, r, \tilde{x}^3)$, $F_{24}(\tilde{x}^1, r, \tilde{x}^3)$, $c_i = c_i(\tilde{x}^1, r, \tilde{x}^3)$ $(i = 1, \ldots, 4)$ *satisfy the equations*

$$\begin{gathered} \frac{\partial c_2}{\partial r} + \frac{c_2}{r} - \frac{\lambda}{r}\frac{\partial c_1}{\partial \tilde{x}^3} + \frac{\partial F_{24}}{\partial \tilde{x}^1} = 0, \quad \frac{\partial c_1}{\partial \tilde{x}^3} - \frac{\partial F_{13}}{\partial r} + \frac{\partial c_3}{\partial \tilde{x}^1} = 0, \\ \frac{\partial c_4}{\partial r} + \frac{c_4}{r} - \frac{\lambda}{r}\frac{\partial c_3}{\partial \tilde{x}^3} - \frac{\partial F_{24}}{\partial \tilde{x}^3} = 0, \quad \frac{\lambda}{r}\frac{\partial F_{13}}{\partial \tilde{x}^3} - \frac{\partial c_4}{\partial \tilde{x}^1} - \frac{\partial c_2}{\partial \tilde{x}^3} = 0 \end{gathered} \tag{95}$$

and the system

$$\begin{gathered} \frac{\partial c_1}{\partial r} + \frac{\lambda}{r}\frac{\partial c_1}{\partial \tilde{x}^3} - \frac{c_2}{r} = 0, \quad \frac{\partial c_2}{\partial r} + \frac{\lambda}{r}\frac{\partial c_2}{\partial \tilde{x}^3} - \frac{c_1}{r} = 0, \quad \frac{\partial F_{13}}{\partial r} + \frac{\lambda}{r}\frac{\partial F_{13}}{\partial \tilde{x}^3} = 0, \\ \frac{\partial c_3}{\partial r} + \frac{\lambda}{r}\frac{\partial c_3}{\partial \tilde{x}^3} - \frac{c_4}{r} = 0, \quad \frac{\partial c_4}{\partial r} + \frac{\lambda}{r}\frac{\partial c_4}{\partial \tilde{x}^3} - \frac{c_3}{r} = 0, \quad \frac{\partial F_{24}}{\partial r} + \frac{\lambda}{r}\frac{\partial F_{24}}{\partial \tilde{x}^3} = 0 \end{gathered} \tag{96}$$

The transformation of coordinates is defined by formulae (91).

The class $P_{2,6}$ of potentials A_i, that are invariant with respect to the group $G_{2,6}$, consists of the following fields [23]

$$\begin{aligned} &A_1 = A_1(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r), \quad A_2 = C_1 \cosh\varphi + C_2 \sinh\varphi, \\ &A_3 = A_3(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r), \quad A_4 = -C_1 \sinh\varphi - C_2 \cosh\varphi, \end{aligned} \tag{97}$$

where

$$\begin{aligned} C_1 &= a_1(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \cosh \ln r + a_2(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \sinh \ln r, \\ C_2 &= a_1(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \sinh \ln r + a_2(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) \cosh \ln r. \end{aligned} \tag{98}$$

For the field $A_i = (-\Phi, 0, 0, 0)$, $\Phi = \Phi(\tilde{x}^1, \tilde{x}^3 - \lambda \ln r) = \Phi(t_1, t_2)$, we obtain

$$F_{12} = F_{14} = -\frac{\lambda}{r}\Phi_2 e^{-\varphi}, \quad F_{13} = \Phi_2,$$
$$F_{23} = F_{24} = F_{34} = 0 \quad (\Phi_2 = \partial\Phi/\partial t_2). \tag{99}$$

Proposition 44. *Let* $\Phi_{ij} = \partial^2\Phi/\partial t_i \partial t_j$. *If the partial derivatives* Φ_{21} *and* Φ_{22} *are linearly independent, then the Maxwell space defined by the tensor* (99) *admits the two-dimensional group* $G_S = G_{2,6}$.

6.2. **Three-dimensional subgroups.**

6.2.1. *The class* $C_{3,5}$. The algebra $\mathcal{L}_{3,5} = L\{e_{24}, e_1, e_3\}$ corresponds to the group $G_{3,5}$ generated by pseudo-rotations in the plane Ox^2x^4 and by translations along the vectors of the Euclidean plane Ox^1x^3. *The Maxwell space of the class* $C_{3,5}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} &F_{12} = a_1(r) \cosh\varphi + \frac{b_1}{r} \sinh\varphi, \quad F_{13} = b_3, \\ &F_{14} = -a_1(r) \sinh\varphi - \frac{b_1}{r} \cosh\varphi, \quad F_{23} = a_2(r) \cosh\varphi + \frac{b_2}{r} \sinh\varphi, \\ &F_{24} = F_{24}(r) \quad F_{34} = a_2(r) \sinh\varphi + \frac{b_2}{r} \cosh\varphi, \end{aligned} \tag{100}$$

where $a_1(r)$, $a_2(r)$ *Υ* $F_{24}(r)$ *are arbitrary functions,* $b_k = \text{const}$, *and the transformation of coordinates is defined by formulae* (31).

Proposition 45. *If $F_{24}(r) \neq \mathrm{const}$, and the functions $a_1(r)$, $a_2(r)$, $1/r$ are linearly independent, then the Maxwell space defined by the tensor* (100) *admits the three-dimensional group $G_S = G_{3,5}$.*

6.2.2. *The class $C_{3,7}$.* The algebra $\mathcal{L}_{3,7} = L\{e_{24} + \lambda e_3,\, e_1,\, e_2 - e_4\}$ corresponds to the group $G_{3,7}$ generated by hyperbolic helices and by translations along the vectors of an isotropic plane. $C_{3,7} = C_{2,5} \cap C_{2,6}$. *The Maxwell space of the class $C_{3,7}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned}
&F_{12} = F_{14} = \frac{\lambda}{r}\, e^{-\varphi}\, \Phi\left(r\, e^{-\tilde{x}^3/\lambda}\right), \quad F_{13} = \Phi\left(r\, e^{-\tilde{x}^3/\lambda}\right),\\
&F_{24} = \Psi\left(r\, e^{-\tilde{x}^3/\lambda}\right), \quad F_{23} = c_3(r, \tilde{x}^3)\cosh\varphi + c_4(r, \tilde{x}^3)\sinh\varphi,\\
&F_{34} = c_3(r, \tilde{x}^3)\sinh\varphi + c_4(r, \tilde{x}^3)\cosh\varphi,
\end{aligned} \tag{101}$$

where $\Phi(u)$ and $\Psi(u)$ are arbitrary functions of one variable, the functions $c_3 = c_3(r, \tilde{x}^3)$ and $c_4 = c_4(r, \tilde{x}^3)$ are satisfied to equations

$$\begin{aligned}
&\frac{\partial c_4}{\partial r} + \frac{c_4}{r} - \frac{\lambda}{r}\frac{\partial c_3}{\partial \tilde{x}^3} - \frac{\partial F_{24}}{\partial \tilde{x}^3} = 0,\\
&\frac{\partial c_3}{\partial r} + \frac{\lambda}{r}\frac{\partial c_3}{\partial \tilde{x}^3} - \frac{c_4}{r} = 0,\\
&\frac{\partial c_4}{\partial r} + \frac{\lambda}{r}\frac{\partial c_4}{\partial \tilde{x}^3} - \frac{c_3}{r} = 0,
\end{aligned} \tag{102}$$

and the transformation of coordinates is defined by formulae (91)*.*

The class $P_{3,7}$ of potentials A_i that are invariant with respect to the group $G_{3,7}$ consists of the fields defined by (97) and (98), where the functions do not depend on $\tilde{x}^1$. For the field $A_i = (-\Phi, 0, 0, 0)$, $\Phi = \Phi(\tilde{x}^3 - \lambda \ln r) = \Phi(t)$, we obtain

$$F_{12} = F_{14} = -\frac{\lambda}{r}\Phi' e^{-\varphi}, \quad F_{13} = \Phi', \quad F_{23} = F_{24} = F_{34} = 0 \quad (\Phi' = d\Phi/dt). \tag{103}$$

Proposition 46. *If $\Phi'' \neq 0$, then the Maxwell space defined by the tensor* (103) *admits the three-dimensional group $G_S = G_{3,7}$.*

6.2.3. *The class $C_{3,16}$.* The algebra $\mathcal{L}_{3,16} = L\{e_{12} - e_{14},\, e_{24} + \lambda e_1 + \mu e_3,\, e_2 - e_4\}$ corresponds to the group $G_{3,16}$ generated by parabolic rotations, hyperbolic helices, and by translations along an isotropic straight line. For description of the class $C_{3,16}$ we use the change

$$x^1 = \lambda\varphi + \tilde{x}^1, \quad x^2 = r\cosh\varphi, \quad x^3 = \mu\varphi + \tilde{x}^3, \quad x^4 = r\sinh\varphi, \tag{104}$$

which contains, in particular, transformations (25), (31), and (91). *The Maxwell space is defined by the tensor F_{ij} of the form*

$$\begin{aligned}
&F_{12} = c_1\cosh\varphi + c_2\sinh\varphi, \quad F_{14} = -c_1\sinh\varphi - c_2\cosh\varphi,\\
&F_{13} = b_1\tilde{x}^1 + b_2, \quad F_{23} = c_3\cosh\varphi + c_4\sinh\varphi,\\
&F_{34} = c_3\sinh\varphi + c_4\cosh\varphi, \qquad F_{24} = a_2,
\end{aligned} \tag{105}$$

where

$$c_1 = -c_2 = \frac{a_2\tilde{x}^1 + a_3}{r}, \quad c_3 = -\frac{1}{r}\left(\frac{b_1}{2}\left(\tilde{x}^1\right)^2 + b_2\tilde{x}^1 + b_3\right), \quad c_4 = b_1 r - c_3, \tag{106}$$

and the functions $a_k = a_k(r,\, \tilde{x}^3)$ $(k = 2,\, 3)$, $b_l = b_l(r,\, \tilde{x}^3)$ $(l = 1,\, 2,\, 3)$ *are defined by the following formulae:*

a) *for the class* $C_{3,16a}$ *that corresponds to the algebra* $\mathcal{L}_{3,16a} = L\{e_{12} - e_{14},\, e_{24},\, e_2 - e_4\}$ ($\mathcal{L}_{3,16}$ *for* $\lambda = \mu = 0$),

$$\begin{aligned} &a_2 = \Phi_1(\tilde{x}^3), \quad a_3 = \Phi_2(\tilde{x}^3), \quad b_1 = \Phi_1'(\tilde{x}^3), \\ &b_2 = \Phi_2'(\tilde{x}^3), \quad b_3 = \Phi_3(\tilde{x}^3) - \frac{r^2}{2}\Phi_1'(\tilde{x}^3), \end{aligned} \tag{107}$$

where $\Phi_1(\tilde{x}^3)$, $\Phi_2(\tilde{x}^3)$, $\Phi_3(\tilde{x}^3)$ *are arbitrary functions and the change is defined by* (31);

b) *for the class* $C_{3,16b}$ *that corresponds to the algebra* $L\{e_{12} - e_{14},\, e_{24} + \lambda e_1,\, e_2 - e_4\}$ ($\mathcal{L}_{3,16}$ *for* $\lambda \neq 0,\ \mu = 0$),

$$\begin{aligned} &a_2 = \Phi_1(\tilde{x}^3), \quad a_3 = \Phi_2(\tilde{x}^3) - \lambda \ln r\, \Phi_1(\tilde{x}^3), \\ &b_2 = \Phi_2'(\tilde{x}^3) + (\lambda - \lambda \ln r)\, \Phi_1'(\tilde{x}^3), \quad b_1 = \Phi_1'(\tilde{x}^3), \\ &b_3 = \Phi_3(\tilde{x}^3) + \left(\frac{\lambda^2}{2}\ln^2 r - \lambda^2 \ln r - \frac{r^2}{2}\right)\Phi_1'(\tilde{x}^3) - \lambda \ln r\, \Phi_2'(\tilde{x}^3), \end{aligned} \tag{108}$$

and the change is defined by (25) (for $\lambda = 0$ the formulae (108) are transformed in (107));

c) *for the class* $C_{3,16c}$ *that corresponds to the algebra* $\mathcal{L}_{3,16c} = \mathcal{L}_{3,16}$ $(\lambda \neq 0,\ \mu \neq 0)$,

$$\begin{aligned} &a_2 = \Phi_1(u), \quad a_3 = \Phi_2(u) - \frac{\lambda v}{2\mu}\Phi_1(u), \quad b_2 = \Phi_4(u) - \frac{\lambda v}{2\mu}\Phi_3(u), \\ &b_1 = \Phi_3(u), \quad b_3 = \frac{\lambda^2 v^2}{8\mu^2}\Phi_3(u) - \frac{\lambda v}{2\mu}\Phi_4(u) - \frac{r^2}{2}\Phi_3(u) + \Phi_5(u), \end{aligned} \tag{109}$$

where

$$u = \tilde{x}^3 - \mu \ln r, \quad v = \tilde{x}^3 + \mu \ln r, \tag{110}$$

$\Phi_1(u)$, $\Phi_2(u)$, $\Phi_5(u)$ *are arbitrary functions,*

$$\begin{aligned} &\Phi_3(u) = -\frac{1}{\mu}e^{u/\mu}\int \Phi_1'(u)e^{-u/\mu}\, du, \\ &\Phi_4(u) = -\frac{1}{\mu}e^{u/\mu}\int \left(\Phi_2'(u) - \frac{\lambda}{2\mu}\Phi_1(u) + \frac{\lambda}{2}\Phi_3(u)\right)e^{-u/\mu}\, du, \end{aligned} \tag{111}$$

and the change is defined by (104);

d) *for the class* $C_{3,16d}$ *that corresponds to the algebra* $L\{e_{12} - e_{14},\ e_{24} + \mu e_3,\ e_2 - e_4\}$ ($\mathcal{L}_{3,16}$ *for* $\lambda = 0,\ \mu \neq 0$),

$$\begin{aligned} &a_2 = \Phi_1(u), \quad a_3 = \Phi_2(u), \quad b_1 = \Phi_3(u), \\ &b_2 = \Phi_4(u), \quad b_3 = -\frac{r^2}{2}\Phi_3(u) + \Phi_5(u), \end{aligned} \tag{112}$$

where u *is defined in* (110) *and* $\Phi_1(u)$, $\Phi_2(u)$, $\Phi_5(u)$ *are arbitrary functions,*

$$\Phi_3(u) = -\frac{1}{\mu}e^{u/\mu}\int \Phi_1'(u)e^{-u/\mu}\, du, \quad \Phi_4(u) = -\frac{1}{\mu}e^{u/\mu}\int \Phi_2'(u)e^{-u/\mu}\, du, \tag{113}$$

and the change is defined by (91) $(\lambda \mapsto \mu)$

$$x^1 = \tilde{x}^1, \quad x^2 = r\cosh\varphi, \quad x^3 = \mu\varphi + \tilde{x}^3, \quad x^4 = r\sinh\varphi. \tag{114}$$

For $\lambda = 0$ the formulae (109) and (111) are transformed in (112) and (113) .

For classes $C_{3,16a}$ and $C_{3,16b}$ we take $\Phi_1 = \Phi_3 = 0$ and $\Phi_2 = \Phi(\tilde{x}^3)$; we obtain the following examples of the tensors F_{ij}:

$$F_{12} = F_{14} = \frac{1}{r}\Phi(\tilde{x}^3)e^{-\varphi}, \; F_{13} = \Phi'(\tilde{x}^3),$$
$$F_{24} = 0, \; F_{34} = -F_{23} = \frac{1}{r}\tilde{x}^1\Phi'(\tilde{x}^3)e^{-\varphi} \quad (115)$$

and

$$\begin{aligned} F_{12} = F_{14} &= \frac{1}{r}\Phi(\tilde{x}^3)e^{-\varphi}, \quad F_{13} = \Phi'(\tilde{x}^3), \quad F_{24} = 0, \\ F_{34} = -F_{23} &= \frac{1}{r}(\tilde{x}^1 - \lambda \ln r)\Phi'(\tilde{x}^3)e^{-\varphi}. \end{aligned} \quad (116)$$

Proposition 47. *If $\Phi''(\tilde{x}^3) \neq 0$, then the Maxwell space defined by the tensor (115) admits the three-dimensional group $G_S = G_{3,16a}$, and the Maxwell space defined by the tensor (116) admits the group $G_S = G_{3,16b}$.*

For classes $C_{3,16c}$ and $C_{3,16d}$ we take $\Phi_2 = \Phi(u) = \Phi(\tilde{x}^3 - \mu \ln r)$ and $\Phi_1 = \Phi_5 = 0$. Then $\Phi_3 = 0$ and

$$\Phi_4(u) = -\frac{1}{\mu}e^{u/\mu}\int \Phi'(u)e^{-u/\mu}\,du. \quad (117)$$

We obtain the following examples of the tensors F_{ij}:

$$\begin{aligned} F_{12} = F_{14} &= \frac{1}{r}\Phi(\tilde{x}^3 - \mu \ln r)e^{-\varphi}, \quad F_{13} = \Phi_4(\tilde{x}^3 - \mu \ln r), \quad F_{24} = 0, \\ F_{34} = -F_{23} &= \frac{1}{2\mu r}e^{-\varphi}\Phi_4(\tilde{x}^3 - \mu \ln r)\cdot(2\mu\tilde{x}^1 - \lambda\tilde{x}^3 - \lambda\mu \ln r) \end{aligned} \quad (118)$$

and

$$\begin{aligned} F_{12} = F_{14} &= \frac{1}{r}\Phi(\tilde{x}^3 - \mu \ln r)e^{-\varphi}, \quad F_{13} = \Phi_4(\tilde{x}^3 - \mu \ln r), \\ F_{34} = -F_{23} &= \frac{\tilde{x}^1}{2r}e^{-\varphi}\Phi_4(\tilde{x}^3 - \mu \ln r), \quad F_{24} = 0. \end{aligned} \quad (119)$$

Proposition 48. *If $\Phi''(u) \neq 0$, then the Maxwell space defined by the tensor (118) admits the three-dimensional group $G_S = G_{3,16c}$ and the Maxwell space defined by the tensor (119) admits the group $G_S = G_{3,16d}$.*

6.2.4. *The class $C_{3,21}$.* The algebra $\mathcal{L}_{3,21} = L\{e_{12}, e_{14}, e_{24}\}$ corresponds to the group $G_{3,21}$ generated by rotations in the plane Ox^1x^2 and by pseudo-rotations in planes Ox^1x^4 and Ox^2x^4. *The Maxwell space of the class $C_{3,21}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = x^1\,\Phi(x^3), \quad F_{23} = x^2\,\Phi(x^3), \quad F_{34} = x^4\,\Phi(x^3), \quad (120)$$

where $\Phi(x^3)$ is an arbitrary function.

Proposition 49. *If $\Phi(x^3) \neq$ const, then the Maxwell space defined by the tensor (120) admits the three-dimensional group $G_S = G_{3,21}$.*

6.3. **Four-dimensional subgroups.**

6.3.1. *The class* $C_{4,5}$. The algebra $\mathcal{L}_{4,5} = L\{e_{24},\, e_1,\, e_3,\, e_2 + e_4\}$ corresponds to the group $G_{4,5}$ generated by pseudo-rotations in the plane Ox^2x^4 and by translations along the vectors of the isotropic hyperplane. Since $\mathcal{L}_{3,5} \subset \mathcal{L}_{4,5}$, then $C_{4,5} \subset C_{3,5}$. *The Maxwell space of the class* $C_{4,5}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} &F_{12} = -F_{14} = \frac{b_1}{x^2 - x^4}, \quad F_{23} = F_{34} = \frac{b_2}{x^2 - x^4}, \\ &F_{13} = b_3, \quad F_{24} = b_4 \quad (b_k = \mathrm{const}). \end{aligned} \tag{121}$$

Proposition 50. *Suppose one of the following conditions is satisfied: 1)* $b_1 \neq 0$ *and* $b_3 \neq 0$, *2)* $b_1 \neq 0$ *and* $b_4 \neq 0$, *3)* $b_2 \neq 0$ *and* $b_3 \neq 0$, *4)* $b_2 \neq 0$ *and* $b_4 \neq 0$; *then the Maxwell space defined by the tensor* (121) *admits the four-dimensional group* $G_S = G_{4,5}$.

6.3.2. *The class* $C_{4,13}$. The algebra $\mathcal{L}_{4,13} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_1,\ e_3,\ e_2 - e_4\}$ corresponds to the group $G_{4,13}$ generated by parabolic rotations, by hyperbolic helices, and by translations along the vectors of an isotropic plane. Since $\mathcal{L}_{3,16b} \subset \mathcal{L}_{4,13}$, then $C_{4,13} \subset C_{3,16b}$. *The Maxwell space of the class* $C_{4,13}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} &F_{12} = F_{14} = \frac{1}{r}\left(K_1\tilde{x}^1 - K_1\lambda\ln r + K_2\right)e^{-\varphi}, \\ &F_{23} = -F_{34} = \frac{K_3}{r}\,e^{-\varphi}, \quad F_{13} = 0, \quad F_{24} = K_1, \end{aligned} \tag{122}$$

where K_1, K_2, K_3 *are arbitrary constants, and the transformation of coordinates is defined by formulae* (25).

Proposition 51. *If* $K_1 \neq 0$, *then the Maxwell space defined by the tensor* (122) *admits the four-dimensional group* $G_S = G_{4,13}$.

6.3.3. *The class* $C_{4,14}$. The algebra $\mathcal{L}_{4,14} = L\{e_{12} - e_{14},\, e_{24} + \lambda e_3,\, e_1 + \nu e_3,\, e_2 - e_4\}$ corresponds to the group $G_{4,14}$ generated by parabolic rotations, by hyperbolic helices, and by translations along the vectors of an isotropic plane. The group $G_{4,14}$ is not conjugate to $G_{4,13}$. Since $\mathcal{L}_{3,16d} \subset \mathcal{L}_{4,14}$, then $C_{4,14} \subset C_{3,16b}$. *The Maxwell space of the class* $C_{4,14}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} &F_{12} = F_{14} = \frac{1}{r}\left(K_1\tilde{x}^1 - \frac{K_1}{\nu}\left(\tilde{x}^3 - \lambda\ln r\right) + K_2\right)e^{-\varphi}, \quad F_{13} = -\frac{K_2}{\nu}, \\ &F_{23} = -F_{34} = \frac{1}{r}\left(\frac{K_1}{\nu}\tilde{x}^1 - \frac{K_1}{\nu^2}\left(\tilde{x}^3 - \lambda\ln r\right) + K_3\right)e^{-\varphi}, \quad F_{24} = K_1, \end{aligned} \tag{123}$$

where $K_i = \mathrm{const}$ *and the transformation of coordinates is defined by formulae* (91).

Proposition 52. *If* $K_1 \neq 0$, *then the Maxwell space defined by the tensor* (123) *admits the four-dimensional group* $G_S = G_{4,14}$.

For $\nu = 0$, *i. e., for the algebra* $\mathcal{L}_{4,14a} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_3,\ e_1,\ e_2 - e_4\}$, *the class* $C_{4,14a}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} &F_{12} = F_{14} = \frac{K_1}{r}e^{-\varphi}, \quad F_{13} = F_{24} = 0, \\ &F_{23} = -F_{34} = -\frac{1}{r}\Psi(\tilde{x}^3 - \lambda\ln r)e^{-\varphi}, \end{aligned} \tag{124}$$

where $\Psi(u)$ is an arbitrary function.

Proposition 53. *If $\Psi'(u) \neq 0$, then the Maxwell space defined by the tensor (124) admits the four-dimensional group $G_S = G_{4,14a}$.*

6.3.4. *The class $C_{4,15}$.* The algebra $\mathcal{L}_{4,15} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_{24} + \lambda e_1,\, e_2 - e_4\}$ corresponds to the group $G_{4,15}$ generated by two one-dimensional subgroups of parabolic rotations, by hyperbolic helices, and by translations along an isotropic straight line. Since $\mathcal{L}_{3,16b} \subset \mathcal{L}_{4,15}$, then $C_{4,15} \subset C_{3,16b}$. *The Maxwell space of the class $C_{4,15}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} &F_{12} = F_{14} = \frac{1}{r} e^{-\varphi}(K_1 \tilde{x}^1 - \lambda K_1 \ln r + K_2), \quad F_{13} = 0, \\ &F_{23} = -F_{34} = -\frac{1}{r} e^{-\varphi}(K_1 \tilde{x}^3 + K_3), \quad F_{24} = K_1, \end{aligned} \tag{125}$$

where K_i $(i = 1, 2, 3)$ are arbitrary constants, and the transformation of coordinates is defined by formulae (25).

Proposition 54. *If $K_1 \neq 0$, then the Maxwell space defined by the tensor (125) admits the four-dimensional group $G_S = G_{4,15}$.*

6.3.5. *The class $C_{4,19}$.* The algebra $\mathcal{L}_{4,19} = L\{e_{12},\; e_{14},\; e_{24},\; e_3\}$ corresponds to the group $G_{4,19}$ generated by rotations, by pseudo-rotations, and by translations along a space-like straight line. Since $\mathcal{L}_{3,21} \subset \mathcal{L}_{4,19}$, then $C_{4,19} \subset C_{3,21}$. *The Maxwell space of the class $C_{4,19}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = Kx^1,$$
$$F_{23} = Kx^2, \quad F_{34} = Kx^4 \quad (K = \text{const}). \tag{126}$$

Proposition 55. *If $K \neq 0$, then the Maxwell space defined by the tensor (126) admits the four-dimensional group $G_S = G_{4,19}$.*

6.4. **Five- and six-dimensional subgroups.**

6.4.1. *For each algebra*

$$\mathcal{L}_{5,7} = L\{e_{12} - e_{14},\, e_{24},\, e_1,\, e_3,\, e_2 - e_4\},$$

$$\mathcal{L}_{5,8} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_{24} + \lambda e_3,\, e_1,\, e_2 - e_4\},$$

and

$$\mathcal{L}_{6,6} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_{24},\, e_1,\, e_3,\, e_2 - e_4\}$$

there is the class $C_{6,6}$ of Maxwell's spaces defined by the tensor F_{ij} of the form

$$\begin{aligned} &F_{12} = F_{14} = \frac{K_1}{x^2 + x^4}, \quad F_{13} = F_{24} = 0, \\ &F_{23} = -F_{34} = \frac{K_2}{x^2 + x^4}, \quad (K_1,\, K_2 = \text{const}). \end{aligned} \tag{127}$$

Proposition 56. *If $K_1 \neq 0$ or $K_2 \neq 0$, then the Maxwell space defined by the tensor (127) admits the six-dimensional group $G_S = G_{6,6}$.*

6.4.2. *For the algebra* $\mathcal{L}_{6,9} = L\{e_{12},\, e_{14},\, e_{24},\, e_1,\, e_2,\, e_4\}$ *the corresponding class of Maxwell's spaces is empty.*

7. The Maxwell spaces that admit parabolic helices

In this section, we describe classes of the Maxwell spaces that admit parabolic helices and parabolic rotations and were not discussed in the previous sections. See details in [21].

7.1. **Two-dimensional subgroups.**

7.1.1. *The classes* $C_{2,7a}$ *and* $C_{2,7b}$. The algebra $\mathcal{L}_{2,7} = L\{e_{12} - e_{14} + \lambda e_2 + \mu e_3,\ e_2 - e_4\}$ corresponds to the group $G_{2,7}$ generated by parabolic helices and by translations along an isotropic straight line. The algebra $\mathcal{L}_{2,7}$ is an extension of the algebra $\mathcal{L}_{1,4}$ by the vector $\xi = e_2 - e_4$. Therefore, the classes $C_{2,7a}$, $C_{2,7b}$, and $C_{2,7c}$ that corresponding to $\mathcal{L}_{2,7}$ for various values of λ and μ are obtained as restrictions of classes the $C_{1,4a}$, $C_{1,4b}$, and $C_{1,4c}$ by using condition (2) for the vector $\xi = e_2 - e_4$: $\partial_2 F_{ij} - \partial_4 F_{ij} = 0$. *The Maxwell space of the class* $C_{2,7b}$ $(\lambda = 0,\ \mu \neq 0)$ *is defined by the tensor* F_{ij} *of the form* (36)*, where* $C_k = C_k(\tilde{x}^1,\, \tilde{x}^3)$ *are the smooth functions that satisfy* (37) *and satisfy the equations*

$$\begin{gathered}\frac{\mu}{\tilde{x}^1}\frac{\partial C_1}{\partial \tilde{x}^3} + \frac{\partial C_5}{\partial \tilde{x}^3} = 0, \quad \frac{\partial C_1}{\partial \tilde{x}^1} + \frac{\partial C_6}{\partial \tilde{x}^3} = 0,\\ \frac{\mu}{\tilde{x}^1}\frac{\partial C_3}{\partial \tilde{x}^3} - \frac{C_2}{\tilde{x}^1} - \frac{\partial C_2}{\partial \tilde{x}^1} + \frac{\partial C_7}{\partial \tilde{x}^3} = 0, \quad \frac{\mu}{\tilde{x}^1}\frac{\partial C_6}{\partial \tilde{x}^3} - \frac{C_5}{\tilde{x}^1} - \frac{\partial C_5}{\partial \tilde{x}^1}\end{gathered} \tag{128}$$

(the transformation of coordinates is defined by formulae (39)*).*

For $\mu = 0$ the system (128) is simplified and integrated partially. We obtain the result. *The Maxwell space of the class* $C_{2,7a}$ $(\lambda = 0,\ \mu = 0)$ *is defined by the tensor* F_{ij} *of the form* (36)*, where* $C_k = C_k(\tilde{x}^1,\, \tilde{x}^3)$ $(k \neq 5)$ *are the smooth functions satisfying* (37) *and the equations*

$$\frac{\partial C_1}{\partial \tilde{x}^1} + \frac{\partial C_6}{\partial \tilde{x}^3} = 0, \quad \frac{C_2}{\tilde{x}^1} + \frac{\partial C_2}{\partial \tilde{x}^1} - \frac{\partial C_7}{\partial \tilde{x}^3} = 0, \tag{129}$$

and $C_5 = A/\tilde{x}^1$ $(A = \text{const}$*, the transformation of coordinates is defined by formulae* (35)*).*

Proposition 57. *Suppose the functions* $\Phi(t_1,\, t_2) = \Phi\left(x^2 + x^4,\ x^3\right)$, Φ_1, Φ_2, Φ_{12} *and* Φ_{22} *are linearly independent* ($\Phi_k = \partial\Phi/\partial t_k$, $\Phi_{kl} = \partial^2\Phi/\partial t_k \partial t_l$)*; then the Maxwell space defined by the tensor*

$$F_{12} = F_{14} = -\Phi_1 - \frac{\Phi}{x^2 + x^4}, \qquad F_{13} = -\Phi_2,$$
$$F_{23} = -F_{34} = \frac{x^1}{x^2 + x^4}\Phi_2, \qquad F_{24} = 0$$

admits the two-dimensional group $G_S = G_{2,7a}$.

In particular, this condition holds for the function $\Phi(t_1,\, t_2) = (t_1^2 + t_2^3)t_2$.

Proposition 58. *Let*

$$\Phi(t_1,t_2)=\Phi\left(x^2+x^4,\ x^3+\frac{\mu x^1}{x^2+x^4}\right),$$
$$\Phi_k=\partial\Phi/\partial t_k,\quad \Phi_{kl}=\partial^2\Phi/\partial t_k\partial t_l.$$

Suppose the partials Φ_2, Φ_{12}, and Φ_{22} are linearly independent; then the Maxwell space defined by the tensor F_{ij} of the form

$$F_{12}=F_{14}=F_{24}=0,\quad F_{13}=\frac{\mu}{x^2+x^4}\Phi_2,$$
$$F_{23}=-F_{34}=\Phi_1-\frac{\mu x^1}{(x^2+x^4)^2}\Phi_2,$$

admits the two-dimensional group $G_S=G_{2,7b}$.

In particular, this condition holds for the function $\Phi(t_1,t_2)=(t_1^2+t_2)t_2$.

7.1.2. *The class $C_{2,7c}$.* For $\lambda\neq 0$ and $\mu=0$ we have the following result. *The Maxwell space of the class $C_{2,7c}$ is defined by the tensor F_{ij} of the form* (36)*, where the functions $C_k=C_k(\tilde{x}^1,\,\tilde{x}^3)$ $(k\neq 6)$ satisfy* (37) *and also satisfy the equations*

$$\frac{C_1}{\lambda}-2\lambda\frac{\partial C_3}{\partial\tilde{x}^1}-\frac{\partial C_7}{\partial\tilde{x}^3}=0,\quad 2\lambda\frac{\partial C_1}{\partial\tilde{x}^1}+\frac{\partial C_5}{\partial\tilde{x}^3}=0,\tag{130}$$

and $C_6=\mathrm{const}$ (the coordinates $\{\tilde{x}^i\}$ are related with $\{x^i\}$ by means of formulae (41)*).*

Proposition 59. *Let*

$$\Phi(t_1,t_2)=\Phi\left(2\lambda x^1+\left(x^2+x^4\right)^2,\ x^3\right),$$
$$\Phi_k=\partial\Phi/\partial t_k,\quad \Phi_{kl}=\partial^2\Phi/\partial t_k\partial t_l.$$

Suppose the partials Φ_1, Φ_{11}, and Φ_{12} are linearly independent; then the Maxwell space defined by the tensor F_{ij} of the form

$$F_{12}=F_{14}=F_{24}=0,\quad F_{13}=2\lambda\Phi_1,\quad F_{23}=-F_{34}=2(x^2+x^4)\Phi_1,$$

admits the two-dimensional group $G_S=G_{2,7c}$.

In particular, this condition holds for the function $\Phi(t_1,\ t_2)=t_1(t_1^2+t_2^2)$.

7.1.3. *The class $C_{2,8}$.* The algebra $\mathcal{L}_{2,8}=L\{e_{12}-e_{14}+\lambda e_2,\ e_3\}$ corresponds to the group $G_{2,8}$ generated by parabolic helices and by translations along a space-like straight line. Since $\mathcal{L}_{1,4c}\subset\mathcal{L}_{2,8}$, then $C_{2,8}\subset C_{1,4c}$. *The Maxwell space of the class $C_{2,8}$ is defined by the tensor F_{ij} of the form* (36)*, where the functions $C_k=C_k(\tilde{x}^1,\,\tilde{x}^4)$ satisfy* (37) *and the following equations,*

$$\begin{gathered}\frac{C_1}{\lambda}+\frac{\tilde{x}^1}{2\lambda}\frac{\partial C_2}{\partial\tilde{x}^4}-2\lambda\frac{\partial C_3}{\partial\tilde{x}^1}=0,\quad \frac{\tilde{x}^1}{2\lambda}\frac{\partial C_1}{\partial\tilde{x}^4}-\lambda\frac{\partial C_3}{\partial\tilde{x}^4}=0,\\ 2\frac{\partial C_1}{\partial\tilde{x}^1}-\frac{\partial C_2}{\partial\tilde{x}^4}=0,\quad \frac{\tilde{x}^1}{2\lambda}\frac{\partial C_5}{\partial\tilde{x}^4}-2\lambda\frac{\partial C_6}{\partial\tilde{x}^1}-\lambda\frac{\partial C_7}{\partial\tilde{x}^4}=0,\end{gathered}\tag{131}$$

and the transformation of the coordinates is defined by formulae (41).

Proposition 60. *The Maxwell space defined by the tensor F_{ij} is of the form*

$$
\begin{aligned}
&F_{12} = F_{14} = F_{24} = 0, \quad F_{23} = 2(x^2 + x^4)\Phi_1 + \left(x^1 + \frac{(x^2+x^4)^2}{\lambda} \right) \Phi_2, \\
&F_{13} = 2\lambda\Phi_1 + (x^2 + x^4)\Phi_2, \\
&F_{34} = -2(x^2 + x^4)\Phi_1 - \left(\lambda + x^1 + \frac{(x^2+x^4)^2}{\lambda} \right) \Phi_2,
\end{aligned}
$$

where

$$
\Phi(t_1,\, t_2) = \Phi\left(2\lambda x^1 + \left(x^2+x^4\right)^2,\ \lambda x^4 + x^1(x^2+x^4) + \frac{1}{3\lambda}\left(x^2+x^4\right)^3 \right),
$$

admits the two-dimensional group $G_S = G_{2,8}$ whenever the partials $\Phi_k = \partial\Phi/\partial t_k$ and $\Phi_{kl} = \partial^2\Phi/\partial t_k \partial t_l$ are linearly independent.

In particular, this condition holds for the function $\Phi(t_1,\, t_2) = t_1(t_1^2 + t_2^2)$.

7.1.4. *The class $C_{2,11}$.* The algebra $\mathcal{L}_{2,11} = L\{e_{12} - e_{14} + \lambda e_1 + \mu e_3,\, e_{23} + e_{34} - \mu e_1 + \lambda e_3\}$ $(\lambda = 0, \mu \neq 0 \sim \lambda \neq 0, \mu = 0)$ corresponds to the group $G_{2,11}$ generated by two one-dimensional subgroups of parabolic helices. *The Maxwell space of the class $C_{2,11}$ is defined by the tensor F_{ij} of the form* (36)*, where the functions $C_k = C_k(\tilde{x}^1,\, \tilde{x}^3,\, \tilde{x}^4)$ satisfy* (37)*,* (38b)*,* (38e)*,* (38g)*,* (40)*, and the equations*

$$\frac{(\tilde{x}^1)^2 - \mu^2}{\tilde{x}^1}\frac{\partial C_5}{\partial \tilde{x}^3} - \tilde{x}^1\tilde{x}^3\frac{\partial C_5}{\partial \tilde{x}^4} = 0, \tag{132a}$$

$$\frac{(\tilde{x}^1)^2 - \mu^2}{\tilde{x}^1}\frac{\partial C_6}{\partial \tilde{x}^3} - \tilde{x}^1\tilde{x}^3\frac{\partial C_6}{\partial \tilde{x}^4} - C_1 + \frac{\mu}{\tilde{x}^1}C_5 = 0, \tag{132b}$$

$$\frac{(\tilde{x}^1)^2 - \mu^2}{\tilde{x}^1}\frac{\partial C_7}{\partial \tilde{x}^3} - \tilde{x}^1\tilde{x}^3\frac{\partial C_7}{\partial \tilde{x}^4} + C_2 - \frac{\mu}{\tilde{x}^1}C_6 = 0, \tag{132c}$$

$$\frac{(\tilde{x}^1)^2 - \mu^2}{\tilde{x}^1}\frac{\partial C_1}{\partial \tilde{x}^3} - \tilde{x}^1\tilde{x}^3\frac{\partial C_1}{\partial \tilde{x}^4} = 0, \tag{132d}$$

$$\frac{(\tilde{x}^1)^2 - \mu^2}{\tilde{x}^1}\frac{\partial C_2}{\partial \tilde{x}^3} - \tilde{x}^1\tilde{x}^3\frac{\partial C_2}{\partial \tilde{x}^4} + C_5 + \frac{\mu}{\tilde{x}^1}C_1 = 0, \tag{132e}$$

$$\frac{(\tilde{x}^1)^2 - \mu^2}{\tilde{x}^1}\frac{\partial C_3}{\partial \tilde{x}^3} - \tilde{x}^1\tilde{x}^3\frac{\partial C_3}{\partial \tilde{x}^4} + C_6 + \frac{\mu}{\tilde{x}^1}C_2 = 0 \tag{132f}$$

(the transformation of coordinates is defined by formulae (39)*).*

Proposition 61. *The Maxwell space defined by the tensor F_{ij} of the form*

$$F_{12}=\frac{1}{\mu^2-(x^2+x^4)^2}\left\{\mu^3\left(\mu x^2-x^1x^3\right)+\left(x^2+x^4\right)\left[\mu^4-\mu^2\left(x^1\right)^2\right]+\right.$$
$$+\left(x^2+x^4\right)^2\left(4\mu\, x^1x^3-5\mu^2x^2\right)+$$
$$\left(x^2+x^4\right)^3\left[\frac{5}{2}\left(x^1\right)^2+\frac{3}{2}\left(x^3\right)^2-2\mu^2\right]+$$
$$\left.+4x^2\left(x^2+x^4\right)^4+\left(x^2+x^4\right)^5\right\},$$
$$F_{13}=-\left(x^2+x^4\right)\left(\mu\, x^1+x^3\left(x^2+x^4\right)\right),$$
$$F_{14}=F_{12}-\left(x^2+x^4\right)\left[\mu^2-\left(x^2+x^4\right)^2\right],$$
$$F_{24}=-\left(x^2+x^4\right)\left(\mu\, x^3+x^1\left(x^2+x^4\right)\right),$$
$$F_{34}=-F_{23}=\frac{1}{\mu^2-(x^2+x^4)^2}\cdot$$
$$\left\{\mu^3x^2\left(x^2+x^4\right)+\frac{\mu}{2}\left[\left(x^1\right)^2+\left(x^3\right)^2\right]\left(x^2+x^4\right)^2+\right.$$
$$\left.+\left(x^1x^3-\mu\, x^2\right)\left(x^2+x^4\right)^3\right\}.$$

admits the two-dimensional group $G_S=G_{2,11}$.

7.1.5. *The class $C_{2,11a}$.* The algebra $\mathcal{L}_{2,11a}=L\{e_{12}-e_{14},\, e_{23}+e_{34}\}$ ($\mathcal{L}_{2,11}$ for $\lambda=\mu=0$) corresponds to the group $G_{2,11a}$ generated by two one-dimensional subgroups of parabolic rotations. *The Maxwell space of the class $C_{2,11a}$ is defined by the tensor F_{ij} of the form (36), where the functions $C_k=C_k(\tilde x^1,\,\tilde x^3,\,\tilde x^4)$ satisfy (37), (38), and the following equations*

$$\begin{aligned}
&\frac{\partial C_5}{\partial\tilde x^3}-\tilde x^3\frac{\partial C_5}{\partial\tilde x^4}=0,\;\tilde x^1\left(\frac{\partial C_6}{\partial\tilde x^3}-\tilde x^3\frac{\partial C_6}{\partial\tilde x^4}\right)-C_1=0,\\
&\tilde x^1\left(\frac{\partial C_7}{\partial\tilde x^3}-\tilde x^3\frac{\partial C_7}{\partial\tilde x^4}\right)+C_2=0,\quad\frac{\partial C_1}{\partial\tilde x^3}-\tilde x^3\frac{\partial C_1}{\partial\tilde x^4}=0,\\
&\tilde x^1\left(\frac{\partial C_2}{\partial\tilde x^3}-\tilde x^3\frac{\partial C_2}{\partial\tilde x^4}\right)+C_5=0,\\
&\tilde x^1\left(\frac{\partial C_3}{\partial\tilde x^3}-\tilde x^3\frac{\partial C_3}{\partial\tilde x^4}\right)+C_6=0
\end{aligned}\tag{133}$$

(the transformation of coordinates is defined by formulae (35)).

Proposition 62. *Let*

$$\Phi=\Phi(t_1,\,t_2)=\Phi\left(x^2+x^4,\,\frac{1}{2}\left((x^1)^2+(x^3)^2\right)+x^2(x^2+x^4)\right),$$

$\Phi_k = \partial\Phi/\partial t_k$, $\Phi_{kl} = \partial^2\Phi/\partial t_k \partial t_l$. *Suppose the functions* Φ, Φ_1, Φ_2, *and* Φ_{22} *are linearly independent; then the Maxwell space defined by the tensor* F_{ij} *of the form*

$$F_{12} = \frac{1}{x^2+x^4}\Phi + \Phi_1 + \left(\frac{(x^1)^2}{x^2+x^4} + 2x^2 + x^4\right)\Phi_2,$$
$$F_{13} = x^3\Phi_2, \quad F_{24} = x^1\Phi_2,$$
$$F_{14} = \frac{1}{x^2+x^4}\Phi + \Phi_1 + \left(\frac{(x^1)^2}{x^2+x^4} + x^2\right)\Phi_2,$$
$$F_{23} = -F_{34} = -\frac{x^1x^3}{x^2+x^4}\Phi_2$$

admits the two-dimensional group $G_S = G_{2,11a}$.

7.1.6. *The class* $C_{2,12}$. The algebra $\mathcal{L}_{2,12} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_3\}$ corresponds to the group $G_{2,12}$ generated by parabolic rotations and by hyperbolic helices. *The Maxwell space of the class* $C_{2,12}$ *is defined by the tensor* F_{ij} *of the form* (36), *where the functions* $C_k = C_k(\tilde{x}^1, \tilde{x}^3, \tilde{x}^4)$ *satisfy* (37), (38) *and to the following equations*

$$\tilde{x}^1\frac{\partial C_1}{\partial \tilde{x}^1} + \lambda\frac{\partial C_1}{\partial \tilde{x}^3} + \left(\tilde{x}^1\right)^2\frac{\partial C_1}{\partial \tilde{x}^4} - C_1 = 0, \tag{134a}$$
$$\tilde{x}^1\frac{\partial C_2}{\partial \tilde{x}^1} + \lambda\frac{\partial C_2}{\partial \tilde{x}^3} + \left(\tilde{x}^1\right)^2\frac{\partial C_2}{\partial \tilde{x}^4} = 0, \tag{134b}$$
$$\tilde{x}^1\frac{\partial C_3}{\partial \tilde{x}^1} + \lambda\frac{\partial C_3}{\partial \tilde{x}^3} + \left(\tilde{x}^1\right)^2\frac{\partial C_3}{\partial \tilde{x}^4} - C_4 = 0, \tag{134c}$$
$$\tilde{x}^1\frac{\partial C_5}{\partial \tilde{x}^1} + \lambda\frac{\partial C_5}{\partial \tilde{x}^3} + \left(\tilde{x}^1\right)^2\frac{\partial C_5}{\partial \tilde{x}^4} - C_5 = 0, \tag{134d}$$
$$\tilde{x}^1\frac{\partial C_6}{\partial \tilde{x}^1} + \lambda\frac{\partial C_6}{\partial \tilde{x}^3} + \left(\tilde{x}^1\right)^2\frac{\partial C_6}{\partial \tilde{x}^4} = 0, \tag{134e}$$
$$\tilde{x}^1\frac{\partial C_7}{\partial \tilde{x}^1} + \lambda\frac{\partial C_7}{\partial \tilde{x}^3} + \left(\tilde{x}^1\right)^2\frac{\partial C_7}{\partial \tilde{x}^4} + C_8 = 0. \tag{134f}$$

(the transformation of coordinates is defined by formulae (35)).

7.1.7. *The class* $C_{2,12a}$. The algebra $\mathcal{L}_{2,12a} = L\{e_{12} - e_{14},\ e_{24}\}$ ($\mathcal{L}_{2,12}$ for $\lambda = 0$) corresponds to the group $G_{2,12a}$, generated by parabolic rotations and by pseudo-rotations *The Maxwell space of the class* $C_{2,12a}$ *is defined by the tensor* F_{ij} *of the form* (36), *where functions*

$$\begin{aligned}
&C_1 = \tilde{x}^1\Phi_1\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right), \quad C_2 = \Phi_2\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right),\\
&C_3 = -\frac{\tilde{x}^1}{2}\Phi_1\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right) + \frac{1}{\tilde{x}^1}\Phi_3\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right),\\
&C_5 = \tilde{x}^1\Phi_4\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right), \quad C_6 = \Phi_5\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right),\\
&C_7 = -\frac{\tilde{x}^1}{2}\Phi_4\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right) + \frac{1}{\tilde{x}^1}\Phi_6\left(\tilde{x}^3, \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2\right),\\
&C_4 = -C_1 - C_3, \quad C_8 = C_5 + C_7
\end{aligned} \tag{135}$$

satisfy (38) *(the transformation of coordinates is defined by formulae* (35)).

The potential of the form

$$A_i = \left(\Phi,\ -\frac{x^1}{x^2+x^4}\Phi,\ 0,\ -\frac{x^1}{x^2+x^4}\Phi\right),$$
$$\Phi = \Phi(t_1,t_2) = \Phi(x^3,\ (x^1)^2+(x^2)^2-(x^4)^2)$$

belongs to the class $P_{2,12a}$; it generates the following tensor F_{ij}:

$$\begin{aligned} F_{12} &= -\frac{1}{x^2+x^4}\Phi - \left(\frac{2(x^1)^2}{x^2+x^4} + 2(x^2)^2\right)\Phi_2, \\ F_{14} &= -\frac{1}{x^2+x^4}\Phi - \left(\frac{2(x^1)^2}{x^2+x^4} - 2(x^4)^2\right)\Phi_2, \\ F_{13} &= -\Phi_1, \quad F_{24} = -2x^1\Phi_2, \quad F_{23} = -F_{34} = \frac{x^1}{x^2+x^4}\Phi_1 \end{aligned} \tag{136}$$

($\Phi_k = \partial\Phi/\partial t_k$, $\Phi_{kl} = \partial^2\Phi/\partial t_k\partial t_l$).

Proposition 63. *The Maxwell space defined by the tensor* F_{ij} *of the form* (136) *admits the two-dimensional group* $G_S = G_{2,12a}$ *whenever the functions* $x^1\Phi_{11} - 2x^1x^3\Phi_{12}$, $F_{12} + x^2\Phi_{11} - 2x^2x^3\Phi_{12}$, *and* $2x^3x^4\Phi_{12} - x^4\Phi_{11}$ *are linearly independent.*

In particular, this condition holds for the function $\Phi = t_1t_2$.

7.2. **Three-dimensional subgroups.**

7.2.1. *The class* $C_{3,8}$. The algebra $\mathcal{L}_{3,8} = L\{e_{12} - e_{14} + \lambda e_2,\ e_3,\ e_2 - e_4\}$ corresponds to the group $G_{3,8}$ generated by parabolic helices and by translations along the vectors of the isotropic plane. *The Maxwell space of the class* $C_{3,8}$ *is defined by the tensor* F_{ij} *of the form* (36), *where*

$$C_1 = a_1, \quad C_3 = \frac{a_1}{2\lambda^2}\tilde{x}^1 + a_2, \quad C_6 = a_3 \quad (a_i = \text{const}), \tag{137}$$

and the functions $C_k = C_k(\tilde{x}^1)$ $(k = 2,\ 4,\ 5,\ 7,\ 8)$ *satisfy* (37) *(the coordinates* $\{\tilde{x}^i\}$ *and* $\{x^i\}$ *are related by* (41)).

Let $a_1 = A$, $a_2 = a_3 = 0$, $C_k = 0$ $(k = 2,\ 4,\ 5,\ 7,\ 8)$; taking into account the change (41), we obtain

$$\begin{aligned} &F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = \frac{A}{\lambda}(x^2+x^4), \\ &F_{23} = \frac{A}{\lambda^2}(\lambda^2 + \lambda x^1 + (x^2+x^4)^2), \quad F_{34} = -A - F_{23}. \end{aligned} \tag{138}$$

Proposition 64. *If* $A \neq 0$, *then the Maxwell space defined by the tensor* F_{ij} *of form* (138) *admits the three-dimensional group* $G_S = G_{3,8}$.

7.2.2. *The class* $C_{3,9}$. The algebra $\mathcal{L}_{3,9} = L\{e_{12}-e_{14}+\lambda e_2+\mu e_3,\ e_1,\ e_2-e_4\}$ corresponds to the group $G_{3,9}$ generated by parabolic rotations or parabolic helices with various axes and by translations along the vectors of an isotropic plane. We consider three cases.

a) $\lambda = \mu = 0$. *The Maxwell space of the class* $C_{3,9a}$ *is defined by the tensor* F_{ij} *of the form*

$$F_{12} = F_{14} = \Phi(x^2 + x^4), \quad F_{13} = F_{24} = 0, \quad F_{34} = -F_{23} = \Psi(x^2 + x^4,\ x^3). \tag{139}$$

where $\Phi(u)$ *Υ* $\Psi(u, v)$ *are arbitrary functions.*

Proposition 65. *If* $\Phi(u) \neq \text{const}$ *and* $\partial\Psi/\partial x^3 \neq 0$, *then the Maxwell space defined by the tensor* F_{ij} *of the form* (139) *admits the three-dimensional group* $G_S = G_{3,9a}$.

b) $\lambda = 0$, $\mu \neq 0$. *The Maxwell space of the class* $C_{3,9b}$ *is defined by the tensor* F_{ij} *of the form* (36), *where*

$$\begin{aligned}
{}_1 &= -\frac{a_1}{\mu}\tilde{x}^1 + a_2, \quad C_2 = \frac{\tilde{x}^3}{\mu}\left(-\frac{a_1}{\mu}\tilde{x}^1 + a_2\right) + \mu\Phi(\tilde{x}^1), \\
C_3 &= \frac{(\tilde{x}^3)^2}{2\mu^2}\left(-\frac{a_1}{\mu}\tilde{x}^1 + a_2\right) + \tilde{x}^3\Phi(\tilde{x}^1) + \Psi_1(\tilde{x}^1), \\
C_4 &= -C_1 - C_3, \quad C_5 = a_1, \quad C_6 = \frac{a_1}{\mu}\tilde{x}^3 - \mu^2\Phi'(\tilde{x}^1), \\
C_7 &= -\frac{a_1}{2\mu^2}(\tilde{x}^3)^2 + \mu\tilde{x}^3\Phi'(\tilde{x}^1) + \Psi_2(\tilde{x}^1), \quad C_8 = C_5 + C_7;
\end{aligned} \tag{140}$$

$a_1,\ a_2 = \text{const}$, *and* $\Phi(\tilde{x}^1)$, $\Psi_1(\tilde{x}^1)$, $\Psi_2(\tilde{x}^1)$ *are arbitrary functions (the coordinates* $\{\tilde{x}^i\}$ *and* $\{x^i\}$ *are related by* (39)).

Set $a_1 = a_2 = 0$ in (140) and let $\Psi_1(\tilde{x}^1) = \Psi_2(\tilde{x}^1) = 0$; taking into account change (39), we obtain

$$\begin{aligned}
&F_{12} = F_{14} = \mu x^3\Phi'(x^2 + x^4), \quad F_{13} = \mu\Phi(x^2 + x^4), \\
&F_{23} = -F_{34} = x^3\Phi(x^2 + x^4), \quad F_{24} = -\mu^2\Phi'(x^2 + x^4).
\end{aligned} \tag{141}$$

Proposition 66. *If the functions* $\Phi(\tilde{x}^1)$ *and* $\Phi'(\tilde{x}^1)$ *are linearly independent, then the Maxwell space defined by the tensor* F_{ij} *of the form* (140) *admits the three-dimensional group* $G_S = G_{3,9b}$.

c) $\lambda \neq 0$, $\mu = 0$. *The Maxwell space of the class* $C_{3,9c}$ *is defined by the tensor* F_{ij} *of the form* (36), *where* $C_2 = C_2(\tilde{x}^3)$, $C_3 = C_3(\tilde{x}^3)$, *and* $C_7 = C_7(\tilde{x}^3)$ *are arbitrary functions and*

$$\begin{aligned}
&C_1 = \lambda C_7'(\tilde{x}^3), \quad C_4 = -\lambda C_7'(\tilde{x}^3) - C_3(\tilde{x}^3), \quad C_5 = A, \\
&C_6 = B, \quad C_8 = C_7(\tilde{x}^3) + A \quad (A, B = \text{const})
\end{aligned} \tag{142}$$

(the transformation of coordinates is defined by formulae (41) *and the prime denotes differentiation with respect to* $\tilde{x}^3$.)

Let $C_2 = \Phi(\tilde{x}^3)$, $C_3 = C_7 = A = B = 0$; taking into account the change (41), we obtain

$$F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = \Phi(x^3), \quad F_{23} = -F_{34} = \frac{x^2 + x^4}{\lambda}\Phi(x^3). \tag{143}$$

Proposition 67. *If $\Phi'(x^3) \neq 0$, then the Maxwell space defined by the tensor F_{ij} of the form (142) admits the three-dimensional group $G_S = G_{3,9c}$.*

7.2.3. *The class $C_{3,10}$.* The algebra $\mathcal{L}_{3,10} = L\{e_{12} - e_{14} + \lambda e_2,\ e_1 + \mu e_3,\ e_2 - e_4\}$ ($\lambda \neq 0$, $\mu \neq 0$) is an extension of the algebra $\mathcal{L}_{2,7c} = L\{e_{12} - e_{14} + \lambda e_2,\ e_2 - e_4\}$ that corresponds to the class $C_{2,7c}$; hence $C_{3,10} \subset C_{2,7c}$. *The Maxwell space of the class $C_{3,10}$ is defined by the tensor F_{ij} of the form (36), where $C_k = C_k(\mu\tilde{x}^1 - 2\lambda\tilde{x}^3)$ $(k = 2, 3, 7)$ are arbitrary functions of one variable and the others are expressed from formulae*

$$\begin{aligned} &C_1 = 2\lambda^2 \left(\mu C_3' - C_7'\right), \quad C_4 = -C_1 - C_3, \quad C_5 = A + \mu C_1, \\ &C_6 = B, \quad C_8 = C_5 + C_7 \quad (A, B = \mathrm{const}) \end{aligned} \tag{144}$$

(the transformation of the coordinates is defined by formulae (41), the prime denotes differentiation).

Let $C_2 = \Phi(\mu\tilde{x}^1 - 2\lambda\tilde{x}^3)$, and $C_3 = C_7 = A = B = 0$; taking into account the change (41), we obtain

$$F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = \Phi(u), \quad F_{23} = -F_{34} = \frac{x^2 + x^4}{\lambda}\Phi(u), \tag{145}$$

where $u = \mu\tilde{x}^1 - 2\lambda\tilde{x}^3 = 2\lambda(\mu x^1 - x^3) + (x^2 + x^4)^2$.

Proposition 68. *If $\Phi'(u) \neq 0$, then the Maxwell space defined by the tensor F_{ij} of the form (144) admits the three-dimensional group $G_S = G_{3,10}$.*

7.2.4. *The class $C_{3,10a}$.* The algebra $\mathcal{L}_{3,10a} = L\{e_{12} - e_{14},\ e_1 + \mu e_3,\ e_2 - e_4\}$ ($\mathcal{L}_{3,10}$ for $\lambda = 0$ and $\mu \neq 0$) is an extension of the algebra $\mathcal{L}_{2,7a} = L\{e_{12} - e_{14},\ e_2 - e_4\}$ that corresponds to the class $C_{2,7a}$; thus $C_{3,10a} \subset C_{2,7a}$. *The Maxwell space of the class $C_{3,10a}$ is defined by the tensor F_{ij} of the form (36), where*

$$\begin{aligned} &C_1 = \frac{a_1}{\mu\tilde{x}^1} + a_2, \quad C_2 = \frac{\tilde{x}^3 C_1 - b_1(\tilde{x}^1)}{\mu\tilde{x}^1}, \quad C_3 = \frac{\tilde{x}^3 C_2}{\mu\tilde{x}^1} + b_2(\tilde{x}^1), \\ &C_4 = -C_1 - C_3, \quad C_5 = \frac{a_1}{\tilde{x}^1}, \quad C_6 = \frac{a_1\tilde{x}^3}{\mu\left(\tilde{x}^1\right)^2} b_1'(\tilde{x}^1), \\ &C_7 = -\frac{a_1\left(\tilde{x}^3\right)^2}{2\mu^2\left(\tilde{x}^1\right)^3} - \frac{\tilde{x}^3 b_1'(\tilde{x}^1)}{\mu\tilde{x}^1} + b_3(\tilde{x}^1), \quad C_8 = C_5 + C_7, \end{aligned} \tag{146}$$

$a_1, a_2 = \mathrm{const}$, $b_1(\tilde{x}^1)$, $b_2(\tilde{x}^1)$ and $b_3(\tilde{x}^1)$ are arbitrary functions (the transformation of coordinates is defined by formulae (35), prime denotes differentiation with respect to $\tilde{x}^1$).

Let in (146) $a_1 = 0$, $a_2 = A$, $b_1 = b_2 = b_3 = 0$; taking into account the change (35), we obtain

$$\begin{aligned} &F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = \frac{A}{\mu}\frac{x^3 - \mu x^1}{x^2 + x^4}, \\ &F_{23} = \frac{A}{2\mu^2}\frac{(x^3 - \mu x^1)^2}{(x^2 + x^4)^2}, \quad F_{34} = -F_{23} - A. \end{aligned} \tag{147}$$

Proposition 69. *If $A \neq 0$, then the Maxwell space defined by the tensor F_{ij} of the form (147) admits the three-dimensional group $G_S = G_{3,10a}$.*

7.2.5. *The class* $C_{3,15}$. The algebra $\mathcal{L}_{3,15} = L\{e_{12} - e_{14},\ e_{24},\ e_3\}$ is an extension of the algebra $\mathcal{L}_{2,12a}$, hence $C_{3,15} \subset C_{2,12a}$. *The Maxwell space of the class* $C_{3,15}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} &F_{12} = -\frac{C_5}{2}\left(\tilde{x}^2\right)^2 - C_6\tilde{x}^2 + C_7, \quad F_{13} = 0, \quad F_{24} = C_5\tilde{x}^2 + C_6, \\ &F_{14} = -\frac{C_5}{2}\left(\tilde{x}^2\right)^2 - C_6\tilde{x}^2 + C_8, \quad F_{23} = -F_{34} = \frac{A}{\tilde{x}^1} \quad (A = \mathrm{const}), \end{aligned} \tag{148}$$

where the functions $C_k = C_k(\tilde{x}^1,\ \tilde{x}^4)$ $(k = 5,\ \ldots,\ 8)$ *satisfy equation* (38d)*, the second equation in* (37)*, and the system*

$$\begin{aligned} &\tilde{x}^1\frac{\partial C_5}{\partial \tilde{x}^1} + \left(\tilde{x}^1\right)^2\frac{\partial C_5}{\partial \tilde{x}^4} - C_5 = 0, \\ &\tilde{x}^1\frac{\partial C_6}{\partial \tilde{x}^1} + \left(\tilde{x}^1\right)^2\frac{\partial C_6}{\partial \tilde{x}^4} = 0, \\ &\tilde{x}^1\frac{\partial C_7}{\partial \tilde{x}^1} + \left(\tilde{x}^1\right)^2\frac{\partial C_7}{\partial \tilde{x}^4} + C_8 = 0; \end{aligned} \tag{149}$$

the transformation of coordinates is defined by formulae (35).

Let the function Φ in (136) be independent of x^3; then we get instead (136) the tensor F_{ij} of the form

$$\begin{aligned} &F_{12} = -\frac{1}{x^2 + x^4}\Phi - \left(\frac{2(x^1)^2}{x^2 + x^4} + 2(x^2)^2\right)\Phi', \quad F_{13} = F_{23} = F_{34} = 0, \\ &F_{14} = -\frac{1}{x^2 + x^4}\Phi - \left(\frac{2(x^1)^2}{x^2 + x^4} - 2(x^4)^2\right)\Phi', \quad F_{24} = -2x^1\Phi', \end{aligned} \tag{150}$$

where $\Phi = \Phi(t) = \Phi((x^1)^2 + (x^2)^2 - (x^4)^2)$ is an arbitrary function.

Proposition 70. *If the functions* $\Phi(t)$ *and* $\Phi'(t)$ *are linearly independent, then the Maxwell space defined by the tensor* (150) *admits the three-dimensional group* $G_S = G_{3,15}$.

7.2.6. *The class* $C_{3,17}$. The algebra $\mathcal{L}_{3,17} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{24}\}$ is an extension of the algebra $\mathcal{L}_{2,12a}$, hence $C_{3,17} \subset C_{2,12a}$. *The Maxwell space of the class* $C_{3,17}$ *is defined by the tensor* F_{ij} *of the form* (36)*, where*

$$\begin{aligned} &C_1 = \frac{Ay_1}{y_4^2}, \quad C_3 = -\frac{A\left(y_1^2 + y_3^2\right)}{2y_1y_4^2} - B\frac{y_3}{y_1}, \quad C_4 = -C_1 - C_3, \\ &C_6 = \frac{Ay_3}{y_4^2} + B, \quad C_2 = C_5 = C_7 = C_8 = 0 \quad (A,\ B = \mathrm{const}), \end{aligned} \tag{151}$$

and

$$y_1 = \tilde{x}^1, \quad y_3 = \tilde{x}^3, \quad y_4 = \tilde{x}^4 - \frac{1}{2}\left(\tilde{x}^1\right)^2 + \frac{1}{2}\left(\tilde{x}^3\right)^2;$$

the transformation of coordinates is defined by formulae (35).

The potential of the form

$$\begin{aligned} &A_i = \left(\Phi,\ -\frac{x^1}{x^2 + x^4}\Phi,\ 0,\ -\frac{x^1}{x^2 + x^4}\Phi\right), \\ &\Phi = \Phi(t) = \Phi((x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2) \end{aligned}$$

belongs to the class $P_{3,17}$; it generates the following tensor F_{ij}:

$$\begin{aligned} F_{12} &= -\frac{1}{x^2+x^4}\Phi - \left(\frac{2(x^1)^2}{x^2+x^4} + 2x^2\right)\Phi', \\ F_{14} &= -\frac{1}{x^2+x^4}\Phi - \left(\frac{2(x^1)^2}{x^2+x^4} - 2x^4\right)\Phi', \\ F_{13} &= -2x^3\Phi', \quad F_{24} = -2x^1\Phi', \quad F_{23} = -F_{34} = \frac{2x^1x^3}{x^2+x^4}\Phi'. \end{aligned} \tag{152}$$

Proposition 71. *If $\Phi''(t) \neq 0$, then the Maxwell space defined by the tensor* (152) *admits the three-dimensional group $G_S = G_{3,17}$.*

7.2.7. *The class $C_{3,18}$.* The algebra $\mathcal{L}_{3,18} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda(e_2 - e_4)\}$ is an extension of the algebra $\mathcal{L}_{2,11a}$, thus $C_{3,18} \subset C_{2,11a}$. *The Maxwell space of the class $C_{3,18}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} F_{12} &= F_{14} = \frac{x^1}{x^2+x^4}\Phi(x^2+x^4), \quad F_{24} = \Phi(x^2+x^4), \\ F_{23} &= -F_{34} = -\frac{x^3}{x^2+x^4}\Phi(x^2+x^4), \quad F_{13} = 0, \end{aligned} \tag{153}$$

where $\Phi(t)$ is an arbitrary function.

Proposition 72. *If $\Phi'(t) \neq 0$, then the Maxwell space defined by the tensor* (153) *admits the three-dimensional group $G_S = G_{3,18}$.*

7.3. **Four- and five-dimensional subgroups.**

7.3.1. *The class $C_{4,9}$.* The algebra $\mathcal{L}_{4,9} = L\{e_{12}-e_{14}+\lambda e_2,\ e_1,\ e_3,\ e_2-e_4\}$ is an extension of the algebra $\mathcal{L}_{3,8}$, hence $C_{4,9} \subset C_{3,8}$. *The Maxwell space of the class $C_{4,9}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} F_{12} &= -\frac{b_3}{2}\left(\tilde{x}^2\right)^2 - b_4\tilde{x}^2 + b_5, \quad F_{13} = b_1, \quad F_{14} = F_{12} + b_3, \\ F_{23} &= -F_{34} = b_1\tilde{x}^2 + b_2, \quad F_{24} = b_3\tilde{x}^2 + b_4 \quad (b_k = \text{const}), \end{aligned} \tag{154}$$

where $\tilde{x}^2 = (x^2 + x^4)/\lambda$.

Let us take in (154) $b_3 = B = \text{const},\ b_1 = b_2 = b_4 = b_5 = 0$:

$$\begin{aligned} F_{12} &= -\frac{B}{2\lambda^2}\left(x^2+x^4\right)^2, \quad F_{13} = F_{23} = F_{34} = 0, \\ F_{14} &= F_{12} + B, \quad F_{24} = \frac{B}{\lambda}(x^2+x^4). \end{aligned} \tag{155}$$

Proposition 73. *If $B \neq 0$, then the Maxwell space defined by the tensor* (155) *admits the four-dimensional group $G_S = G_{4,9}$.*

7.3.2. *The class $C_{4,9a}$.* The algebra $\mathcal{L}_{4,9a} = L\{e_{12} - e_{14},\ e_1,\ e_3,\ e_2 - e_4\}$ is an extension of the algebra $\mathcal{L}_{2,7}$ (for $\lambda = \mu = 0$), thus $C_{4,9a} \subset C_{2,7a}$. *The Maxwell space of the class $C_{4,9a}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = \Phi(x^2+x^4), \quad F_{13} = F_{24} = 0, \quad F_{23} = -F_{34} = \Psi(x^2+x^4), \tag{156}$$

where $\Phi(u)$ and $\Psi(u)$ are arbitrary functions.

Proposition 74. *If the functions* Φ, Ψ *and* $(x^2+x^4)\Phi'$ *are linearly independent, then the Maxwell space defined by the tensor* (156) *admits the five-dimensional group* G_S *that corresponds to the algebra*

$$L\{e_{12}-e_{14},\ e_{23}+e_{34},\ e_1,\ e_3,\ e_2-e_4\}. \tag{157}$$

There are no Maxwell's spaces with the symmetry group $G_S = G_{4,9a}$.

7.3.3. *The classes* $C_{4,12a}$ *and* $C_{4,12b}$. The algebra

$$\mathcal{L}_{4,12} = L\{e_{12}-e_{14}+\mu e_3,\ e_{23}+e_{34}+\nu e_2,\ e_1,\ e_2-e_4\}$$

is an extension of the algebra $\mathcal{L}_{3,9}$ (for $\lambda = 0$). Therefore, the corresponding classes (for various μ and ν) are contained in $C_{3,9a}$ or $C_{3,9b}$.

a) $\mu \neq 0$, $\nu = 0$. *The Maxwell space of the class* $C_{4,12a}$ *that corresponds to the algebra* $\mathcal{L}_{4,12a} = L\{e_{12}-e_{14}+\mu e_3,\ e_{23}+e_{34},\ e_1,\ e_2-e_4\}$ *(*$\mathcal{L}_{4,12}$ *for* $\nu = 0$*) is defined by the tensor* F_{ij} *of the form* (156).

Proposition 75. *There are no Maxwell's spaces with the symmetry group* $G_S = G_{4,12a}$.

b) $\mu = 0$, $\nu \neq 0$. *The Maxwell space of the class* $C_{4,12b}$ *that corresponds to the algebra* $\mathcal{L}_{4,12b} = L\{e_{12}-e_{14},\ e_{23}+e_{34}+\nu e_2,\ e_1,\ e_2-e_4\}$ *(*$\mathcal{L}_{4,12}$ *for* $\mu = 0$*) is defined by the tensor* F_{ij} *of the form.*

$$\begin{aligned} &F_{12} = F_{14} = A = \text{const}, \quad F_{13} = F_{24} = 0,\\ &F_{34} = -F_{23} = \Psi\left(x^3 - \frac{1}{2\nu}\left(x^2+x^4\right)^2\right). \end{aligned} \tag{158}$$

where $\Psi(t)$ *is an arbitrary function.*

Proposition 76. *If* $A \neq 0$ *and* $\Psi'(t) \neq 0$, *then the Maxwell space defined by the tensor* (158) *admits the four-dimensional group* $G_S = G_{4,12b}$.

c) $\mu \neq 0$ and $\nu \neq 0$. In this case, the algebra is $\mathcal{L}_{4,12}$ and the class of the Maxwell spaces $C_{6,3}$ is defined by (72).

Proposition 77. *There are no Maxwell's spaces with the symmetry group* $G_S = G_{4,12c}$ *(*$G_{4,12}$ *for* $\mu \neq 0$ *and* $\nu \neq 0$*).*

7.3.4. *The class* $C_{4,20}$. The algebra $\mathcal{L}_{4,20} = L\{e_{12}-e_{14},\ e_{23}+e_{34},\ e_{13},\ e_{24}\}$ is an extension of the algebra $\mathcal{L}_{3,17}$, thus $C_{4,20} \subset C_{3,17}$. *The Maxwell space of the class* $C_{4,20}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} &F_{12} = F_{14} = \frac{Bx^1}{x^2+x^4}, \quad F_{23} = -F_{34} = -\frac{Bx^3}{x^2+x^4},\\ &F_{13} = 0, \quad F_{24} = B \quad (B = \text{const} \neq 0). \end{aligned} \tag{159}$$

Proposition 78. *The Maxwell space defined by the tensor* F_{ij} *of the form* (159) *admits the five-dimensional group* G_S *that corresponds to the algebra*

$$\mathcal{L}_{5,9} = L\{e_{12}-e_{14},\ e_{23}+e_{34},\ e_{13},\ e_{24},\ e_2-e_4\}.$$

There are no Maxwell's spaces with the symmetry group $G_S = G_{4,20}$.

7.3.5. *The class* $C_{5,5}$. The algebra $\mathcal{L}_{5,5} = L\{e_{12} - e_{14},\ e_{23} + e_{34} + \lambda e_2,\ e_2 - e_4,\ e_1,\ e_3\}$ is an extension of the algebra $\mathcal{L}_{4,9a}$, thence $C_{5,5} \subset C_{4,9a}$.

Proposition 79. *For $\lambda = 0$ the class of the Maxwell spaces $C_{5,5}$ coincides with $C_{4,9a}$ and is defined by (156); for $\lambda \neq 0$ it coincides with $C_{6,3}$ and is defined by (72).*

8. The classes of the Maxwell spaces that admit proportional bi-rotations

In this section, we describe classes of the Maxwell spaces that admit proportional bi-rotations and were not included in previous sections. These classes were found by E. G. Morokhova [19]. We use the change defined in Eq. (43).

8.1. **Two-dimensional subgroups.**

8.1.1. *The class* $C_{2,9}$. The algebra $\mathcal{L}_{2,9} = L\{e_{13} + \lambda e_{24},\ e_2 - e_4\}$ corresponds to the group $G_{2,9}$ generated by proportional bi-rotations and by translations along an isotropic straight line. As $\mathcal{L}_{1,5} \subset \mathcal{L}_{2,9}$, then the class $C_{2,9}$ is a subclass of the class $C_{1,5}$. *The Maxwell space of the class $C_{2,9}$ is defined by the tensor F_{ij} of the form (44) – (45) if the following equations*

$$
\begin{aligned}
&\frac{\partial F_{13}}{\partial \rho} - \frac{1}{\lambda\rho}\frac{\partial F_{13}}{\partial \theta} = 0, \quad \frac{\partial}{\partial \rho}(c_1 + c_2) = \frac{\partial}{\partial \rho}(c_3 + c_4) = 0,\\
&\frac{\partial c_1}{\partial \rho} - \frac{1}{\lambda\rho}\frac{\partial c_1}{\partial \theta} - \frac{c_2}{\rho} - \frac{c_3}{\lambda\rho} = 0, \quad \frac{\partial c_2}{\partial \rho} - \frac{1}{\lambda\rho}\frac{\partial c_2}{\partial \theta} - \frac{c_1}{\rho} - \frac{c_4}{\lambda\rho} = 0,\\
&\frac{\partial c_3}{\partial \rho} + \frac{1}{\lambda\rho}\frac{\partial c_3}{\partial \theta} + \frac{c_4}{\rho} - \frac{c_1}{\lambda\rho} = 0, \quad \frac{\partial c_4}{\partial \rho} + \frac{1}{\lambda\rho}\frac{\partial c_4}{\partial \theta} + \frac{c_3}{\rho} - \frac{c_2}{\lambda\rho} = 0.
\end{aligned}
\tag{160}
$$

are satisfied.

We may define the part of the class $P_{2,9}$ of potentials A_i as follows,

$$A_1 = A_3 = 0, \quad A_2 = -A_4 = \rho\Phi(r, \lambda\theta + \ln\rho), \tag{161}$$

where Φ is an arbitrary function of two variables. For the case $\Phi = \Phi(r)$ we obtain the tensor F_{ij} of the form

$$
\begin{aligned}
&F_{12} = -F_{14} = \rho e^{\lambda\varphi}\Phi'(r)\cos(\theta - \varphi), \quad F_{13} = 0,\\
&F_{23} = F_{34} = -\rho e^{\lambda\varphi}\Phi'(r)\sin(\theta - \varphi), \quad F_{24} = 2\Phi(r).
\end{aligned}
\tag{162}
$$

Proposition 80. *If $\Phi'(r) \neq 0$, then the Maxwell space defined by the tensor (162) admits the two-dimensional group $G_S = G_{2,9}$.*

8.1.2. *The class* $C_{2,10}$. The algebra $\mathcal{L}_{2,10} = L\{e_{13},\ e_{24}\} = L\{e_{13} + \lambda e_{24},\ e_{24}\}$ corresponds to the group $G_{2,10}$ generated by proportional bi-rotations and pseudo-rotations. As $\mathcal{L}_{1,5} \subset \mathcal{L}_{2,10}$, then $C_{2,10} \subset C_{1,5}$. *The Maxwell space of the class $C_{2,10}$ is defined by the tensor F_{ij} of the form (44) and*

$$F_{13} = F_{13}(\rho, r), \quad F_{24} = F_{24}(\rho, r), \tag{163}$$

where

$$
\begin{aligned}
&c_1 = -k_3\sin\theta + k_4\cos\theta, \quad c_2 = -k_1\sin\theta + k_2\cos\theta,\\
&c_3 = k_3\cos\theta + k_4\sin\theta, \qquad c_4 = k_1\cos\theta + k_2\sin\theta,
\end{aligned}
\tag{164}
$$

and the functions $F_{13}(\rho, r)$, $F_{24}(\rho, r)$, *and* $k_i = k_i(\rho, r)$ *satisfy the equations*

$$\frac{\partial F_{24}}{\partial r} + \frac{\partial k_1}{\partial \rho} + \frac{k_1}{\rho} = 0, \quad \frac{\partial F_{13}}{\partial \rho} - \frac{\partial k_4}{\partial r} - \frac{k_4}{r} = 0, \quad k_2 = \frac{K}{r\rho} \quad (K = \mathrm{const}). \tag{165}$$

The part of the class $P_{2,10}$ of potentials that admits the group $G_{2,10}$ is defined as follows:

$$A_i = (0,\, \Phi(r, \rho)e^{\lambda\varphi},\, 0,\, -\Phi(r, \rho)e^{\lambda\varphi}),$$

where $\Phi = \Phi(r, \rho)$ is an arbitrary function. This potential generates the tensor F_{ij} such that

$$\begin{aligned} &F_{12} = -F_{14} = e^{\lambda\varphi}\Phi'_r(r, \rho)\cos(\theta - \varphi), \quad F_{13} = 0, \\ &F_{23} = F_{34} = -e^{\lambda\varphi}\Phi'_r(r, \rho)\sin(\theta - \varphi), \quad F_{24} = -\left(\Phi'_\rho(r, \rho) + \frac{1}{\rho}\Phi(r, \rho)\right). \end{aligned} \tag{166}$$

Proposition 81. *If the conditions*

$$\Phi''_{r\rho} + \frac{1}{\rho}\Phi'_r \neq 0, \quad \Phi''_{\rho\rho} + \frac{1}{\rho}\Phi'_\rho - \frac{1}{\rho^2}\Phi \neq 0,$$

are satisfied, then the Maxwell space defined by the tensor in (166) *admits the two-dimensional group* $G_S = G_{2,10}$.

8.2. **Three-dimensional subgroups.**

8.2.1. *The class* $C_{3,11}$. The algebra $\mathcal{L}_{3,11} = L\{e_{13} + \lambda e_{24},\, e_1,\, e_3\}$ corresponds to the group $G_{3,11}$ generated by proportional bi-rotations and by translations along the vectors of two-dimensional Euclidian plane. *The Maxwell space of the class* $C_{3,11}$ *is defined by the tensor* F_{ij} *of the form* (44) *and*

$$F_{13} = \mathrm{const}, \quad F_{24} = F_{24}(\rho), \tag{167}$$

where $F_{24}(\rho)$, $c_2 = c_2(\rho)$, *and* $c_4 = c_4(\rho)$ *are arbitrary functions and*

$$c_1 = \lambda(c_4 + \rho c'_4), \quad c_3 = -\lambda(c_2 + \rho c'_2). \tag{168}$$

Let us take in (167), (168), and (44) $c_2 = \Phi(\rho)$, $c_4 = 0$, $F_{24} = \Psi(\rho)$; we get an example of the tensor F_{ij} that defines the Maxwell space of the class $C_{3,11}$:

$$\begin{aligned} &F_{12} = -\Phi(\rho)\sinh\lambda\varphi\, \sin\varphi - \lambda(\Phi(\rho) + \rho\Phi'(\rho))\cosh\lambda\varphi\, \cos\varphi, \\ &F_{14} = \Phi(\rho)\cosh\lambda\varphi\, \sin\varphi + \lambda(\Phi(\rho) + \rho\Phi'(\rho))\sinh\lambda\varphi\, \cos\varphi, \\ &F_{23} = \Phi(\rho)\sinh\lambda\varphi\, \cos\varphi - \lambda(\Phi(\rho) + \rho\Phi'(\rho))\cosh\lambda\varphi\, \sin\varphi, \\ &F_{34} = \Phi(\rho)\cosh\lambda\varphi\, \cos\varphi - \lambda(\Phi(\rho) + \rho\Phi'(\rho))\sinh\lambda\varphi\, \sin\varphi, \\ &F_{13} = \mathrm{const}, \quad F_{24} = \Psi(\rho). \end{aligned} \tag{169}$$

Proposition 82. *If* $\Phi(\rho) \neq \mathrm{const}$ *and* $\Psi(\rho) \neq \mathrm{const}$, *then the Maxwell space defined by the tensor* (169) *admits the three-dimensional group* $G_S = G_{3,11}$.

8.2.2. *The class* $C_{3,12}$. The algebra $\mathcal{L}_{3,12} = L\{e_{13} + \lambda e_{24}, e_2, e_4\}$ corresponds to the group $G_{3,12}$ generated by proportional bi-rotations and by translations along the vectors of two-dimensional pseudo-Euclidian plane. *The Maxwell space of the class* $C_{3,12}$ *is defined by the tensor* F_{ij} *of the form* (44) *and*

$$F_{13} = F_{13}(r), \quad F_{24} = \text{const}, \tag{170}$$

where

$$\begin{aligned} c_1 &= \sin\theta(t_1\cosh\lambda\theta + t_2\sinh\lambda\theta) - \cos\theta(t_3\cosh\lambda\theta + t_4\sinh\lambda\theta),\\ c_2 &= -\sin\theta(t_1\sinh\lambda\theta + t_2\cosh\lambda\theta) + \cos\theta(t_3\sinh\lambda\theta + t_4\cosh\lambda\theta),\\ c_3 &= -\cos\theta(t_1\cosh\lambda\theta + t_2\sinh\lambda\theta) - \sin\theta(t_3\cosh\lambda\theta + t_4\sinh\lambda\theta),\\ c_4 &= \cos\theta(t_1\sinh\lambda\theta + t_2\cosh\lambda\theta) + \sin\theta(t_3\sinh\lambda\theta + t_4\cosh\lambda\theta), \end{aligned} \tag{171}$$

moreover, the functions $F_{13}(r)$, $t_3 = t_3(r)$, *and* $t_4 = t_4(r)$ *are arbitrary and*

$$t_1 = -\frac{1}{\lambda}(t_4 + rt_4'), \quad t_2 = -\frac{1}{\lambda}(t_3 + rt_3'). \tag{172}$$

Proposition 83. *If* $F_{13} \neq \text{const}$ *and* $t_3 \neq \text{const}$ *(or* $t_4 \neq \text{const}$*), then the Maxwell space defined by the tensor* F_{ij} *according to formulae* (44), (170), (171), *and* (172) *admits the three-dimensional group* $G_S = G_{3,12}$.

8.2.3. *The class* $C_{3,13}$. The algebra $\mathcal{L}_{3,13} = L\{e_{13}, e_{24}, e_2 - e_4\}$ is an extension of the algebra $\mathcal{L}_{2,10}$, thus $C_{3,13} \subset C_{2,10}$. *The Maxwell space of the class* $C_{3,13}$ *is defined by the tensor* F_{ij} *of the form* (44), (164) *and*

$$F_{13} = \Phi_1(r), \quad F_{24} = \Phi_2(r), \tag{173}$$

where

$$k_1 = -\frac{1}{2}\Phi_2'(r)\rho^2 + \Phi_3(r), \quad k_2 = -k_4 = \frac{K}{r\rho}, \quad k_3 = -\frac{1}{2}\Phi_2'(r)\rho^2 - \Phi_3(r) \tag{174}$$

($K = \text{const}$, $\Phi_1(r)$, $\Phi_2(r)$, $\Phi_3(r)$ *are arbitrary functions).*

8.3. Four-dimensional subgroups.

8.3.1. *The class* $C_{4,7}$. The algebra $\mathcal{L}_{4,7} = L\{e_{13} + \lambda e_{24}, e_1, e_3, e_2 + e_4\}$ is an extension of the algebra $\mathcal{L}_{3,11}$, hence $C_{4,7} \subset C_{3,11}$. *The Maxwell space of the class* $C_{4,7}$ *is defined by the tensor* F_{ij} *of the form*

$$\begin{aligned} -F_{12} = F_{14} &= (c_1\sin\varphi - c_2\cos\varphi)\, e^{\lambda\varphi}, \quad F_{13} = B = \text{const},\\ F_{23} = F_{34} &= (c_1\cos\varphi + c_2\sin\varphi)\, e^{\lambda\varphi}, \quad F_{24} = E = \text{const}, \end{aligned} \tag{175}$$

where $c_1 = c_1(\rho)$, $c_2 = c_2(\rho)$ *are arbitrary functions).*

Proposition 84. *Suppose two following conditions are satisfied:*
1) $c_1 \neq \text{const}$ *or* $c_2 \neq \text{const}$,
2) $B \neq 0$ *or* $E \neq 0$.
Then the Maxwell space defined by the tensor F_{ij} *of the form* (175) *admits the four-dimensional group* $G_S = G_{4,7}$.

8.3.2. *The class $C_{4,10}$.* The algebra $\mathcal{L}_{4,10} = L\{e_{13},\, e_{24},\, e_1,\, e_3,\}$ contains $\mathcal{L}_{2,10}$ as a subalgebra, therefore the class $C_{4,10}$ is a subclass of $C_{2,10}$. *The Maxwell space of the class $C_{4,10}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0, \quad F_{13} = B = \text{const}, \quad F_{24} = F_{24}(\rho) \tag{176}$$

($F_{24}(\rho)$ is an arbitrary function).

Proposition 85. *If $B \neq 0$ and $F_{24}(\rho) \neq \text{const}$, then the Maxwell space defined by the tensor F_{ij} of the form (176) admits the four-dimensional group $G_S = G_{4,10}$.*

8.3.3. *The class $C_{4,11}$.* The algebra $\mathcal{L}_{4,11} = L\{e_{13},\, e_{24},\, e_2,\, e_4\}$ is also an extension of the algebra $\mathcal{L}_{2,10}$, hence $C_{4,11} \subset C_{2,10}$. *The Maxwell space of the class $C_{4,11}$ is defined by the tensor F_{ij} of the form*

$$F_{12} = F_{14} = F_{23} = F_{34} = 0, \quad F_{13} = F_{13}(r), \quad F_{24} = E = \text{const}. \tag{177}$$

($F_{13}(r)$ is an arbitrary function).

Proposition 86. *If $E \neq 0$ and $F_{13}(r) \neq \text{const}$, then the Maxwell space, defined by the tensor F_{ij} of the form (177) admits the four-dimensional group $G_S = G_{4,11}$.*

9. Other classes

In this section, we describe classes of Maxwell spaces which were not described in the previous sections.

9.0.4. *The classes $C_{3,14a}$, $C_{3,14b}$ and $C_{3,14c}$.* The algebra

$$\mathcal{L}_{3,14} = L\{e_{12} - e_{14} + \lambda e_1 + \mu e_3,\, e_{23} + e_{34} + \nu e_1 + \lambda e_3,\, e_2 - e_4\}$$

corresponds to the group $G_{3,14}$ generated by two one-dimensional subgroups of parabolic helices and by translations along an isotropic straight line. We describe the corresponding classes in three particular cases.

a) $\lambda = \mu = \nu = 0$. *The Maxwell space of the class $C_{3,14a}$ that corresponds to the algebra $\mathcal{L}_{3,14a} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_2 - e_4\}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned}
F_{13} &= \frac{K_1 x^1 + K_2 x^3 + K_3}{(x^2 + x^4)^2}, \quad F_{24} = \frac{K_2 x^1 - K_1 x^3}{(x^2 + x^4)^2} + \Phi_1(x^2 + x^4),\\
F_{14} &= \frac{K_2\left((x^1)^2 - (x^3)^2\right) - 2K_1 x^1 x^3 - 2K_3 x^3}{2\,(x^2 + x^4)^3} + \frac{x^1 \Phi_1(x^2 + x^4)}{x^2 + x^4} +\\
&\quad + \Phi_2(x^2 + x^4), \quad F_{12} = F_{14} + \frac{K_2}{x^2 + x^4},\\
F_{34} &= \frac{K_1\left((x^1)^2 - (x^3)^2\right) + 2K_2 x^1 x^3 + 2K_3 x^1}{2\,(x^2 + x^4)^3} + \frac{x^3 \Phi_1(x^2 + x^4)}{x^2 + x^4} +\\
&\quad + \Phi_3(x^2 + x^4), \quad F_{23} = -F_{34} + \frac{K_1}{x^2 + x^4},
\end{aligned} \tag{178}$$

where K_1, K_2, $K_3 = \text{const}$, and $\Phi_1(t)$, $\Phi_2(t)$, and $\Phi_3(t)$ are arbitrary functions of one variable.

b) $\lambda \neq 0$ and $\mu = \nu = 0$. *The Maxwell space of the class $C_{3,14b}$ that corresponds to the algebra $\mathcal{L}_{3,14b} = L\{e_{12} - e_{14} + \lambda e_1,\, e_{23} + e_{34} + \lambda e_3,\, e_2 - e_4\}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned}
&F_{12} = F_{14} + \frac{K_2}{x^2 + x^4 - \lambda}, \quad F_{13} = \frac{K_1 x^1 + K_2 x^3 + K_3}{\left(x^2 + x^4\right)^2 - \lambda^2},\\
&F_{14} = \frac{K_2\left(x^1\right)^2}{2\left(x^2 + x^4 - \lambda\right)^3} - \frac{2K_1 x^1 x^3 + K_2\left(x^3\right)^2 + 2K_3 x^3}{2\left(x^2 + x^4 + \lambda\right)^2\left(x^2 + x^4 - \lambda\right)} +\\
&\quad + \frac{x^1 \Phi_1(x^2 + x^4)}{x^2 + x^4 - \lambda} + \Phi_2(x^2 + x^4),\\
&F_{24} = \frac{K_2 x^1}{(x^2 + x^4 - \lambda)^2} - \frac{K_1 x^3}{\left(x^2 + x^4 + \lambda\right)^2} + \Phi_1(x^2 + x^4),\\
&F_{34} = -\frac{K_1\left(x^3\right)^2}{2\left(x^2 + x^4 + \lambda\right)^3} + \frac{K_1\left(x^1\right)^2 + 2K_2 x^1 x^3 + 2K_3 x^1}{2\left(x^2 + x^4 - \lambda\right)^2\left(x^2 + x^4 + \lambda\right)} +\\
&\quad + \frac{x^3 \Phi_1(x^2 + x^4)}{x^2 + x^4 + \lambda} + \Phi_3(x^2 + x^4), \ F_{23} = -F_{34} + \frac{K_1}{x^2 + x^4 + \lambda}.
\end{aligned} \tag{179}$$

Formulae (179) transforms to (178) if $\lambda = 0$.

c) $\lambda = 0$ and $\mu = \nu \neq 0$. *The Maxwell space of the class $C_{3,14c}$ that corresponds to the algebra $\mathcal{L}_{3,14c} = L\{e_{12} - e_{14} + \mu e_3,\, e_{23} + e_{34} + \mu e_1,\, e_2 - e_4\}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned}
&F_{12} = F_{14} + \mu\Psi_1(u) + u\Psi_2(u), \quad F_{13} = x^1\Psi_1(u) + x^3\Psi_2(u) + C_1(u),\\
&F_{23} = -F_{34} + u\Psi_1(u) - \mu\Psi_2(u), \quad F_{24} = x^1\Psi_2(u) - x^3\Psi_1(u) + C_2(u),\\
&F_{14} = F_{14}(x^1, x^3, u), \quad F_{34} = F_{34}(x^1, x^3, u) \ \ (u = x^2 + x^4),
\end{aligned} \tag{180}$$

where

$$\begin{aligned}
&\Psi_1(u) = \frac{(K_1 + 2K_2\mu)\left(u^2 - \mu^2\right) - 2K_3\mu u}{\left(u^2 + \mu^2\right)^2},\\
&\Psi_2(u) = -\frac{2K_1\mu u + 4K_2\mu^2 u + K_3\left(u^2 - \mu^2\right)}{\left(u^2 + \mu^2\right)^2},\\
&C_1(u) = \frac{K_4 - 2\mu\Phi(u)}{u^2 + \mu^2}, \quad C_2(u) = \Phi'(u)
\end{aligned} \tag{181}$$

($K_1, \ldots, K_4 =$ const, $\Phi(u)$ is an arbitrary function), and the components F_{14}, F_{34} satisfy the equations

$$\begin{aligned}
&\partial_1 F_{14} = \partial_3 F_{34} = -\frac{\mu F_{13} - u F_{24}}{u^2 + \mu^2},\\
&\partial_3 F_{14} = -\partial_1 F_{34} = \frac{u F_{13} + \mu F_{24}}{u^2 + \mu^2}.
\end{aligned} \tag{182}$$

9.0.5. *The class $C_{3,20}$.* The algebra $\mathcal{L}_{3,20} = L\{e_{12},\, e_{13},\, e_{23}\}$ corresponds to the three-dimensional group $G_{3,20}$ of rotations around the origin in the subspace of the Minkowski space $\mathbb{R}_0^3 = \{x \in \mathbb{R}_1^4 : x^4 = 0\}$. *The Maxwell space of the class $C_{3,20}$ is defined by the tensor F_{ij} of the form*

$$\begin{aligned} &F_{12} = -Ax^3/r^3,\ F_{13} = Ax^2/r^3,\ F_{23} = -Ax^1/r^3\ (A = \text{const}),\\ &F_{14} = -x^1\Phi(r,\, x^4),\quad F_{24} = -x^2\Phi(r,\, x^4),\quad F_{34} = -x^1\Phi(r,\, x^4). \end{aligned} \tag{183}$$

9.0.6. *The class $C_{4,16}$.* For the algebra

$$\mathcal{L}_{4,16} = L\{e_{12} - e_{14} + \lambda e_3,\, e_{23} + e_{34} + \lambda e_1,\, e_{13},\, e_2 - e_4\}$$

we obtain the empty class $C_{4,16}$.

9.0.7. *The class $C_{6,1}$.* The algebra $\mathcal{L}_{6,1} = L\{e_{12}, e_{13}, e_{23}, e_{14}, e_{24}, e_{34}\}$ corresponds to the Lorentz group $G_{6,1}$. *The class of Maxwell spaces that corresponds to the Lorentz group is empty.*

Acknowledgements. The author thanks A. V. Bolsinov and A. T. Fomenko for their constructive remarks, which helped to improve the formulation of the initial problem.

References

[1] *L. D. Landau and E. M. Lifshits,* Field Theory [in Russian], Moscow: Nauka, 1967.

[2] *A. S. Ivanova and M. A. Parinov,* First Integrals of Lorentz Equations for Some Classes of Electromagnetic Fields, *Proc. of the Steklov Inst. of Math.*, Vol. 236, pp. 186–192 (2002).

[3] *Janner A., Ascher E.,* Space-time symmetry of linearly polarized electromagnetic plane waves, *Lettere nuovo Cimento*, Vol. 2, n. 15, pp. 703–705 (1969).

[4] *Janner A., Ascher E.,* Space-time symmetry of transverse electromagnetic plane waves, *Helv. phys. acta.*, Vol. 43, n. 3, pp. 296–303 (1970).

[5] *Janner A., Ascher E.,* Relativistic symmetry groups of uniform electromagnetic fields, *Physica*, Vol. 48, n. 3, pp. 425–446 (1970).

[6] *Bacry H., Combe Ph., Sorba P.,* Connected subgroups of the Poincare group. I, *Rep. of Math. Phys.*, Vol. 5, n. 2, pp. 145–186 (1974).

[7] *Bacry H., Combe Ph., Sorba P.,* Connected subgroups of the Poincare group. II, *Rep. of Math. Phys.*, Vol. 5, n. 3, pp. 361–392 (1974).

[8] *Combe Ph., Sorba P.,* Electromagnetic fields with symmetry, *Physica*, Vol. A80, n. 3, pp. 271–286 (1975).

[9] *I. V. Bel'ko,* Subgroups of Lorentz – Poincare Group [in Russian], *Izv. Akad. Nauk Bel. SSR,* n. 1, pp. 5–13 (1971).

[10] *M. A. Parinov,* The Problem of Group Classification of Electromagnetic Fields [in Russian], *Sovremennye metody teorii funktsii i smezhnye problemy: Voronezhskaya zimnyaya matematicheskaya shkola*, Voronezh:VSU, 1999, p. 156.

[11] *M. A. Parinov,* A Group Classification of Maxwell Spaces [in Russian], *Sovremennyi analiz i ego prilozheniya: Voronezhskaya zimnyaya matematicheskaya shkola*, Voronezh:VSU, 2000, pp. 129–130.

[12] *A. L. Besse,* Einstein Manifolds, Berlin: Springer-Verlag, 1987.

[13] *O. G. Belova, A. N. Zarembo, M. A. Parinov, O. O. Sergeeva and Yu. G. Ugarova,* Classification of Static Electromagnetic Fields over Subgroups of the Poincare Group [in Russian], *Nauch. Tr. Ivan. Gos. Un-ta. Ser. Mat.*, Vol. 3, pp. 11–22 (2000).

[14] *A. I. Vorob'ev,* Group Classification of Maxwell Spaces Admitting Hyperbolic Helices [in Russian], *Nauch. Tr. Ivan. Gos. Un-ta. Ser. Mat.*, Vol. 4, pp. 35–42 (2001).

[15] *Vorob'ev A. I.,* On the Classification of Maxwell Spaces Admitting Hyperbolic Helices and First Integrals of Lorentz Equations, *Proc. of Intern. Conf. on Dynam. Systems and Diff. Equations*, Vol. 10, Part 4, pp. 39–47 (2003).

[16] *N. A. Kosheleva, A. K. Kuramshina and M. A. Parinov,* Group Classification of Maxwell Spaces Admitting Elliptic Helices [in Russian], *Nauch. Tr. Ivan. Gos. Un-ta. Ser. Mat.*, Vol. 4, pp. 73–82 (2001).

[17] *D. A. L'vov and M. A. Parinov,* Group Classification of Maxwell Spaces Admitting Parabolic Rotations [in Russian], *Nauch. Tr. Ivan. Gos. Un-ta. Ser. Mat.*, Vol. 5, pp. 51–62 (2002).

[18] *E. V. Morozova and M. A. Parinov,* Group Classification of Maxwell Spaces Admitting Translations along Isotropic Straight Lines [in Russian], *Nauch. Tr. Ivan. Gos. Un-ta. Ser. Mat.*, Vol. 4, pp. 87–94 (2001).

[19] *E. G. Morokhova,* Group Classification of Maxwell Spaces Admitting Proportional Birotation [in Russian], *Nauch. Tr. Ivan. Gos. Un-ta. Ser. Mat.*, Vol. 5, pp. 63–70 (2002).

[20] *M. A. Parinov,* Classes of Maxwell Spaces Admitting Translations, Proc. of Intern. Conf. on Dynam. Systems and Diff. Equations, Vol. 10, Part 4, pp. 157–166 (2003).

[21] *M. A. Parinov,* Einstein – Maxwell Spaces and Lorentz Equations [in Russian], Ivanovo: IvSU, 2003.

[22] *N. S. Polezhaeva and M. A. Parinov,* Group Classification of 4-potentials Admitting Parabolic Rotations [in Russian], *Available from VINITI*, Ivanovo: IvSU, 2003, n. 1489-V2003.

[23] *A. I. Vorob'ev,* Classification of Potential Stuctures Invariant Relatively Hyperbolic Helices [in Russian], *Matematika i eye Prilozheniya: Jour. Ivan. Mat. Ob-va*, n. 1, pp. 41–50 (2004).

Ivanovo State University. 153025 Russia, Ivanovo, Ermaka str., 39. IvSU, Chair of Geometry.

E-mail address: parinov@ivanovo.ac.ru