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by

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# Recursion operators for vacuum Einstein equations with symmetries

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**Abstract**

Direct and inverse recursion operator is derived for the vacuum Einstein equations for metrics with two commuting Killing vectors that are orthogonal to a foliation by 2-dimensional leaves.

## 1 Introduction

In the past decade, inverse recursion operators became a subject of a number of papers [7, 8, 14, 15]. In particular, the work of Guthrie [7] opened a new perspective on recursion operators by essentially identifying them with auto-Bäcklund transformations for linearized equation [18].

It is known for a long time that recursion operators of integrable systems are obtainable from their Lax pairs (see [6] and references therein) and ZCR's (see [10, 24]). However, recently it became clear that zero curvature representations are related much closer to inverse recursion operators than to their 'direct' counterparts [19, 20]. Examples that have been already published elsewhere include the Korteweg–de Vries and Tzitzéica equation [19] and the stationary Nizhnik–Novikov–Veselov equation [20]. In the present paper, the methods of [19, 20] are applied to equations of General Relativity.

## 2 Recursion operators

Let $\mathcal{E} = \{F^l = 0\}$ be a system of PDE's in unknown functions $u^k$ of two independent variables $x, y$. We assume that $F^l$ are functions of $x, y, u^k$ and a finite number of the derivatives $u^k_{ij} = \partial^{i+j} u^k / \partial x^i \partial y^j$ $(u^k_{00} = u^k)$. Consider the infinite-dimensional jet space $J^\infty$ with local coordinates $x, y, u^k_{ij}$ along with the commuting vector fields $D_x = \partial/\partial x + \sum_{ij} u^k_{i+1,j}(\partial/\partial u^k_{ij})$, $D_y = \partial/\partial y + \sum_{ij} u^k_{i,j+1}(\partial/\partial u^k_{ij})$, called *total derivatives.* The submanifold $E$ determined by equations $F^l = 0$ and their differential consequences $D_x F^l = 0$, $D_y F^l = 0$, $D_x^2 F^l = 0$, $D_x D_y F^l = 0$, $D_y^2 F^l = 0$, $\dots$, is called the *equation manifold* (and is an underlying space of the diffiety structure [12] employed in [19]). In this context, infinitesimal symmetries (more precisely, their generating functions) are functions $U^k$ defined on $E$ such that $\sum_{k,i,j}(\partial F^l/\partial u^k_{ij}) D^i_x D^j_y U^k = 0$.

It is then natural to consider the jet space with coordinates $x, y$, $u^k_{ij}$, $U^k_{ij}$, and denote

$$LF^l = \sum_{k,i,j} \frac{\partial F^l}{\partial u^k_{ij}} U^k_{ij}. \qquad (1)$$

The system $L\mathcal{E} := \{F^l = 0, LF^l = 0\}$ on unknowns $u^k, U^k$ will be called the *linearized equation*. Now, Guthrie's recursion operators [7] may be interpreted as auto-Bäcklund transformations of the linearized equation $L\mathcal{E}$ that keep variables $u^k$ unchanged [19].

In now standard formalism [21], recursion operators are pseudodifferential operators, characterized by the occurrence of inverses of total derivatives $D_x^{-1}$. Under Guthrie's approach, $p = D_x^{-1} f$ is introduced as an auxiliary nonlocal variable satisfying

$$p_x = f, \qquad p_y = g, \qquad (2)$$

provided such a $g$ exists, and it actually does without known exception; see Sergyeyev [23] for a proof in case of evolution systems. Thus, $p$ is a potential of a conservation law $f\,dx + g\,dy$ of the linearized equation $L\mathcal{E}$.

For the inverse recursion operators, the nonlocalities tend to be genuinely nonabelian pseudopotentials related to a zero curvature representation of the system in question. Let $\mathfrak{g}$ be a matrix Lie algebra. Let $\alpha = A\,dx + B\,dy$ be a $\mathfrak{g}$-valued zero curvature representation (ZCR) for the system $\mathcal{E}$. This means that $A, B$ are $\mathfrak{g}$-valued functions on the equation submanifold $E$ and $D_y A - D_x B + [A, B] = 0$ holds on $E$. Let us introduce the associated pseudopotential $P$ as a $\mathfrak{g}$-valued solution of the compatible system

$$P_x = [A, P] + LA, \qquad P_y = [B, P] + LB. \qquad (3)$$

A recursion operator $R$ is then a linear operator in $U^k$ and $P$ such that $U' = R(U, P)$ solves the linearized system $L\mathcal{E}$ whenever $U$ does and $P$ satisfies (3) (see [19, 20]). In this way, the inverse recursion operator can be found without previous knowledge of the direct recursion operator. A remarkable aspect of this approach is that $R(U, P)$ tends to be a very simple expression.

For the above scheme to work, it is not necessary that the ZCR $\alpha$ a priori depends on the "spectral parameter." However, if $R$ is a recursion operator related to the ZCR $\alpha$ as above, then $(R^{-1} + \mu\,\mathrm{Id})^{-1}$ is another recursion operator, associated with a ZCR $\alpha_\mu$ which depends on $\mu$.

# 3 The results

We consider vacuum Einstein equations for a space-time with two commuting Killing vectors that are orthogonal to a foliation by 2-dimensional surfaces [4, 5]. Our presentation will be restricted to the case when both Killing vectors are space-like. The case when one of the Killing vectors is time-like is equivalent to ours via an appropriate complex transformation of coordinates.

As is well known, there exist coordinates $x, y, z^1, z^2$ such that the metric in question can be written in the form $ds^2 = 2f(x,y)\,dx\,dy + g_{ij}(x,y)\,dz^i\,dz^j$ (the Lewis [13] metric). The vacuum Einstein equations essentially reduce to

$$(\sqrt{\det g}\; g_x g^{-1})_y + (\sqrt{\det g}\; g_y g^{-1})_x = 0, \tag{4}$$

while $f$ can be obtained by quadrature. Using the standard normalization $\det g = (x+y)^2$ compatible with Eq. (4), we parametrize $g$ as follows: $g_{11} = (x+y)/u$, $g_{12} = (x+y)v/u$, $g_{22} = (x+y)(u^2+v^2)/u$. Equation (4) then becomes

$$\begin{aligned} u_{xy} &= \frac{u_x u_y - v_x v_y}{u} - \frac{1}{2}\frac{u_x + u_y}{x+y}, \\ v_{xy} &= \frac{v_x u_y + u_x v_y}{u} - \frac{1}{2}\frac{v_x + v_y}{x+y}. \end{aligned} \tag{5}$$

As is well known, Eq. (5) has a ZCR and a Bäcklund transformation [1, 2, 3, 9, 16, 17, 22]. The ZCR reads

$$\begin{aligned} A &= \frac{1}{2}\begin{pmatrix} -(\theta+1)u_x/u & (\theta+1)v_x/u^2 \\ (\theta-1)v_x & (\theta+1)u_x/u \end{pmatrix}, \\ B &= \frac{1}{2\theta}\begin{pmatrix} -(\theta+1)u_y/u & (\theta+1)v_y/u^2 \\ (-\theta+1)v_y & (\theta+1)u_y/u \end{pmatrix}, \end{aligned}$$

where $\theta = \sqrt{(\mu+y)/(\mu-x)}$, $\mu$ being the spectral parameter.

The main result of this paper, obtained by the methods of [19, 20], is as follows: If nonlocal variables $p_{11}, p_{12}, p_{21}$ satisfy

$$\begin{aligned} p_{11,x} &= -\frac{\theta-1}{2} v_x p_{12} + \frac{\theta+1}{2}\frac{v_x}{u^2} p_{21} - \frac{\theta+1}{2}\frac{1}{u} U_x + \frac{\theta+1}{2}\frac{u_x}{u^2} U, \\ p_{12,x} &= -(\theta+1)\frac{v_x}{u^2} p_{11} - (\theta+1)\frac{u_x}{u} p_{12} \\ &\quad - (\theta+1)\frac{v_x}{u^3} U + \frac{\theta+1}{2}\frac{1}{u^2} V_x, \\ p_{21,x} &= (\theta-1) v_x p_{11} + (\theta+1)\frac{u_x}{u} p_{21} + \frac{\theta-1}{2} V_x, \\ p_{11,y} &= \frac{\theta-1}{2\theta} v_y p_{12} + \frac{\theta+1}{2\theta}\frac{v_y}{u^2} p_{21} + \frac{\theta+1}{2\theta}\frac{u_y}{u^2} U - \frac{\theta+1}{2\theta}\frac{1}{u} U_y, \\ p_{12,y} &= -\frac{\theta+1}{\theta}\frac{v_y}{u^2} p_{11} - \frac{\theta+1}{\theta}\frac{u_y}{u} p_{12} \\ &\quad - \frac{\theta+1}{\theta}\frac{v_y}{u^3} U + \frac{\theta+1}{2\theta}\frac{1}{u^2} V_y, \\ p_{21,y} &= -\frac{\theta-1}{\theta} v_y p_{11} + \frac{\theta+1}{\theta}\frac{u_y}{u} p_{21} - \frac{\theta-1}{2\theta} V_y, \end{aligned} \tag{6}$$

then

$$U' = 2\frac{u}{\sqrt{(\mu - x)(\mu + y)}}p_{11} + \frac{1}{\sqrt{(\mu - x)(\mu + y)}}U,$$
$$V' = -\frac{u^2}{\sqrt{(\mu - x)(\mu + y)}}p_{12} - \frac{1}{\sqrt{(\mu - x)(\mu + y)}}p_{21} \tag{7}$$

is a recursion operator for Eq. (5), namely, it sends symmetries to symmetries if the latter are viewed as solutions of the linearized system

$$\begin{aligned} U_{xy} &= \left(\frac{u_y}{u} - \frac{1}{2(x+y)}\right)U_x + \left(\frac{u_x}{u} - \frac{1}{2(x+y)}\right)U_y \\ &\quad - \frac{u_x u_y - v_x v_y}{u^2}U - \frac{v_y}{u}V_x - \frac{v_x}{u}V_y, \\ V_{xy} &= \frac{v_y}{u}U_x + \frac{v_x}{u}U_y - \frac{v_x u_y + u_x v_y}{u^2}U \\ &\quad + \left(\frac{u_y}{u} - \frac{1}{2(x+y)}\right)V_x + \left(\frac{u_x}{u} - \frac{1}{2(x+y)}\right)V_y. \end{aligned}$$

The ‘direct’ recursion operator for this equation seems to be missing in the literature; we can obtain it by inverting the operator (7), the result being

$$\begin{aligned} U' &= uvp_1 - up_2 + (y - x)U, \\ V' &= -\tfrac{1}{2}(u^2 - v^2)p_1 - vp_2 - \tfrac{1}{2}p_3 + (y - x)V, \end{aligned}$$

where $p_1, p_2, p_3$ satisfy

$$\begin{aligned} p_{1,x} &= (x+y)\left(-2\frac{v_x}{u^3}U + \frac{1}{u^2}V_x\right), \\ p_{2,x} &= (x+y)\left(-\frac{uu_x + 2vv_x}{u^3}U + \frac{1}{u}U_x + \frac{v_x}{u^2}V + \frac{v}{u^2}V_x\right), \\ p_{3,x} &= (x+y)\left(2\frac{(uu_x + vv_x)v}{u^3}U - 2\frac{v}{u}U_x \right. \\ &\qquad\qquad \left. - 2\frac{uu_x + vv_x}{u^2}V + \frac{u^2 - v^2}{u^2}V_x\right), \\ p_{1,y} &= (x+y)\left(2\frac{v_y}{u^3}U - \frac{1}{u^2}V_y\right), \\ p_{2,y} &= (x+y)\left(\frac{uu_y + 2vv_y}{u^3}U - \frac{1}{u}U_y - \frac{v_y}{u^2}V - \frac{v}{u^2}V_y\right), \\ p_{3,y} &= (x+y)\left(-2\frac{(uu_y + vv_y)v}{u^3}U + 2\frac{v}{u}U_y \right. \\ &\qquad\qquad \left. + 2\frac{uu_y + vv_y}{u^2}V - \frac{u^2 - v^2}{u^2}V_y\right). \end{aligned}$$

It is readily seen that $p_i$ are potentials of the linearizations [18] of the three obvious conservation laws of Eq. (4).

Quite unusually, neither of the recursion operators found generates an infinite series of local symmetries (and no such series is known). The action of our operators on the infinite-dimensional Geroch group of nonlocal symmetries [5, 11] remains to be investigated.

It is convenient to rewrite system (6) in triangular form. To achieve this, we introduce the Riccati pseudopotential $q$ by

$$\begin{aligned} q_x &= \frac{\theta-1}{2} v_x q^2 - (\theta+1)\frac{u_x}{u} q - \frac{\theta+1}{2}\frac{v_x}{u^2}, \\ q_y &= -\frac{\theta-1}{2\theta} v_y q^2 - \frac{\theta+1}{2\theta}\frac{u_y}{u} q - \frac{\theta+1}{2\theta}\frac{v_y}{u^2} \end{aligned}$$

and a nonlocal potential $r$ by

$$\begin{aligned} r_x &= (\theta-1) v_x q - (\theta+1)\frac{u_x}{u}, \\ r_y &= -\frac{\theta-1}{\theta} v_y q - \frac{\theta+1}{\theta}\frac{u_y}{u}. \end{aligned}$$

Then the inverse recursion operator assumes the form

$$\begin{aligned} U' &= \frac{1}{\sqrt{(\mu-x)(\mu+y)}}\left(2uQ - 2\frac{uq}{\mathrm{e}^r} R + U\right), \\ V' &= \frac{1}{\sqrt{(\mu-x)(\mu+y)}}\left(-u^2\mathrm{e}^r P - 2u^2 q Q + \frac{u^2q^2-1}{\mathrm{e}^r} R\right), \end{aligned}$$

where $P, Q, R$ are supposed to satisfy

$$\begin{aligned} P_x &= (\theta+1)\frac{q}{u}\mathrm{e}^{-r} U_x + \left(\frac{\theta+1}{2}\frac{1}{u^2} - \frac{\theta-1}{2} q^2\right)\mathrm{e}^{-r} V_x \\ &\quad - (\theta+1)\left(\frac{q u_x}{u^2} + \frac{v_x}{u^3}\right)\mathrm{e}^{-r} U, \\ Q_x &= -\frac{\theta+1}{2}\frac{1}{u} U_x + \frac{\theta-1}{2} q V_x + \frac{\theta+1}{2}\frac{u_x}{u^2} U - \frac{\theta-1}{2} v_x \mathrm{e}^r P, \\ R_x &= \frac{\theta-1}{2}\mathrm{e}^r V_x + (\theta-1) v_x \mathrm{e}^r Q, \\ P_y &= \frac{\theta+1}{\theta}\frac{q}{u}\mathrm{e}^{-r} U_y + \left(\frac{\theta+1}{2\theta}\frac{1}{u^2} + \frac{\theta-1}{2\theta} q^2\right)\mathrm{e}^{-r} V_y \\ &\quad - \frac{\theta+1}{\theta}\left(\frac{q u_y}{u^2} + \frac{v_y}{u^3}\right)\mathrm{e}^{-r} U, \\ Q_y &= -\frac{\theta+1}{2\theta}\frac{1}{u} U_y - \frac{\theta-1}{2\theta} q V_y + \frac{\theta+1}{2\theta}\frac{u_y}{u^2} U + \frac{\theta-1}{2\theta} v_y \mathrm{e}^r P, \\ R_y &= -\frac{\theta-1}{2\theta}\mathrm{e}^r V_y - \frac{\theta-1}{\theta} v_y \mathrm{e}^r Q. \end{aligned}$$

This form of the inverse recursion operator is better adapted to generation of symmetries, which is, however, beyond the scope of this paper.

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