The Diffiety Institute Preprint Series

Preprint DIPS–6/2003 **October 7, 2003**

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The long exact sequence of a covering: three applications

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Abstract. We construct the long exact sequence of $\mathscr{C}$-cohomology associated to a covering over a differential equation $\mathscr{E}$ and use this sequence to solve the following problems: (1) infinitesimal description of smooth irreducible families of coverings; (2) reconstruction of nonlocal symmetries by their shadows; (3) description of action of recursion operators on (nonlocal) symmetries.

Introduction

Cohomological methods proved to be quite useful and fruitful in studying invariant properties of PDE's [5, 6, 9]. Here we want to demonstrate the power of these techniques by applying a very simple cohomological construction to three problems arising in nonlocal theory of PDE's. These problems are (more detailed statements see below):

(1) How to describe smooth families of coverings?
(2) Can a nonlocal symmetry be reconstructed by its shadow?
(3) What happens when we apply a recursion operator to a nonlocal symmetry?

The first question was answered in [1] and here we just repeat the corresponding construction (Subsection 2.1). Problem 2 was solved by N. Khor′kova long ago and a solution given in local coordinate language can be found in [7]. In Subsection 2.2 we give a new, very short cohomological proof. Finally, there were no general answer to the last question and we give a solution in Subsection 2.3.

1. The long exact sequence of a covering

Let $\mathscr{E} \subset J^\infty(\pi)$ be an infinitely prolonged differential equation understood as a submanifold in the manifold of infinite jets, $\pi_\infty\colon \mathscr{E} \to M$ be the natural projection. Consider a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ and the structure element $U_\tau \in \mathrm{D}^{\mathrm{v}}(\Lambda^1(\tilde{\mathscr{E}})) \subset \mathrm{D}(\Lambda^1(\tilde{\mathscr{E}}))$, where (and below) $\mathrm{D}(\Lambda^i(\tilde{\mathscr{E}}))$ denotes the module of $\Lambda^i(\tilde{\mathscr{E}})$-valued derivations $C^\infty(\tilde{\mathscr{E}}) \to \Lambda^i(\tilde{\mathscr{E}})$ while $\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))$ consists of π_∞-vertical derivation. Then (see [3]) the *$\mathscr{C}$-complex*

$$0 \to \mathrm{D}^{\mathrm{v}}(\tilde{\mathscr{E}}) \xrightarrow{\partial_\tau} \mathrm{D}^{\mathrm{v}}(\Lambda^1(\tilde{\mathscr{E}})) \to \cdots \to \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) \xrightarrow{\partial_\tau} \mathrm{D}^{\mathrm{v}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) \to \cdots \qquad (1)$$

arises, where the differential $\partial_\tau = [\![U_\tau, \cdot\,]\!]$ is defined by the *Frölicher-Nijenhuis bracket*. The corresponding cohomology is denoted by $H^i_{\mathscr{C}}(\tilde{\mathscr{E}})$ and called the *$\mathscr{C}$-cohomology* of the covering τ.

2000 *Mathematics Subject Classification.* 37K10,58J70.
Key words and phrases. Coverings, $\mathscr{C}$-cohomology, nonlocal symmetries, recursion operators.
Partially supported by the University of Twente, the Netherlands.

Consider in $\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))$ the submodule

$$\mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}})) = \{X \in \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}) \mid X(C^\infty(\mathscr{E})) = 0\},$$

where $C^\infty(\mathscr{E})$ is understood as a subalgebra in $C^\infty(\tilde{\mathscr{E}})$. Let also

$$\mathrm{D}^{\mathrm{s}}(\Lambda^i(\tilde{\mathscr{E}})) = \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))/\mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}}))$$

be the quotient module. By basic properties of the Frölicher-Nijenhuis bracket, $\partial_\tau(\mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}}))) \subset \mathrm{D}^{\mathrm{g}}(\Lambda^{i+1}(\tilde{\mathscr{E}}))$ and thus the short exact sequence of complexes

$$\begin{tikzcd}
 & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \\
0 \arrow[r] & \mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}})) \arrow[r] \arrow[d, "\partial_\tau"] & \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) \arrow[r] \arrow[d, "\partial_\tau"] & \mathrm{D}^{\mathrm{s}}(\Lambda^i(\tilde{\mathscr{E}})) \arrow[r] \arrow[d, "\partial_\tau"] & 0 \\
0 \arrow[r] & \mathrm{D}^{\mathrm{g}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) \arrow[r] \arrow[d] & \mathrm{D}^{\mathrm{v}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) \arrow[r] \arrow[d] & \mathrm{D}^{\mathrm{s}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) \arrow[r] \arrow[d] & 0 \\
 & \dots & \dots & \dots &
\end{tikzcd} \tag{2}$$

is defined (we preserve the notation ∂_τ for the differential in both quotient and subcomplexes). Denote by $H^i_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^i_{\mathrm{g}}(\tilde{\mathscr{E}})$ the cohomology groups of the quotient and subcomplexes, respectively.

Definition 1. The groups $H^i_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^i_{\mathrm{g}}(\tilde{\mathscr{E}})$ are called *shadow* and *gauge* $\mathscr{C}$-cohomologies of the covering τ, respectively. The cohomological sequence

$$0 \to H^0_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^1_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots$$
$$\cdots \to H^i_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^i_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^i_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{i+1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots \tag{3}$$

corresponding to (2) is called the *long exact sequence* of the covering τ.

Remark 1. Recall (see [3]) that the modules $\Lambda^i(\tilde{\mathscr{E}})$ split into the direct sum

$$\Lambda^i(\tilde{\mathscr{E}}) = \oplus_{p+q=i}\Lambda^{p,q}(\tilde{\mathscr{E}}), \quad \Lambda^{p,q}(\tilde{\mathscr{E}}) = \mathscr{C}\Lambda^p(\tilde{\mathscr{E}}) \otimes \Lambda^q_h(\tilde{\mathscr{E}}), \tag{4}$$

where $\mathscr{C}\Lambda^p(\tilde{\mathscr{E}})$ and $\Lambda^q_h(\tilde{\mathscr{E}})$ are modules of *Cartan* and *horizontal* forms, respectively. To (4) there correspond the splitting

$$\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) = \oplus_{p+q=i}\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q}(\tilde{\mathscr{E}}))$$

and the differential ∂_τ takes elements $\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q}(\tilde{\mathscr{E}}))$ to $\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q+1}(\tilde{\mathscr{E}}))$. This leads to the cohomology groups $H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}})$, $H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}})$ and (3) actually splits into the series of the following sequences

$$0 \to H^{p,0}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{p,0}_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^{p,0}_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{p,1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots$$
$$\cdots \to H^{p,q}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{p,q+1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots$$

for all $p \geq 0$. The groups $H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}})$ are identified with the *horizontal cohomology* of $\tilde{\mathscr{E}}$ with coefficients in $\mathrm{D}^{\mathrm{v}}(\mathscr{C}\Lambda^p(\tilde{\mathscr{E}}))$, [8].

2. Applications

Before discussing particular applications of the above constructions, let us describe the geometrical meaning of some of the groups in (3) that we shall need below.

$H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$: elements of this group are identified with *nonlocal τ-symmetries*;

$H^0_{\mathrm{g}}(\tilde{\mathscr{E}})$: these are *gauge symmetries* in τ, i.e., τ-vertical symmetries (the corresponding diffeomorphisms, if they exist, are automorphisms of τ);

$H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$: this cohomology group consists of *τ-shadows* of nonlocal symmetries, i.e., π_∞-vertical derivations $X\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ preserving the Cartan distributions. In local coordinates, shadows are described by vector-functions $\varphi = (\varphi^1, \dots \varphi^m)$, $\varphi^l \in C^\infty(\tilde{\mathscr{E}})$, $m = \dim \pi$, satisfying the equation $\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$, where $\tilde{\ell}_{\mathscr{E}}$ is the linearization of $\mathscr{E}$ naturally lifted to $\tilde{\mathscr{E}}$;

$H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$: these are equivalence classes of nontrivial *infinitesimal deformations* of U_τ, i.e., of the element defining the basic geometrical structure on $\tilde{\mathscr{E}}$; a deformation is infinitesimally trivial if and only if its cohomological class in $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$ vanishes. On the other hand, elements of $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$ act on $H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) = \mathrm{sym}_\tau \mathscr{E}$ by contraction: $R(X) = \mathrm{i}_X(R)$, $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$, $R \in H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$. Nontrivial actions may correspond to elements of $H^{1,0}_{\mathscr{C}}(\tilde{\mathscr{E}})$ only;

$H^1_{\mathrm{g}}(\tilde{\mathscr{E}})$: those cohomological classes are *τ-vertical*, or *gauge*, deformations. They deform the covering structure itself only and do not change the structure of the underlying equation $\mathscr{E}$;

$H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$: similar to $H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$, these are also shadows of $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$, i.e., classes of π_∞-vertical derivations $C^\infty(\mathscr{E}) \to \Lambda^1(\tilde{\mathscr{E}})$ that preserve the Cartan distributions. Elements of $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}})$ have the following local description: $\mu = (\mu^1, \dots, \mu^m)$, $\mu^l \in \mathscr{C}\Lambda^1(\tilde{\mathscr{E}})$ lies in $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}})$ if and only if $\ell_{\mathscr{E}}(\mu) = 0$.

2.1. **Irreducible families of coverings.** Essentially, we repeat here the main result given in [1]. Let us first recall the following construction, [7]. Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and X be a symmetry of $\mathscr{E}$ possessing a one-parameter group of transformations $A_t\colon \mathscr{E} \to \mathscr{E}$. Consider an arbitrary lift of A_t to $\tilde{\mathscr{E}}$ such that the diagram

$$
\begin{array}{ccc}
\tilde{\mathscr{E}} & \xrightarrow{\tilde{A}_t} & \tilde{\mathscr{E}} \\
{\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\
\mathscr{E} & \xrightarrow{A_t} & \mathscr{E}
\end{array}
\qquad (5)
$$

is commutative. Let us define on $\tilde{\mathscr{E}}$ a t-parameter family of distributions $\tilde{\mathscr{C}}_t$ by setting

$$(\tilde{\mathscr{C}}_t)_{\tilde{\theta}} = (\tilde{A}_t)_*^{-1} \tilde{\mathscr{C}}_{\tilde{A}_t^{-1}\tilde{\theta}}, \quad \tilde{\theta} \in \tilde{\mathscr{E}}.$$

All distributions $\tilde{\mathscr{C}}_t$ are integrable, i.e., $[\tilde{\mathscr{C}}_t, \tilde{\mathscr{C}}_t] \subset \tilde{\mathscr{C}}_t$, and thus $\tilde{\mathscr{E}}_t = (\tilde{\mathscr{E}}, \tilde{\mathscr{C}}_t)$ covers $\mathscr{E}$ by means of τ. Denote this covering by $\tau_t\colon \tilde{\mathscr{E}}_t \to \mathscr{E}$ and notice that if $\tilde{A}'_t$ is another lift satisfying (5) then for all t the covering τ_t is either equivalent to τ'_t or not. If for all t (sufficiently small) τ_t and τ'_t are equivalent then this means that X can be lifted to a symmetry of $\tilde{\mathscr{E}}$. Thus we obtain

Proposition 1. *Let $\tau\colon \tilde{\mathscr{E}}$ be a covering and X be a symmetry of $\mathscr{E}$ possessing a one-parameter group of transformations and such that it cannot be lifted to a symmetry of $\tilde{\mathscr{E}}$. Then X generates a one-family of (equivalence classes of) coverings $\tau_t\colon \tilde{\mathscr{E}}_t \to \tilde{\mathscr{E}}$ such that*

(1) *$\tilde{\mathscr{E}}_t$ and $\tilde{\mathscr{E}}_{t'}$ isomorphic as manifolds with distributions;*
(2) *τ_t and $\tau_{t'}$ are pair-wise inequivalent for sufficiently small t and t'.*

We say that a family τ_t satisfying Properties 1 and 2 above is *irreducible*.

Remark 2. Let $\mathfrak{g}$ be a finite-dimensional Lie subalgebra in the algebra of classical symmetries of $\mathscr{E}$. Then we obtain a G/H-irreducible family, where G is the Lie group corresponding to $\mathfrak{g}$ and H is the stabilizer of τ under the above described action.

Remark 3. Proposition 1 means that if we have a covering τ and a shadow that cannot be reconstructed to a symmetry in this covering[1] then, under certain conditions, this shadow generates an irreducible family of coverings. As we shal see below, this is, in a sense, a general way to obtain irreducible families.

Consider an irreducible family of coverings $\tau_t\colon \tilde{\mathscr{E}} \to \mathscr{E}$, $\tau_0 = \tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$, and the following part of (3)

$$H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^1_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^1_{\mathscr{C}}(\tilde{\mathscr{E}}).$$

The family τ_t may be considered as a deformation of τ; hence its infinitesimal part μ lies in $H^1_{\mathrm{g}}(\tilde{\mathscr{E}})$. By Property 1 from Proposition 1, the corresponding deformation of $\tilde{\mathscr{E}}$ is trivial and thus $\alpha(\mu) = 0$. By the exactness, $\mu = \partial(\varphi)$, $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$. The deformation τ_t is infinitesimally nontrivial if and only if $\mu \neq 0$ and again by the exactness if and only if $\varphi \neq \beta(X)$, $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$, i.e., if and only if φ is not reconstructible to a τ-symmetry. To state the final result, it remains to note that the action of ∂ is given by $[\![U_\tau, \cdot]\!]$.

Theorem 1. *Any irreducible family of coverings τ_t, $\tau_0 = \tau$, must be infinitesimally generated by a τ-shadow that cannot be reconstructed to a nonlocal τ-symmetry.*

Remark 4. Of course, not to any shadow φ there corresponds an irreducible family of coverings. Anyway, we can construct a *formal deformation* $\exp(t\varphi)U_\tau$ and use it in some applications.

2.2. **Reconstruction of shadows.** Consider again a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ and a shadow $\varphi\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$. We say that φ is *reconstructible* in τ if there exists a symmetry $X\colon C^\infty(\tilde{\mathscr{E}}) \to C^\infty(\tilde{\mathscr{E}})$ such that $X\big|_{C^\infty(\mathscr{E})} = \varphi$. Of course, given a covering τ, not any shadow can be reconstructed in this covering, but a weaker result was found by N. Khor′kova and proved in [7]:

Theorem 2. *Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and φ be a τ-shadow. Then there exists a covering $\tilde{\tau}\colon \tilde{\tilde{\mathscr{E}}} \to \tilde{\mathscr{E}}$ such that φ is reconstructible in the composition covering $\tau\circ\tilde{\tau}\colon \tilde{\tilde{\mathscr{E}}} \to \mathscr{E}$.*

[1]For the problem of reconstruction see Subsection 2.2.

Remark 5. How the theorem works in practice may be seen in the classical example of the KdV equation

$$u_t = uu_x + u_{xxx}. \tag{6}$$

It is known that this equation possesses two infinite series of symmetries: a local one, independent of x and t and consisting of higher KdV equations, and another, (x,t)-dependent series whose first two terms are local (the Galilean boost and the scaling symmetry) while all others are nonlocal. Take the first of the them. It has the form $\psi_5 = tu_{xxxxx} + \dots$ and contains the nonlocal variable w^1 defined by the relations

$$w^1_x = u, \qquad w^1_t = \frac{1}{2}u^2 + u_{xx}. \tag{7}$$

Actually, ψ_5 is not a symmetry but only a shadow in the covering corresponding to (7). To reconstruct the symmetry, one needs to find the coefficient at $\partial/\partial w^1$. But when doing this, a new nonlocal variable arises that must satisfy the relation $w^2_x = u^2$, etc. As it was shown in [2], this process "stops" at infinity only: to reconstruct the initial shadow, one has to add all nonlocal variables of the form $w^j_x = c^j$, where c^j is the density of the jth conservation law. As we shall see, this situation is somewhat typical.

Proof of Theorem 2. Consider the following part of (3)

$$H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^1_{\mathrm{g}}(\tilde{\mathscr{E}}).$$

Let $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$ be a shadow. It is reconstructible if there exists a symmetry $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$ such that $\beta(X) = \varphi$. Due to the exactness, this is equivalent to $\partial(\varphi) = 0$. Thus the element $\omega_\varphi = \partial(\varphi)$ is the obstruction to reconstructibility of the shadow φ. So, the intermediate result is

Proposition 2. *A shadow $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$ is reconstructible in the covering τ if and only if the obstruction $\omega_\varphi \in H^1_{\mathrm{g}}(\tilde{\mathscr{E}})$ vanishes.*

The obstruction ω_φ is a vector-valued horizontal 1-form on $\tilde{\mathscr{E}}$, the number of its components ω^l_φ equals the dimension of the covering τ. Each form ω^l_φ is closed and thus determines a 1-dimensional covering τ^l over $\tilde{\mathscr{E}}$, see [7]. This covering is trivial if and only if this form is exact (with respect to the horizontal de Rham differential) and $\omega_\varphi = 0$ if and only if all ω^l_φ are exact. Thus, if we choose those ω^l_φ whose horizontal cohomology classes are nontrivial and take the Whitney product $\tilde{\tau}\colon \tilde{\tilde{\mathscr{E}}} \to \tilde{\mathscr{E}}$ of the corresponding coverings then the obstruction ω_φ will vanish in this covering, but a new obstruction of the same nature may arise. If the latter vanishes the shadow is reconstructible in $\tilde{\tau}$, otherwise we must repeat the construction, etc. Eventually, we shall either stop at some finite step, or shall arrive to an infinite covering. In both cases, there will be no obstruction to reconstruct the shadow φ. □

Remark 6. From the proof of the theorem it can be seen that the shadow φ completely determines the "minimal" covering, where it reconstructs to a symmetry. In the case of a two-dimensional base M (i.e., when the equation $\mathscr{E}$ is in two independent variables) the cohomology classes of the forms ω^l_φ are identified with *conservation laws* of $\mathscr{E}$, generally nonlocal. But in some cases (for the KdV and similar equations) the reconstruction procedure deals with local conservation laws only, [2].

Remark 7. Though the covering, where the shadow at hand reconstructs, is well defined (but not unique) algorithmically, the symmetry that corresponds to this shadow is definitely not unique: even when covering $\tilde{\tau}$ is chosen, a symmetry corresponding to φ is defined up to elements of $H^0_{\mathrm{g}}(\tilde{\tilde{\mathscr{E}}})$, i.e., up to gauge symmetries in $\tilde{\tau}$. This means that to deal with nonlocal symmetries is not the same as to deal with their shadows.

2.3. **Action of recursion operators.** We now pass to the last topic of these notes. Let us first briefly recall the cohomological theory of recursion operators as it was exposed in [5]. Consider an equation $\mathscr{E}$ and the $\mathscr{C}$-complex (1) associated to it[2]. Then, as it was already mentioned, the group $H^0_{\mathscr{C}}(\mathscr{E})$ is identified with the Lie algebra $\operatorname{sym}(\mathscr{E})$ of higher symmetries of $\mathscr{E}$. The contraction (or inner product) operation determines an action of elements of $H^1_{\mathscr{C}}(\mathscr{E})$ on $\operatorname{sym}(\mathscr{E})$ and nontrivial actions may correspond to the elements of $H^{1,0}_{\mathscr{C}}(\mathscr{E})$ only[3]. Locally, these elements are represented as Cartan vector-forms of degree 1, $\omega = (\omega_1, \dots, \omega_m)$, $\omega_l \in \mathscr{C}\Lambda^1(\mathscr{E})$, $m = \dim \pi$, satisfying the equation

$$\ell_{\mathscr{E}}(\omega) = 0 \tag{8}$$

and solving (8) and applying these solutions to the known symmetries we, in principal, can generate new symmetries. The same remains valid for nonlocal symmetries if we take $\tilde{\mathscr{E}}$ instead of $\mathscr{E}$. It is natural to anticipate that in this way we find recursion operators for symmetries of the equation at hand.

But in practice the picture is more complicated. If, for example, we realize the above procedure for the KdV equation (6) we shall obtain trivial solutions only. On the other hand, this equation possesses the recursion operator

$$R = D_x^2 + \frac{2}{3}u + \frac{1}{3}D_x^{-1},$$

where D_x is the *total derivative* with respect to x. Of course, this operator could not be found by the above described method because of the nonlocal term D_x^{-1}. Such a term may arise in the covering given by (7), but if we try the same in this nonlocal setting the result will be negative again.

What actually gives nontrivial results is the following (see all the examples in [5]). Let us show how it works for the KdV equation. Namely, we do the following:

(1) We take the covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ given by (7). Note that $\tilde{\mathscr{E}}$ is isomorphic to the infinite prolongation of the equation

$$w_t = w_{xxx} + \frac{1}{2}u_x^2.$$

(2) Then we consider the module of Cartan 1-forms on $\tilde{\mathscr{E}}$. It is generated by the forms

$$\omega_k = du_k - u_{k+1}\,dx - D_x^k(u_3 + u_1)\,dt, \quad \omega_{-1} = dw^1 - u\,dx - \big(u_2 + \frac{1}{2}u^2\big)\,dt,$$

where $k = 0, 1, \dots$ and $u_k = u\underbrace{x \dots x}_{k \text{ times}}$.

[2] Its terms are $\mathrm{D^v}(\Lambda^i(\mathscr{E}))$.

[3] Note also that the contraction determines a structure of an associative algebra with unit on $H^1_{\mathscr{C}}(\mathscr{E})$.

(3) The next step is to solve the equation

$$\tilde{\ell}_{\mathscr{E}}(\Omega) = 0, \tag{9}$$

where $\tilde{\ell}_{\mathscr{E}}$ is the lift of the linearization to τ and $\Omega = \sum_{k\geq -1} f_k\omega_k$.

(4) Solving (9), we obtain two independent solutions: $\Omega_1 = \omega_0$ and $\Omega_2 = \omega_2 + \frac{2}{3}u\omega_0 + \frac{1}{3}\omega_{-1}$. The first one corresponds to the identical action while the second one gives the classical recursion operator for the KdV equation (see above).

So, from this scheme we see that construction of recursion operators amounts to computation of the group $H_1^{1,0}(\tilde{\mathscr{E}})$. Indeed, we have the following

Theorem 3. *Let $\mathscr{E}$ be an equation and $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering. Then the contraction operation generates the action*

$$R\colon H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \to H^0_{\mathrm{s}}(\tilde{\mathscr{E}}),$$

where $R \in H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$, i.e., elements of $H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$ take τ-symmetries to τ-shadows[4].

Remark 8. The result of this action, in general, is really a shadow. For example, applying the recursion operator of the KdV equation to the scaling symmetry $\psi_3 = tu_{xxx} + \dots$, we obtain the element ψ_5 (see above) that reconstructs to a symmetry in the infinite-dimensional covering, where all conservation laws of the equation are "killed".

Proof. The proof is very simple: consider the standard contraction

$$\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) \lrcorner \mathrm{D}^{\mathrm{v}}(\Lambda^j(\tilde{\mathscr{E}})) \to \mathrm{D}^{\mathrm{v}}(\Lambda^{i+j-1}(\tilde{\mathscr{E}})).$$

It remains to note that if $X \in \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))$ and $Y \in \mathrm{D}^{\mathrm{s}}(\Lambda^j(\tilde{\mathscr{E}}))$ then $X \lrcorner Y \in \mathrm{D}^{\mathrm{s}}(\Lambda^{i+j-1}(\tilde{\mathscr{E}}))$ and that the differential in (1) preserves the inner product and thus the latter is inherited in the cohomology groups. □

Remark 9. In some cases action of recursion operator on local symmetries leads to local symmetries again. For example, this is the case for classical hierarchies of integrable equations. Some remarks on how to establish locality may be found in [4].

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