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by

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Abstract

We present a criterion of reducibility of a zero curvature representation to a solvable subalgebra, hence to a chain of conservation laws. Namely, we show that reducibility is equivalent to the existence of a section of the generalized Riccati covering. Results are applied to conversion between Guthrie's and Olver's form of recursion operators.

1 Introduction

To establish integrability of a nonlinear partial differential equation in the sense of soliton theory [1, 37], at least in two dimensions, one usually looks for a zero curvature representation (ZCR) [43], possibly in the form of a Lax pair [17]. If depending on a non-removable (spectral) parameter, a ZCR may serve as a starting point of methods to derive infinitely many independent conservation laws and large classes of exact solutions.

However, certain ZCR's do not imply integrability because of specific degeneracy, which does not even rule out possible dependence on one or more nonremovable parameters. E.g., Calogero and Nucci [3] gave a formula to assign a Lax pair to any nonlinear system possessing a single conservation law, arguing that such systems are too abundant to be all integrable. Recently Sakovich [33] observed that the Calogero–Nucci examples can be singled out by properties of their associated cyclic bases. In particular, the 'bad' ZCR's fail to generate an integrable hierarchy.

In this paper we postulate that a ZCR is degenerate if it takes values in a solvable Lie algebra or is gauge equivalent to such. Even though some researchers are inclined to admit the relevance of such ZCR's to integrability, our results below seem to support the opposite opinion.

ZCR's taking values in an abelian algebra are well known to be equivalent to a set of local conservation laws (see [1, Sect. 3.2.c]). Using the Lie theorem on finite-dimensional representations of solvable algebras, we show in Sect. 4 rather easily that every ZCR that takes values in a solvable algebra is equivalent to a 'chain' of nonlocal conservation laws. This simple result renders, e.g., attempts to generate infinitely many independent conservation laws out of a degenerate ZCR rather unrealistic.

In Sect. 5 we address the problem of detecting reducibility of a ZCR to a subalgebra, in particular, to a solvable one. Purely algebraic criteria are insufficient since the Lie algebra a ZCR takes values in may be altered by gauge transformation. On the other hand, when trying to find the reducing gauge matrix directly, we face a rather large underdetermined differential system. Our idea is to employ an appropriate matrix decomposition, namely, the Gram or Gauss decomposition. Earlier these decompositions were applied by Dodd and Paul [6, 7] in the context of Bäcklund transformations. A remarkable connection between decompositions and integrable systems emerged in numerical analysis [4, 5, 41].

The last section is devoted to recursion operators, direct and inverse, for symmetries of integrable systems [2, 14, 28, 39]. In Olver's [27, 28] formalism, a recursion operator is a linear integro-differential operator Ψ, which maps symmetries to symmetries. The standard way of inverting Ψ consists in finding differential operators K, L such that $\Psi = L \circ K^{-1}$; then $\Psi^{-1} = K \circ L^{-1}$. However, one encounters the problem of writing the inverse L^{-1} as an integro-differential operator. In the scalar case, L may be put in the form $L = q_n D q_{n-1} D \cdots q_1 D q_0$, where the coefficients q_i are expressible as quotients of wronskians of independent solutions v_i of $L(v) = 0$ (see [31] for a simple derivation of this classical result, equivalent to decomposability into first-order factors; see [44] for the matrix case). In our context, q_i are nonlocal functions and finding them is considered to be the most difficult part of the whole procedure. Once q_i are found, one can invert L simply as $L^{-1} = q_0^{-1} D^{-1} q_1^{-1} \cdots D^{-1} q_{n-1}^{-1} D^{-1} q_n^{-1}$. This is essentially the general scheme behind the works [12, 18, 19, 20, 29, 32].

Guthrie's [11] recursion operators resemble Bäcklund autotransformations for the linearized system and indeed can be interpreted this way (see [22]); their inversion is quite straightforward and does not require the introduction of new nonlocalities. Moreover, Guthrie's operators do not suffer from the known abnormities, related to the fact that $D^{-1} \circ D = \mathrm{id}$ fails to hold ([10, 34]). Let us also remind the reader that computing the 'inverse' Guthrie operator starting from a known ZCR may turn out to be easier than computing the 'direct' one (see [24]). The conversion from Olver's to Guthrie's form was explained by Guthrie [11] himself, the result being further strengthened by Sergyeyev [35]. Concerning the backward conversion, the x-part of a Guthrie operator can be written as an integro-differential operator if the ZCR underlying it is lower triangular. A non-parametric ZCR can be made lower triangular at the cost of the introduction of appropriate nonlocalities. To introduce only few (respected) nonlocalities, we take into account a particular observation (already exploited in [24]) about the structure of Guthrie's recursion operators of integrable systems.

2 Preliminaries

Let E be a system of nonlinear partial differential equations (PDE)

$$F^l = 0 \tag{1}$$

on a number of functions u^k in two independent variables x, y. Here each F^l is a smooth function depending on a finite number of variables $x, y, u^k, u^k_x, u^k_y, \ldots, u^k_I, \ldots$, where I stands for a symmetric multiindex over the two-element set of indices $\{x, y\}$. Besides the *local* variables x, y, u^k, u^k_I, we shall also need non-local variables or pseudopotentials [40], which may be introduced as additional variables satisfying a system of equations

$$z^i_x = f^i, \quad z^i_y = g^i, \tag{2}$$

where f^i, g^i are functions depending on a finite number of local variables as well as the pseudopotentials z^j; we require that the system (2) be compatible as a consequence of (1).

Within their geometric theory of systems of PDE's, Krasil'shchik and Vinogradov [15] introduced the notion of a covering, which separates the invariant content of nonlocality from its coordinate presentation. Pseudopotentials then correspond to a particular but arbitrary choice of coordinates along the fibres of the covering in question. We recall the basic facts below; details we had to leave aside may be found in [15] and also in [16, Ch. 6]. Readers interested mainly in practical computations may skip the rest of this section.

Let J^∞ be an infinite jet space equipped with local jet coordinates x, y, u^k, u^k_I; the functions F^l then may be interpreted as functions defined on J^∞. Since all our considerations are local, we simply let J^∞ be the space of jets of sections of the trivial fibred manifold $Y \times M \to M$, where $M = \mathbb{R}^2$, with x, y being coordinates on $\mathbb{R}^2$ and u^k being coordinates on Y. On J^∞, we have two distinguished commuting vector fields

$$D_x = \frac{\partial}{\partial x} + \sum_{k,I} u^k_{Ix} \frac{\partial}{\partial u^k_I}, \qquad D_y = \frac{\partial}{\partial y} + \sum_{k,I} u^k_{Iy} \frac{\partial}{\partial u^k_I},$$

which are called *total derivatives.*

The *equation manifold* $\mathcal{E}$ associated with system (1) is defined to be the submanifold in J^∞ determined by the infinite system of equations $F^l = 0$ and $D_I F^l = 0$ for I running through all symmetric multiindices in x, y. The total derivatives D_x, D_y are tangent to $\mathcal{E}$, therefore they admit a restriction to $\mathcal{E}$. In what follows, equations will be identified with equation manifolds equipped with the restricted total derivatives; this approach is indeed very practical and suitable for all needs to be encountered below.

Mappings between equation manifolds that commute with projections to the base manifold M and preserve the total derivatives will be called *morphisms* of equations; they map solutions to solutions (we shall not use the general morphisms of *diffieties* which need not commute with the projections and only preserve the distributions generated by the total derivatives). Bijective morphisms are called *isomorphisms*; their inverses are isomorphisms, too.

A *covering* over an equation $\mathcal{E}$ consists of another equation $\mathcal{E}'$ and a surjective morphism $\mathcal{E}' \to \mathcal{E}$.

The system formed by Equation (1) and the $2k$ additional equations (2) generates a covering, where $\mathcal{E}'$ is the trivial vector bundle $\mathcal{E} \times \mathbb{R}^k$ and $z^1, \dots, z^k$ provide coordinates along $\mathbb{R}^k$. In particular, the projection preserves the coordinates x, y. If f^i, g^i are functions defined on E' such that the vector fields

$$D'_x = D_x + \sum_{i=1}^{k} f^i \frac{\partial}{\partial z^i}, \qquad D'_y = D_y + \sum_{i=1}^{k} g^i \frac{\partial}{\partial z^i} \tag{3}$$

commute (which is a geometric way of saying that Equations (2) are compatible), then $\mathcal{E}'$ equipped with the vector fields (3) is a k-dimensional covering over $\mathcal{E}$. Recall from [15] that every finite-dimensional covering is locally of this form.

Two coverings $\mathcal{E}'$ and $\mathcal{E}''$ are said to be *isomorphic over* $\mathcal{E}$ if there exists an isomorphism of the equations $\mathcal{E}' \cong \mathcal{E}''$ that commutes with the projections to $\mathcal{E}$. Isomorphic coverings result from invertible transformations of nonlocal variables. A k-dimensional covering is said to be *trivial* if it is isomorphic to one with $f^i = g^i = 0$; such a covering is essentially a family of identical copies of $\mathcal{E}$.

The simplest yet useful covering (2) may be associated with a single nontrivial conservation law $\alpha = f\,dx + g\,dy$, i.e., a pair of functions f, g defined on $\mathcal{E}$ and satisfying $D_y f = D_x g$ on $\mathcal{E}$:

Definition 1 A *one-dimensional abelian covering* associated with a conservation law $\alpha = f\,dx + g\,dy$ is defined to be the trivial vector bundle $\mathcal{E} \times \mathbb{R} \to \mathcal{E}$, equipped with total derivatives

$$D'_x = D_x + f\frac{\partial}{\partial z}, \qquad D'_y = D_y + g\frac{\partial}{\partial z},$$

where z denotes the coordinate along $\mathbb{R}$.

As f, g do not depend on z, the vector fields D'_x, D'_y on $\mathcal{E}'$ commute if and only if $D_y f = D_x g$. The variable z is called the *potential* of the conservation law α. We have $D'_x z = f$, $D'_y z = g$ or briefly $z_x = f$, $z_y = g$.

Recall that a conservation law is said to be *trivial* if there exists a (local) function h on $\mathcal{E}$ such that $f = D_x h$, $g = D_y h$. A covering associated to a trivial conservation law is isomorphic to a trivial covering through the invertible change of variables $z = z' + h$.

A covering $\bar{\mathcal{E}} \to \mathcal{E}$ is said to be *trivializing* for a conservation law $\alpha = f\,dx + g\,dy$, if the pullback $\bar{\alpha}$ of α along the projection $\bar{\mathcal{E}} \to \mathcal{E}$ is a trivial conservation law on $\bar{\mathcal{E}}$. Obviously, the one-dimensional abelian covering associated with the conservation law α trivializes α.

A general n-dimensional abelian covering is obtained by repeating the construction of the one-dimensional abelian covering (cf. [40, Sect. IV]):

Definition 2 An n-dimensional covering $\widetilde{\mathcal{E}}$ over $\mathcal{E}$ is said to be *abelian*, if

(1) either $n = 1$ and $\widetilde{\mathcal{E}}$ is a one-dimensional abelian covering over $\mathcal{E}$ in the sense of Definition 1;

(2) or $\widetilde{\mathcal{E}}$ is a one-dimensional abelian covering over an $(n-1)$-dimensional abelian covering $\mathcal{E}'$ over $\mathcal{E}$.

Let us remark that Khorkova [13] introduced the *universal abelian covering*, which need not be finite-dimensional.

3 Zero-curvature representations

Simplest pseudopotentials that are not potentials are associated with non-degenerate zero-curvature representations. Let $\mathfrak{g}$ be a matrix Lie algebra (recall that according to the Ado theorem every finite-dimensional Lie algebra has a matrix representation). By a $\mathfrak{g}$-*valued zero-curvature representation* (ZCR) for $\mathcal{E}$ we mean a $\mathfrak{g}$-valued one-form $\alpha = A\,dx + B\,dt$ defined on $\mathcal{E}$ such that

$$D_y A - D_x B + [A, B] = 0 \tag{4}$$

holds on $\mathcal{E}$, which means that (4) holds as a consequence of system (1) (we do not insist that (4) necessarily reproduces system (1), which is normally required in integrability theory).

Let $\mathcal{G}$ be the connected and simply connected matrix Lie group associated with $\mathfrak{g}$. Then for an arbitrary $\mathcal{G}$-valued function S, the form $\alpha^S = A^S\,dx + B^S\,dt$, where

$$A^S = D_x S S^{-1} + S A S^{-1}, \qquad B^S = D_y S S^{-1} + S B S^{-1} \tag{5}$$

is another ZCR, which is said to be *gauge equivalent* to the former.

A ZCR is said to be *trivial* if it is gauge equivalent to zero, i.e., if $A = D_x S S^{-1}$, $B = D_y S S^{-1}$. A covering $\bar{\mathcal{E}} \to \mathcal{E}$ is said to *trivialize* a ZCR $\alpha = A\,dx + B\,dy$ if the pullback $\bar{\alpha}$ of α along the projection $\bar{\mathcal{E}} \to \mathcal{E}$ is a trivial ZCR.

A trivializing covering for the ZCR α can be obtained in the following way.

Proposition 3 *For every $\mathfrak{g}$-valued ZCR α on $\mathcal{E}$ there exists a covering $\pi_\alpha : \widetilde{\mathcal{E}}_\alpha \to \mathcal{E}$ that trivializes α.*

Proof Let $\alpha = A\,dx + B\,dy$ be a ZCR, where A and B are $n \times n$ matrices belonging to the algebra $\mathfrak{g}$. Put $\widetilde{\mathcal{E}}_\alpha = \mathcal{E} \times G$, where G is the matrix Lie group associated with $\mathfrak{g}$. Given an element $C \in \mathfrak{g}$, let us denote by ξ_C the right-invariant vector field on G corresponding to C. Given a $\mathfrak{g}$-valued function C on $\mathcal{E}$, let us denote by Ξ_C the unique vector field on $\widetilde{\mathcal{E}}_\alpha$ with the $\mathcal{E}$-component zero and the G-component equal to ξ_C, at each point of $\widetilde{\mathcal{E}}_\alpha$. Considering the vector fields

$$\widetilde{D}_x = D_x + \Xi_A, \qquad \widetilde{D}_y = D_y + \Xi_B$$

on $\widetilde{\mathcal{E}}_\alpha$, where D_x, D_y are the total derivatives on $\mathcal{E}$, let us show that $\widetilde{D}_x, \widetilde{D}_y$ are the total derivatives for a trivializing covering $\pi_\alpha : \widetilde{\mathcal{E}}_\alpha \to \mathcal{E}$ of α.

Let $A = (a_{ij})$, $B = (b_{ij})$. Let us first consider $G = \mathrm{GL}_n$ with its natural parametrization $\mathrm{GL}_n = \{(z_{ij}) \mid \det z_{ij} \ne 0\}$. We have

$$\Xi_A = \sum_{i,j,l} a_{ij} z_{jl} \frac{\partial}{\partial z_{il}}, \qquad \Xi_B = \sum_{i,j,l} b_{ij} z_{jl} \frac{\partial}{\partial z_{il}}.$$

Then $\widetilde{D}_x, \widetilde{D}_y$ commute since

$$\begin{aligned}
[\widetilde{D}_x, \widetilde{D}_y] &= [D_x, D_y] + [D_x, \Xi_B] - [\Xi_A, D_y] + [\Xi_A, \Xi_B] \\
&= \Xi_{D_x B - D_y A - [A,B]} \\
&= 0.
\end{aligned}$$

The same holds for arbitrary $G \subseteq \mathrm{GL}_n$, since the vector fields Ξ_A, Ξ_B are tangent to G whenever A, B belong to $\mathfrak{g}$.

Now denote by Θ the projection $\widetilde{\mathcal{E}}_\alpha = \mathcal{E} \times G \to G$ viewed as a matrix-valued function on $\widetilde{\mathcal{E}}_\alpha$. Then $D_x\Theta = 0$ and therefore

$$(\widetilde{D}_x\Theta)_{\mu\nu} = (\Xi_A\Theta)_{\mu\nu} = \sum_{i,j,l} a_{ij} z_{jl} \frac{\partial}{\partial z_{il}} z_{\mu\nu} = \sum_j a_{\mu j} z_{j\nu} = (A\Theta)_{\mu\nu}.$$

Thus, $\widetilde{D}_x\Theta \cdot \Theta^{-1} = A$ and similarly $\widetilde{D}_y\Theta \cdot \Theta^{-1} = B$, whence the pullback of α on $\widetilde{\mathcal{E}}_\alpha$ is trivial.

The system (2) corresponding to $\widetilde{\mathcal{E}}_\alpha$ can be compactly written in terms of a single matrix Θ as

$$\Theta_x = A\Theta, \qquad \Theta_y = B\Theta. \tag{6}$$

Under the gauge transformation (5), the matrix Θ becomes $S\Theta$. The coverings $\widetilde{\mathcal{E}}_\alpha$ and $\widetilde{\mathcal{E}}_{\alpha^S}$ are isomorphic via $\Theta \mapsto S\Theta$.

The trivializing covering π_α just constructed has the following factorization property:

Proposition 4 *Let $p : \mathcal{E}' \to \mathcal{E}$ over M be a trivializing covering for a ZCR α on $\mathcal{E}$. Then there exists a morphism $p^\sharp : \mathcal{E}' \to \widetilde{\mathcal{E}}_\alpha$ such that $\pi_\alpha \circ p^\sharp = p$.*

Proof Let $\alpha = A\,dx + B\,dy$. Since p is over M, we have $p^*\alpha = p^*A\,dx + p^*B\,dy$. By assumption this is a trivial ZCR, whence $p^*A = D'_x S S^{-1}$ and $p^*B = D'_y S S^{-1}$ for a suitable G-valued function S on $\mathcal{E}'$. Recall that fibres of the covering $\widetilde{\mathcal{E}}_\alpha$ are diffeomorphic to the Lie group G. Therefore we can define a mapping $p^\sharp : \mathcal{E}' \to \widetilde{\mathcal{E}}_\alpha$ by the formula $\Theta \circ p^\sharp = S$, where, as above, Θ denotes the projection $\widetilde{\mathcal{E}}_\alpha = \mathcal{E} \times G \to G$. The mapping $p^\sharp$ is a morphism, since $(\Theta \circ p^\sharp)_x = S_x = AS = A\Theta \circ p^\sharp$.

4 Lower triangular ZCR's

Let $\mathfrak{t}_n$ denote the algebra of matrices

$$\begin{pmatrix} a_{11} & 0 & \cdot & \cdot & 0 \\ a_{21} & a_{22} & 0 & \cdot & \cdot \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdot & a_{nn} \end{pmatrix}. \tag{7}$$

Denote by $\mathfrak{t}_n^{(k)}$, $k \geq 1$, the derived algebra formed by matrices satisfying $a_{ij} = 0$ whenever $i - j < k$.

ZCR's with values in $\mathfrak{t}_n$ are, in a sense, equivalent to an abelian covering.

Proposition 5 *Every $\mathfrak{t}_n$-valued ZCR can be trivialized by means of an abelian covering of dimension $\leq \frac{1}{2}n(n+1)$.*

Proof Let $\alpha = A\,dx + B\,dy$ be a ZCR such that matrices A and B are lower triangular. We shall construct an abelian covering $\mathcal{E}^{(n-1)}$ in n steps.

It follows from Equation (4) that $\gamma_1 = a_{11}\,dx + b_{11}\,dy$, $\gamma_2 = a_{22}\,dx + b_{22}\,dy$, $\ldots$, $\gamma_n = a_{nn}\,dx + b_{nn}\,dy$ are conservation laws. Let us denote by $\mathcal{E}^{(0)}$ the associated abelian covering with potentials $h_1, \ldots, h_n$ satisfying

$$h_{i,x} = a_{ii}, \quad h_{i,y} = b_{ii} \qquad \text{for } i = 1, \ldots, n.$$

Then

$$H = \begin{pmatrix} \mathrm{e}^{-h_1} & 0 & 0 & \cdot & 0 \\ 0 & \mathrm{e}^{-h_2} & 0 & \cdot & 0 \\ 0 & 0 & \mathrm{e}^{-h_3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \mathrm{e}^{-h_n} \end{pmatrix},$$

is a matrix defined on $\mathcal{E}^{(0)}$, with the property that all diagonal entries of the gauge equivalent matrix $A' = A^H$ vanish:

$$A' = \begin{pmatrix} 0 & \cdot & \cdot & 0 & 0 \\ a'_{21} & 0 & \cdot & \cdot & 0 \\ a'_{31} & a'_{32} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & a'_{n,n-1} & 0 \end{pmatrix}, \tag{8}$$

and similarly for B'. Hence, A', B' take values in $t_n^{(1)}$.

By the same Equation (4), $\gamma'_2 = a'_{21}\,dx + b'_{21}\,dy$, $\gamma'_3 = a'_{32}\,dx + b'_{32}\,dy$, $\ldots$, $\gamma'_n = a'_{n-1,n}\,dx + b'_{n-1,n}\,dy$ are conservation laws on $\mathcal{E}^{(0)}$. Let us introduce a covering $\mathcal{E}'$ over $\mathcal{E}^{(0)}$ with potentials $h'_2, \ldots, h'_n$ satisfying

$$h'_{i,x} = a'_{i,i-1}, \quad h'_{i,y} = b'_{i,i-1} \quad \text{for } i = 2, \ldots, n.$$

Denoting

$$H' = \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 \\ -h'_2 & 1 & \cdot & \cdot & 0 \\ 0 & -h'_3 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & -h'_n & 1 \end{pmatrix},$$

we see that the gauge equivalent matrices $A'' = A'^{H'}$ and $B'' = B'^{H'}$ take values in $t_n^{(2)}$ now. Compared with (8), A'' and B'' have one more subdiagonal of zeroes. The next step is similar: $\gamma''_3 = a''_{31}\,dx + b''_{31}\,dy$, $\gamma''_4 = a''_{42}\,dx + b''_{42}\,dy$, $\ldots$, $\gamma''_n = a''_{n-2,n}\,dx + b''_{n-2,n}\,dy$ are conservation laws on $\mathcal{E}'$. Let us introduce a covering $\mathcal{E}''$ over $\mathcal{E}'$ with potentials $h''_2, \ldots, h''_n$ satisfying

$$h''_{i,x} = a''_{i,i-2}, \quad h''_{i,y} = b''_{i,i-2} \quad \text{for } i = 3, \ldots, n.$$

Denoting

$$H'' = \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 \\ -h''_3 & 0 & 1 & \cdot & \cdot & 0 \\ 0 & -h''_4 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & -h''_n & 0 & 1 \end{pmatrix}$$

we observe that $A''' = A''^{H''}$, $B''' = B''^{H''}$ take values in $t_n^{(3)}$, and so on. Continuing the process until $A^{(n)}$, $B^{(n)}$ become zero, we end up with a sequence of $\frac{1}{2}n(n+1)$

conservation laws

$$
\begin{array}{ccccc}
\gamma_1 & \gamma_2 & \gamma_3 & \dots & \gamma_n \\
 & \gamma_2' & \gamma_3' & \dots & \gamma_n' \\
 & & \gamma_3'' & \dots & \gamma_n'' \\
 & & & \dots & \\
 & & & \gamma_{n-1}^{(n-2)} & \gamma_n^{(n-2)} \\
 & & & & \gamma_n^{(n-1)},
\end{array}
\tag{9}
$$

where (a) $\gamma_1, \dots, \gamma_n$ are conservation laws on $\mathcal{E}$; (b) $\gamma_{n-\iota+1}^{(n-\iota)}, \dots, \gamma_n^{(n-\iota)}$ are conservation laws defined on the abelian covering $\mathcal{E}^{(n-\iota-1)}$ associated with the conservation laws of all the previous levels.

Finally, $\alpha^{HH'\dots H^{(n-1)}} = \alpha^{(n)} = 0$, where each $H^{(\iota)}$ is defined on $\mathcal{E}^{(\iota)}$. Summing up, the covering $\mathcal{E}^{(n-1)}$ trivializes α.

The sequence (9) will be called an *n-fold chain of conservation laws.*

Proposition 6 *Let α be a $\mathfrak{t}_n$-valued ZCR, then the associated covering π_α is isomorphic to an abelian covering of dimension $\leq \frac{1}{2}n(n+1)$.*

Proof According to Proposition 5, there is an abelian covering $p : \mathcal{E}^{(n-1)} \to \mathcal{E}$ that is trivializing for α; namely, we have $\alpha^K = 0$, where $K = HH' \cdots H^{(n-1)}$ (see proof of Proposition 5). Hence, $\alpha = 0^{K^{-1}}$ and, according to Proposition 4, there is a morphism $p^\sharp : \mathcal{E}^{(n-1)} \to \widetilde{\mathcal{E}}_\alpha$, given by $\Theta = K^{-1}$. Here Θ represents the totality of coordinates along the fibres of the covering $\widetilde{\mathcal{E}}_\alpha$, while K is parametrised by coordinates $h_s^{(\iota)}$ along the fibres of the covering $\mathcal{E}^{(n-1)}$. It follows that $p^\sharp$ is bijective on the fibres, hence isomorphism.

5 Reducibility

A $\mathfrak{g}$-valued ZCR is said to be *reducible* if it is gauge equivalent to a ZCR taking values in a proper subalgebra $\mathfrak{h} \subset \mathfrak{g}$; otherwise it is said to be *irreducible*.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. We present a simple criterion for reducibility of a $\mathfrak{g}$-valued ZCR to $\mathfrak{h}$. Let $\mathcal{H} \subset \mathcal{G}$ be the Lie subgroup corresponding to the subalgebra $\mathfrak{h}$. We call $\mathcal{H}$ a *right factor* if there exists a submanifold $\mathcal{K} \subset \mathcal{G}$ (possibly with singularities) such that the multiplication map

$$
\mu : \mathcal{K} \times \mathcal{H} \to \mathcal{G}, \qquad (K, H) \mapsto KH \tag{10}
$$

is a surjective local diffeomorphism. The manifold $\mathcal{K}$ will be called a *cofactor*. By surjectivity, every element $S \in \mathcal{G}$ can be decomposed as a product $S = KH$, where $K \in \mathcal{K}$ and $H \in \mathcal{H}$, possibly non-uniquely. The map μ being a local diffeomorphism, $\mathcal{K}$ has the minimal possible dimension $\dim \mathcal{K} = \dim \mathcal{G} - \dim \mathcal{H}$. If $\mathcal{H}$ is closed, then the assignment $K \mapsto K\mathcal{H}$ defines a local diffeomorphism of $\mathcal{K}$ onto the homogeneous space $\mathcal{G}/\mathcal{H}$.

Proposition 7 *Under the above notation, a $\mathfrak{g}$-valued ZCR α on $\mathcal{E}$ is reducible to the subalgebra $\mathfrak{h}$ if and only if there exists a local $\mathcal{K}$-valued matrix function K on $\mathcal{E}$ such that α^K lies in $\mathfrak{h}$.*

Proof The gauge equivalence with respect to $H \in \mathcal{H}$ preserves the subalgebra $\mathfrak{h}$. Therefore, the gauge-equivalent ZCR $\alpha^S = (\alpha^K)^H$ lies in $\mathfrak{h}$ if and only if α^K lies in $\mathfrak{h}$.

Otherwise said, if a ZCR is reducible to $\mathfrak{h}$, then the corresponding gauge matrix can be found in $\mathcal{K}$. Understandably, different choices of the cofactor $\mathcal{K}$ may lead to different reducibility criteria.

In this paper we are primarily interested in detecting reducibility to a solvable subalgebra. By the well-known Lie theorem ([9, Sect. 9.2]), every finite-dimensional representation of a solvable Lie algebra is equivalent to a representation by lower triangular matrices. Hence, every ZCR reducible to a solvable subalgebra is reducible to $\mathfrak{t}_n$ (and can be trivialized using an abelian covering according to Proposition 5).

Let us therefore apply Proposition 7 to $\mathfrak{h} = \mathfrak{t}_n$. There are two standard ways to make $\mathfrak{t}_n$ a right factor in $\mathfrak{gl}_n$.

The QR or Gram decomposition is an alternative formulation of the famous Gram–Schmidt ortogonalization algorithm. Namely, every $n \times n$ complex matrix A can be decomposed as a product $A = QR$, where $Q \in \mathrm{SU}_n$ and $R \in \mathfrak{t}_n$ [25, Ch. 6, Sect. 1.9]. Proposition 7 then yields

Proposition 8 *A real (complex) ZCR α on $\mathcal{E}$ is reducible to lower triangular if and only if there exists an SO_n-valued (SU_n-valued) function K on $\mathcal{E}$ such that α^K is lower triangular.*

However, the factors Q and R are unique up to a unimodular diagonal multiplier: $QR = Q\Theta \cdot \Theta^{-1}R$, where $\Theta = \operatorname{diag}(\theta_1, \dots, \theta_n) \in \mathrm{S}(\mathrm{U}_1 \times \cdots \times \mathrm{U}_1)$, i.e., $|\theta_\iota| = 1$ and $\prod_{\iota=1}^n \theta_\iota = 1$. Thus, the mapping (10) is not a local diffeomorphism unless it is restricted to a suitable immersion of the orbit space $\mathrm{SU}_n/\mathrm{S}(\mathrm{U}_1 \times \cdots \times \mathrm{U}_1)$ into SU_n. In the real case we have $\theta_\iota = \pm 1$ and we get a 2^{n-1}-to-one local diffeomorphism (10) with $\mathcal{K} = \mathrm{SO}_n$.

The LU or Gauss decomposition can be derived from the Gaussian elimination algorithm. The following result is well known ([25, Ch. 6, Sect. 1.8]):

Proposition 9 *For every non-singular matrix A there exist matrices P, U, L such that $A = PUL$, L is lower triangular, U is upper triangular with diagonal entries equal to 1, and P is a permutation matrix. The matrix P can be omitted if and only if all principal minors of the matrix A are nonzero.*

(Recall that Gaussian elimination may require row swapping, which is where the permutation matrix P comes from.) Let $\mathcal{K}$ denote the set of all products PU where P is a permutation matrix and U is an upper triangular matrix with diagonal entries equal to 1. Then K is a union of $n!$ intersecting submanifolds, labelled by permutation matrices P. Compared to the QR-decomposition, each of the $n!$ submanifolds is easier to parametrize than SO or SU.

Proposition 10 *A ZCR α on $\mathcal{E}$ is reducible to lower triangular if and only if there exists a permutation matrix P and a matrix-valued function*

$$H = \begin{pmatrix} 1 & h_{12} & h_{13} & \cdot & \cdot \\ 0 & 1 & h_{23} & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 1 \end{pmatrix} \tag{11}$$

on $\mathcal{E}$ such that α^{PH} is lower triangular.

Before looking more closely at low values of n, we make a general remark to the effect that every $\mathfrak{gl}_n$-valued ZCR is reducible to $\mathfrak{sl}_n$:

Remark 11 A $\mathfrak{gl}_n$-valued ZCR is decomposable into an $\mathfrak{sl}_n$-valued ZCR (traceless summand) and a conservation law (the trace).

5.1 The case of $n = 2$

When $n = 2$, the reducibility condition corresponding to the QR decomposition is:

Proposition 12 *A $\mathfrak{gl}_2$-valued ZCR*

$$\alpha = A\,dx + B\,dy = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} dx + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} dy$$

is reducible to lower triangular if and only if there exists a function ϕ on $\mathcal{E}$ that is a solution of the system

$$\begin{aligned} D_x\phi &= -a_{12}\cos^2\phi + (a_{11} - a_{22})\sin\phi\cos\phi + a_{21}\sin^2\phi, \\ D_y\phi &= -b_{12}\cos^2\phi + (b_{11} - b_{22})\sin\phi\cos\phi + b_{21}\sin^2\phi. \end{aligned} \tag{12}$$

Proof An arbitrary SO_2 matrix is

$$K = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

By Proposition 8, the ZCR α is reducible to lower triangular if and only if α^K is lower triangular, which is exactly the meaning of conditions (12).

The reducibility conditions corresponding to the LU decomposition are:

Proposition 13 *A $\mathfrak{gl}_2$-valued ZCR*

$$\alpha = A\,dx + B\,dy = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} dx + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} dy$$

on $\mathcal{E}$ is reducible to lower triangular if and only if

1. *either there exists a local function p on $\mathcal{E}$ such that*

$$\begin{aligned} D_x p &= -a_{12} + (a_{11} - a_{22})p + a_{21}p^2, \\ D_y p &= -b_{12} + (b_{11} - b_{22})p + b_{21}p^2; \end{aligned} \tag{13}$$

2. *or A, B are upper triangular:*

$$a_{21} = b_{21} = 0.$$

Proof An arbitrary $\mathcal{K}$-valued function is $K = PU$, where

$$U = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

and P is one of the two 2×2 permutation matrices

$$P_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Subcases 1 and 2 correspond to the choices $P = P_{12}$ and $P = P_{21}$, respectively, and express the conditions that A^{PU}, B^{PU} be lower triangular.

Recall that a *quadratic* or *Riccati pseudopotential* p associated to an $\mathfrak{gl}_2$-valued ZCR α is defined by the compatible system

$$\begin{aligned} p_x &= -a_{12} + (a_{11} - a_{22})p + a_{21}p^2, \\ p_y &= -b_{12} + (b_{11} - b_{22})p + b_{21}p^2, \end{aligned} \tag{14}$$

which are essentially Equations (13). The system (14) being compatible, let us introduce the corresponding one-dimensional *Riccati covering*. Proposition 13 then says that a non-upper-triangular ZCR is reducible to lower triangular if and only if the Riccati covering has a local section.

Remark 14 Obviously, Equations (13) and (12) are not independent, the explicit mapping of their solutions being $p = \tan\phi$ for $\phi \neq (2k+1)\pi/2$. Recently Reyes [30] pointed out a geometric interpretation of the same correspondence in terms of pseudospherical equations.

Example 15 The Burgers equation $u_t = u_{xx} + uu_x$ is well known to be integrable via the Cole–Hopf transformation, which relates its solutions with those of the heat equation [1, Sect. 3.1]. Of the several Lax pairs that have been found all turn out to be degenerate. Let us consider one example [26, 42], where the lower triangular representation could not be obtained by purely Lie algebraic methods:

$$\begin{aligned}\alpha &= \begin{pmatrix} 0 & 1 \\ -\frac14 u_x + \frac{1}{16}(u+\lambda)^2 & 0 \end{pmatrix} dx \\ &\quad + \begin{pmatrix} -\frac14 u_x & \frac12(u-\lambda) \\ -\frac14 u_{xx} - \frac18(u-\lambda)u_x + \frac{1}{32}(u-\lambda)(u+\lambda)^2 & \frac14 u_x \end{pmatrix} dt\end{aligned}$$

In this case, Equations (13) have a local solution $p = 4/(u+\lambda)$, hence

$$\begin{pmatrix} 1 & 4/(u+\lambda) \\ 0 & 1 \end{pmatrix}$$

is a gauge matrix to make the ZCR α lower triangular.

That the Burgers equation has no irreducible $\mathfrak{gl}_2$-valued ZCR follows from the recent classification of second-order evolution equations possessing an $\mathfrak{sl}_2$-valued ZCR [23] and Remark 11. The non-existence of an irreducible ZCR of Wahlquist–Estabrook type for arbitrary n was proved Dodd and Fordy [8] who established solvability of the Wahlquist–Estabrook prolongation algebra of the Burgers (and also of the Kaup) equation.

Example 16 The Calogero–Nucci example [3] of a ZCR that exists for every equation possessing a conservation law $f_t = g_x$:

$$\begin{aligned}&\begin{pmatrix} 0 & 1 \\ \eta\dfrac{f_x}{f} + \lambda f^2 + \eta\mu f - \eta^2 & \dfrac{f_x}{f} + \mu f - 2\eta \end{pmatrix} dx \\ &\quad + \begin{pmatrix} \eta\dfrac{g}{f} + \nu & \dfrac{g}{f} \\ \dfrac{\eta g_x}{f} + \lambda f g + \eta\mu g - \eta^2\dfrac{g}{f} & \dfrac{g_x}{f} + \mu g - \eta\dfrac{g}{f} + \nu \end{pmatrix} dy\end{aligned} \tag{15}$$

where η, λ, μ, ν are arbitrary constants. This ZCR is reducible, which follows from Proposition 13 along with explicit formulas for its reduction. Indeed, we have Subcase 1 again and one easily finds a local solution

$$p = \frac12 \frac{(\mu + \sqrt{\mu^2 + 4\lambda})f - 2\eta}{\lambda f^2 + \eta\mu f - \eta^2}$$

of Equations (13). Hence, the above ZCR is reducible to lower triangular.

Continuing the reduction further, one finally arrives at an abelian subalgebra. Namely, if p is as above and

$$
\begin{aligned}
q &= \frac{\lambda f^2 + \eta\mu f - \eta^2}{\sqrt{\mu^2 + 4\lambda}\, f}, \\
r &= \frac{(\lambda f^2 + \eta\mu f - \eta^2)\big(2\lambda f + (\mu - \sqrt{\mu^2 + 4\lambda})\eta\big)}{2\lambda f + (\mu + \sqrt{\mu^2 + 4\lambda})\eta},
\end{aligned}
$$

then the product of gauge matrices

$$
\begin{pmatrix} \sqrt{r}/f & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}
$$

takes the ZCR to the diagonal form

$$
\begin{aligned}
&\begin{pmatrix} \frac12(\mu - \sqrt{\mu^2 + 4\lambda}) f - \eta & 0 \\ 0 & \frac12(\mu + \sqrt{\mu^2 + 4\lambda}) f - \eta \end{pmatrix} dx \\
&\qquad + \begin{pmatrix} \frac12(\mu - \sqrt{\mu^2 + 4\lambda}) g + \nu & 0 \\ 0 & \frac12(\mu + \sqrt{\mu^2 + 4\lambda}) g + \nu \end{pmatrix} dy,
\end{aligned}
$$

which is manifestly equivalent to the conservation law $f\,dx + g\,dt$.

5.2 The case of $n \geq 3$

For $n \geq 3$, the QR approach is impractical due to relative clumsiness of the parametrisation of SO_n by generalized Euler angles. On the other hand, the LU criteria come out subdivided into as much as $n!$ subcases, one for each of the $n!$ permutation matrices P.

For every n, the case of general position occurs when all principal minors of the gauge matrix K are nonzero. Then the permutation matrix P equals the identity matrix and we can derive explicit formulas that generalize (13) to arbitrary n.

Proposition 17 *A $\mathfrak{gl}_n$-valued ZCR $\alpha = A\,dx + B\,dy$, where $A = (a_{ij})$ and $B = (b_{ij})$, is reducible to lower triangular by means of a gauge matrix with nonzero principal minors* *if and only if the system*

$$\begin{aligned} D_x p_{kl} = & - \sum_{\substack{0\le r\le n-1 \\ i_0<i_1<\dots<i_r=l}} (-1)^r a_{ki_0} p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{r-1} i_r} \\ & - \sum_{\substack{0\le r\le n-1 \\ k<j \\ i_0<i_1<\dots<i_r=l}} (-1)^r p_{kj} a_{j i_0} p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{r-1} i_r}, \\ D_y p_{kl} = & - \sum_{\substack{0\le r\le n-1 \\ i_0<i_1<\dots<i_r=l}} (-1)^r b_{ki_0} p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{r-1} i_r} \\ & - \sum_{\substack{0\le r\le n-1 \\ k<j \\ i_0<i_1<\dots<i_r=l}} (-1)^r p_{kj} b_{j i_0} p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{r-1} i_r} \end{aligned} \tag{16}$$

on $\frac{1}{2}(n-1)n$ *unknown functions* p_{kl}*,* $k < l$*, has a local solution.*

Proof According to Proposition 9, every gauge matrix S with nonzero principal minors decomposes as $S = LU$, with L lower triangular and

$$U = \begin{pmatrix} 1 & p_{12} & p_{13} & \dots & p_{1n} \\ 0 & 1 & p_{23} & \dots & p_{2n} \\ 0 & 0 & 1 & \dots & p_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

The inverse of U is

$$U^{-1} = \begin{pmatrix} 1 & q_{12} & q_{13} & \dots & q_{1n} \\ 0 & 1 & q_{23} & \dots & q_{2n} \\ 0 & 0 & 1 & \dots & q_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where

$$
\begin{aligned}
q_{ij} &= \sum_{\substack{1\le r\le n-1\\ i=i_0<i_1<\cdots<i_r=j}} (-1)^r p_{i_0i_1}p_{i_1i_2}\dots p_{i_{r-1}i_r}\\
&= (-1)^{i+j}\begin{vmatrix}
p_{i,i+1} & p_{i,i+2} & \cdot & \cdot & p_{i,j-1} & p_{i,j}\\
1 & p_{i+1,i+2} & p_{i+1,i+3} & \cdot & \cdot & p_{i+1,j}\\
0 & 1 & p_{i+2,i+3} & p_{i+2,i+4} & \cdot & \cdot\\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot\\
0 & \cdot & \cdot & \cdot & p_{j-2,j-1} & p_{j-2,j}\\
0 & 0 & \cdot & \cdot & 1 & p_{j-1,j}
\end{vmatrix},
\end{aligned}
$$

since $q_{kl} + \sum_{k<i<l} p_{ki}q_{il} + p_{kl} = 0$ whenever $k < l$. Let us consider the gauge equivalent matrix $A^U = U_xU^{-1} + UAU^{-1}$. Terms that contain total derivatives D_xp_{ij} can occur only in the first summand, which is

$$
U_xU^{-1} = \begin{pmatrix}
0 & z_{12} & z_{13} & \dots & z_{1n}\\
0 & 0 & z_{23} & \dots & z_{2n}\\
0 & 0 & 0 & \dots & z_{3n}\\
\vdots & \vdots & \vdots & & \vdots\\
0 & 0 & 0 & \dots & 0
\end{pmatrix},
$$

where

$$
\begin{aligned}
z_{kl} &= \sum_{\substack{1\le r\le n-1\\ k=i_0<i_1<\cdots<i_r=l}} (-1)^{r-1} D_xp_{i_0i_1}\cdot p_{i_1i_2}\dots p_{i_{r-1}i_r}\\
&= (-1)^{k+l+1}\begin{vmatrix}
D_xp_{k,k+1} & D_xp_{k,k+2} & \cdot & D_xp_{k,l-1} & D_xp_{k,l}\\
1 & p_{k+1,k+2} & p_{k+1,k+3} & \cdot & p_{k+1,l}\\
0 & 1 & p_{k+2,k+3} & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & \cdot & 1 & p_{l-1,l}
\end{vmatrix}
\end{aligned}
$$

for all $k < l$. Denoting $A^U =: A' = (a'_{ij})$, we have

$$
a'_{kl} := z_{kl} + a_{kl} + \sum_{j<l} a_{kj}q_{jl} + \sum_{k<ij<l} p_{ki}a_{ij}q_{jl} + \sum_{k<i} p_{ki}a_{il}.
$$

The condition of A' being lower triangular, $a'_{kl} = 0$ for all $k < l$, constitutes a system of equations in total derivatives D_xp_{ij}. The equivalent system resolved with respect to the derivatives is $a'_{kl} + \sum_{k<h<l} a'_{kh}p_{hl} = 0$, since derivatives occur only in the summands containing z_{ij}, which are $z_{kl} + \sum_{k<h<l} z_{kh}p_{hl} = D_xp_{kl}$. The remaining summands then simplify to the expressions given in the statement of the proposition.

5.3 The generalized Riccati covering

A tedious computation shows that Equations (16) are compatible, meaning that there are no integrability conditions resulting from the equalities $D_{xy}p_{kl} = D_{yx}p_{kl}$. This implies the existence of a covering associated with a ZCR which naturally generalizes the Riccati covering. Similar result holds for more general types of decomposition, too.

Let a subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ be a direct complement to the subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ considered throughout this section. Let $\mathrm{pr}_{\mathfrak{k}} : \mathfrak{g} \to k$ be the corresponding projection. Then the condition $\alpha^K \in \mathfrak{h}$ (Proposition 7) can be equivalently written as $\mathrm{pr}_{\mathfrak{k}}\alpha^K = 0$.

Denoting by $\mathcal{K}$ the Lie connected and simply connected matrix Lie group associated with the subalgebra $\mathfrak{k}$, we have

Proposition 18 *Under the above notation, the differential equations*

$$\begin{aligned}\mathrm{pr}_{\mathfrak{k}}(K_x K^{-1} + KAK^{-1}) = 0,\\ \mathrm{pr}_{\mathfrak{k}}(K_y K^{-1} + KBK^{-1}) = 0\end{aligned}\tag{17}$$

on a matrix $K \in \mathcal{K}$ are compatible.

Proof Since $K \in \mathcal{K}$, matrices $K_x K^{-1}, K_y K^{-1}$ belong to $\mathfrak{k}$ and are mapped identically under the projection $\mathrm{pr}_{\mathfrak{k}}$. Hence Equations (17) are differential equations on K and, moreover, can be resolved with respect to K_x, K_y. Let us consider their derivatives

$$\begin{aligned}0 &= D_y \mathrm{pr}_{\mathfrak{k}}(K_x K^{-1} + KAK^{-1})\\ &= \mathrm{pr}_{\mathfrak{k}}(K_{xy} K^{-1} - KBAK^{-1} + KA_y K^{-1}),\\ 0 &= D_x \mathrm{pr}_{\mathfrak{k}}(K_y K^{-1} + KBK^{-1})\\ &= \mathrm{pr}_{\mathfrak{k}}(K_{yx} K^{-1} - KABK^{-1} + KB_x K^{-1}),\end{aligned}$$

where we have made substitutions $\mathrm{pr}_{\mathfrak{k}} K_x K^{-1} \rightsquigarrow -\mathrm{pr}_{\mathfrak{k}} KAK^{-1}$ and $\mathrm{pr}_{\mathfrak{k}} K_y K^{-1} \rightsquigarrow -\mathrm{pr}_{\mathfrak{k}} KBK^{-1}$ according to (17). These equations can also be resolved with respect to K_{xy} and K_{yx}, respectively. Now one can perform the standard check that K_{xy} coincides with K_{yx}:

$$\mathrm{pr}_{\mathfrak{k}}(K_{xy} - K_{yx})K^{-1} = \mathrm{pr}_{\mathfrak{k}}(A_y - B_x + AB - BA) = 0.$$

Definition 19 Given a ZCR α of an equation $\mathcal{E}$ and the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ as above, we define the associated *generalized Riccati covering* as $\mathcal{E} \times \mathcal{K} \to \mathcal{E}$, assuming that the corresponding matrix of pseudopotentials $K \in \mathcal{K}$ satisfies Equations (17).

Summing up, we obtain:

Corollary 20 *A $\mathfrak{gl}_n$-valued ZCR α is reducible to lower triangular by means of a gauge matrix from $\mathcal{K}$ if and only if the generalized Riccati covering associated with the decomposition $\mathfrak{gl}_n = \mathfrak{t}_n + \mathfrak{k}$ has a local section.*

Choosing $\mathfrak{k}$ to be the Lie algebra of strictly upper triangular matrices, we have:

Corollary 21 *A $\mathfrak{gl}_n$-valued ZCR α is reducible to lower triangular by means of a gauge matrix with nonzero principal minors if and only if there exists a local solution to Equations (16).*

6 Guthrie's formulation of recursion operators

In 1994, G.A. Guthrie [11] suggested a general definition of a recursion operator, free of some weaknesses of the then standard definition in terms of integro-differential operators. Geometrically, Guthrie's recursion operator for an equation $\mathcal{E}$ is a Bäcklund autotransformation for the linearized equation $V\mathcal{E}$ ([22]).

In geometrical terms, the linearization $V\mathcal{E}$ can be introduced as the vertical vector bundle $V\mathcal{E} \to \mathcal{E}$ with respect to the projection $\mathcal{E} \to M$ on the base manifold.

At the level of systems of PDE, the linearized system is

$$F^l = 0, \qquad \ell_{F^l}[U] = 0, \tag{18}$$

where

$$\ell_F[U] = \sum_{k,I} \frac{\partial F}{\partial u_I^k} U_I^k \tag{19}$$

(cf. the Fréchet derivative [28]), where U^k are coordinates along the fibres of the projection $V\mathcal{E} \to \mathcal{E}$ and serve as additional dependent variables ('velocities' of the u^k's). We assume summation over all k, I such that the functions F^l depend on u_I^k.

Morphisms $\mathcal{E} \to V\mathcal{E}$ that are sections of the bundle $V\mathcal{E} \to \mathcal{E}$ are in one-to-one correspondence with local symmetries of the equation E. Recall that nonlocal symmetries (more precisely, their *shadows* [15]) correspond to morphisms $\widetilde{\mathcal{E}} \to V\mathcal{E}$ over $\mathcal{E}$, where $\widetilde{\mathcal{E}}$ is a covering of the original equation. In full generality, Guthrie's definition includes such a covering.

Let us denote by $\widetilde{V\mathcal{E}} \to \widetilde{\mathcal{E}}$ the pullback of the vertical bundle $V\mathcal{E} \to \mathcal{E}$ along the covering projection $\widetilde{\mathcal{E}} \to \mathcal{E}$. Then nonlocal symmetries correspond to morphisms $\widetilde{\mathcal{E}} \to \widetilde{V\mathcal{E}}$ that are sections of the projection $\widetilde{V\mathcal{E}} \to \widetilde{\mathcal{E}}$. In coordinates, if the covering $\widetilde{\mathcal{E}}$ is determined by equations $z_x^j = f^j$, $z_y^j = g^j$, then its linearization $\widetilde{V\mathcal{E}}$ corresponds to the system

$$F^l = 0, \quad z_x^j = f^j, \quad z_y^j = g^j, \quad \ell_{F^l}[U] = 0. \tag{20}$$

Definition 22 ([11]) A *recursion operator* for the system (1) consisting of equations $F^l = 0$, $l = 1, \dots, s$, is given by the following data:

(1) a $\mathfrak{gl}_n$-valued zero-curvature representation $\bar\alpha = \bar A\,dx + \bar B\,dy$ for $\widetilde{\mathcal{E}}$;

(2) two n-vector-valued functions $A_\circ = (A_\circ^j)$, $B_\circ = (B_\circ^j)$ on $\widetilde{V\mathcal{E}}$ linear on the fibres (i.e., linear in the variables U_I^k) and satisfying

$$(D_y - \bar B)A_\circ = (D_x - \bar A)B_\circ; \tag{21}$$

(3) an $s \times n$-matrix-valued function $\bar C$ on $\widetilde{\mathcal{E}}$;

(4) an s-vector-valued function $C_\circ$ on $\widetilde{V\mathcal{E}}$ linear on the fibres (i.e., linear in the variables U_I^k).

The following condition is supposed to hold: If $U = (U^k)$ satisfies the linearized equation $\widetilde{V\mathcal{E}}$, then so does $U' = L(U)$, where $L(U)^l = \bar C_j^l W^j + C_\circ^l$ and W^j, $j = 1, \dots, n$, are nonlocal variables of the covering

$$W_x^j = \bar A_i^j W^i + A_\circ^j, \qquad W_y^j = \bar B_i^j W^i + B_\circ^j, \tag{22}$$

The recursion operator defined by these data will be denoted as LK^{-1}.

Once $\bar\alpha$ is a ZCR and (21) holds, Equations (22) determine a covering; see [11, Eq. (3.2)].

Recursion operators exhibit the following form of gauge invariance: If S is a function on E with values in $GL(n)$, then the data

$$\begin{array}{rcl} \bar A' &=& \bar A^S = \widetilde D_x S S^{-1} + S\bar A S^{-1}, \qquad A'_\circ = SA_\circ, \\ \bar B' &=& \bar B^S = \widetilde D_y S S^{-1} + S\bar B S^{-1}, \qquad B'_\circ = SB_\circ, \\ \bar C' &=& \bar C S^{-1}, \qquad C'_\circ = C_\circ \end{array} \tag{23}$$

(we assume matrix operations) determine the same recursion operator as a mapping $U \mapsto U'$.

Remark 23 One can put the definitions in a more compact form. Let us consider $(1+n) \times (1+n)$ matrices

$$\hat A = \left(\begin{array}{c|c} 0 & 0 \\ \hline A_\circ & \bar A \end{array}\right), \quad \hat B = \left(\begin{array}{c|c} 0 & 0 \\ \hline B_\circ & \bar B \end{array}\right). \tag{24}$$

Then

$$\hat\alpha = \hat A\,dx + \hat B\,dy$$

is a ZCR for $\widetilde{V\mathcal{E}}$; this follows easily from formulas (21). Moreover, let us introduce the $s \times (1+n)$-matrix

$$\hat{C} = \left(C_\circ \middle| \quad \bar{C} \quad \right).$$

Then the above formulas (23) of gauge invariance assume the compact form

$$\begin{aligned} \hat{A}' &= \hat{A}^{\hat{S}} = \widetilde{D}_x \hat{S} \hat{S}^{-1} + \hat{S} \hat{A} \hat{S}^{-1}, \\ \hat{B}' &= \hat{B}^{\hat{S}} = \widetilde{D}_y \hat{S} \hat{S}^{-1} + \hat{S} \hat{B} \hat{S}^{-1}, \\ \hat{C}' &= \hat{C} \hat{S}^{-1} \end{aligned} \tag{25}$$

where

$$\hat{S} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & S \end{array} \right).$$

It is even possible to define a generalized recursion operator of the system (1) as consisting of a $\mathfrak{gl}_N$-valued zero-curvature representation $\hat{\alpha} = \hat{A}\,dx + \hat{B}\,dy$ for $\widetilde{V\mathcal{E}}$ along with an $s \times N$-matrix-valued function $\hat{C}$ on $\widetilde{V\mathcal{E}}$, subject to the following condition: If

$$\hat{W}^j_x = \hat{A}^j_i \hat{W}^i, \qquad \hat{W}^j_y = \hat{B}^j_i \hat{W}^i, \tag{26}$$

then $U'^l = \hat{C}^l_j \hat{W}^j$ satisfies the linearized equation $\widetilde{V\mathcal{E}}$.

For $\hat{A}$, $\hat{B}$ given by formulas (24), the correspondence between $\hat{W}$ and W is

$$\hat{W} = \left(\begin{array}{c} \gamma \\ \hline \gamma W \end{array} \right),$$

where γ satisfies $\widetilde{D}_x \gamma = \widetilde{D}_y \gamma = 0$. With $\hat{S}$ being an arbitrary matrix, formulas (25) define a generalized gauge invariance of generalized recursion operators.

Coverings (22) with $\bar{\alpha} = 0$ are associated with conservation laws, since for them Eq. (21) reads $D_y A_\circ = D_x B_\circ$. Examples are provided by recursion operators that can be written in the traditional integro-differential form ([27])

$$U'^l = \sum_{i=0}^{r} R^{li}_k D^i_x U^k + C^l_j D^{-1}_x p^j_k U^k.$$

Upon the obvious identification $D_x^I U^k = U_I^k$ and introduction of nonlocal variables $W^j = D_x^{-1} p_k^j U^k$, the Guthrie form of this operator is

$$\begin{aligned} W_x^j &= p_k^{jI} U_I^k, \\ W_y^j &= q_k^{jI} U_I^k, \\ U'^l &= C_j^l W^j + R_k^{lI} U_I^k, \end{aligned}$$

where $p_k^{jI} U_I^k \, dx + q_k^{jI} U_I^k \, dy$ is a conservation law of the linearized equation $V\mathcal{E}$ (typically a linearized conservation law of the equation $\mathcal{E}$; [22]).

Example 24 The Lenard recursion operator $D_{xx} + 4u + 2u_x D_x^{-1}$ for the KdV equation $u_t = u_{xxx} + 6uu_x$ has the following Guthrie form (with $\widetilde{\mathcal{E}} = \mathcal{E}$ and $\widetilde{V\mathcal{E}} = V\mathcal{E}$):

$$\begin{aligned} W_x &= U, \\ W_t &= U_{xx} + 6uU, \\ U' &= U_{xx} + 4uU + 2u_x W. \end{aligned} \tag{27}$$

Indeed, if U satisfies the linearized equation $V\mathcal{E}$, i.e.,

$$U_t = U_{xxx} + 6uU_x + 6u_x U, \tag{28}$$

then so does U' (for the same u).

Here W is a potential of the conservation law $U\,dx + (U_{xx} + 6uU)\,dt$ of $V\mathcal{E}$, which is a linearization of the conservation law $u\,dx + (u_{xx} + 3u^2)\,dt$ of $\mathcal{E}$.

6.1 Inversion of recursion operators

A recursion operator is said to be *invertible* if the morphism L of Definition 22 is a covering. The recursion operator LK^{-1} is then simply a pair of linear coverings $K, L : \mathcal{R} \to \widetilde{V\mathcal{E}}$, its inverse being the recursion operator KL^{-1}. Noninvertible recursion operators do exist, see Remark 27(2).

One immediately sees that a recursion operator and its inverse are built upon one and the same covering $\widetilde{\mathcal{E}}$. In practice usually $\widetilde{\mathcal{E}} = \mathcal{E}$ (hence the covering $\widetilde{\mathcal{E}}$ is almost obsolete in Definition 22); however, one can simplify the ZCR $\bar{\alpha}$ with the aid of a suitably chosen covering. Namely, given a recursion operator

$$\begin{array}{ccc} & \mathcal{R} & \\ {}^{K}\swarrow & & \searrow^{L} \\ V\mathcal{E} & & V\mathcal{E} \end{array}$$

associated with a ZCR $\bar{\alpha}$, the obvious pullback along a covering $p : \widetilde{\mathcal{E}} \to \mathcal{E}$ yields a recursion operator

$$\begin{array}{ccc} & p^*\mathcal{R} & \\ {\scriptstyle p^*K}\swarrow & & \searrow{\scriptstyle p^*L} \\ \widetilde{V\mathcal{E}} = p^*V\mathcal{E} & & p^*V\mathcal{E} = \widetilde{V\mathcal{E}}, \end{array}$$

which is associated with the pullback $p^*\bar{\alpha}$.

For instance, let $\widetilde{\mathcal{E}} \to \mathcal{E}$ be the trivializing covering for $\bar{\alpha}$. Then, after suitable transformation (23), we have $p^*\bar{\alpha} = 0$, whence the recursion operator becomes integro-differential of first order in D^{-1}. Hence a possible way of conversion of recursion operators from Guthrie's to Olver's form, mentioned in the Introduction. This approach was used in the work by Guthrie and Hickman [12] who, by using formal power series in the spectral parameter λ, were able to describe large algebras of nonlocal symmetries of the KdV equation resulting from iterated application of the inverse recursion operator.

Alternatively, $\widetilde{\mathcal{E}} \to \mathcal{E}$ can be a covering such that $p^*\bar{\alpha}$ is strictly lower triangular (belongs to $\mathfrak{t}^{(1)}$). Then the covering (22) will be abelian by a similar argument as in Proposition 5 and the recursion operator will be integro-differential of order $\le s$ in D^{-1}.

Let us now turn back to recursion operators LK^{-1} with a general covering $\bar{\alpha}$. One usually observes that for systems $\mathcal{E}$ integrable in the sense of soliton theory the covering K is of a very special form, which is described in the following proposition:

Proposition 25 *Let $\alpha = A\,dx + B\,dy$ be a $\mathfrak{g}$-valued ZCR of equation $\mathcal{E}$. Then the trivial vector bundle $\mathfrak{g} \times V\mathcal{E} \to V\mathcal{E}$ carries a covering structure determined by the condition that an arbitrary element W of the Lie algebra $\mathfrak{g}$ be subject to equations*

$$W_x = [A, W] + \ell_A[U], \qquad W_y = [B, W] + \ell_B[U]. \tag{29}$$

Otherwise said, the associated ZCR $\bar{\alpha}$ coincides with the adjoint representation of the ZCR α, while $A_\circ = \ell_A[U]$, $B_\circ = \ell_B[U]$.

Proof The validity of formulas (21) follows from the fact that $A \mapsto \ell_A[U]$ is a differentiation.

Taking account of the last proposition, we arrive at the following construction, which converts a recursion operator from Guthrie's form to Olver's form provided the covering K is of the type (29).

Construction 26 Step 1. Construct the generalized Riccati covering (Definition 19) $\mathcal{E}'$ over $\mathcal{E}$ such that $\alpha' := \alpha^H$ is lower triangular, where H is the matrix (11).

Step 2. Let a'_{ii}, b'_{ii} be the diagonal entries of the lower triangular matrices A^H, B^H, respectively. Then $a'_{ii}\,dx + b'_{ii}\,dy$ are conservation laws; if they are nontrivial, then construct the abelian covering $\mathcal{E}''$ over $\mathcal{E}'$ with the corresponding potentials z_i.

Step 3. Compute $S = ZH$, where Z is the diagonal matrix $\mathrm{diag}(\mathrm{e}^{-z_i})$. Obviously, $\alpha'' := \alpha^S$ is then strictly lower triangular, and so is its adjoint representation

$$\overline{\alpha''} = \bar{\alpha}^{\bar{S}},$$

where $\bar{S}$ is the image of S in the adjoint representation of the group G. The x-part of the resulting recursion operator given by formulas (23) will be expressible in terms of inverse total derivatives D_x^{-1}.

Step 4 (optional). Let us consider the compact form (24) of the recursion operator, which now takes values in the algebra $\mathfrak{t}_{n+1}^{(1)}$ of strictly lower triangular matrices of dimension $n+1$. Choosing appropriately a lower triangular gauge matrix $\hat{S}$ with units on the diagonal, one can, in principle, further simplify the formulas.

If omitting Step 2, the recursion operator will be expressible in terms of inverses $(D_x - a'_{ii})^{-1}$.

Remark 27 (1) Let R be a conventional recursion operator of an integrable system, let id denote the identity map. As a rule, the inverse recursion operator $(R + \lambda\,\mathrm{id})^{-1}$ in the Guthrie form includes a λ-dependent ZCR $\bar{\alpha}$. The parameter λ can be usually identified with the spectral parameter of the standard ZCR of the system.

(2) Let us recall that the formulas (29) can serve as a starting point of a method to find the inverse recursion operator of an integrable system without finding the conventional recursion operator first. One simply computes all morphisms $\mathcal{R} \to V\mathcal{E}$, where $\mathcal{R}$ is the covering determined by (29). Recently the procedure has been applied to the stationary Nizhnik–Veselov–Novikov equation, see [24]. Remarkably enough, the so obtained recursion operator turned out to be noninvertible for the zero value of the spectral parameter λ. Two examples of such computation can be found below.

7 Examples

Example 28 Continuing Example 24, let us invert the Lenard operator. The result is, of course, well known (Guthrie and Hickman [12], Lou [20, 21]).

Instead of tedious inversion of the operator given by formulas (27) and (28), we compute it from scratch. We start with the standard $\mathfrak{sl}_2$-valued ZCR

$$\alpha = \begin{pmatrix} 0 & u \\ -1 & 0 \end{pmatrix} dx + \begin{pmatrix} u_x & u_{xx} + 2u \\ -2u & -u_x \end{pmatrix} dy$$

of the KdV equation with the spectral parameter set to zero. Using (22) and (29) with $\mathfrak{sl}_2$ parametrized as

$$\begin{pmatrix} Q & P \\ R & -Q \end{pmatrix},$$

we get the following formulas for the covering K:

$$\begin{aligned}\begin{pmatrix}P\\Q\\R\end{pmatrix}_x &= \begin{pmatrix}0 & -2u & 0\\ 1 & 0 & u\\ 0 & -2 & 0\end{pmatrix}\begin{pmatrix}P\\Q\\R\end{pmatrix} + \begin{pmatrix}U\\0\\0\end{pmatrix},\\ \begin{pmatrix}P\\Q\\R\end{pmatrix}_t &= \begin{pmatrix}2u_x & -2u_{xx}-4u^2 & 0\\ 2u & 0 & u_{xx}+2u^2\\ 0 & -4u & -2u_x\end{pmatrix}\begin{pmatrix}P\\Q\\R\end{pmatrix} + \begin{pmatrix}U_{xx}+4uU\\U_x\\-2U\end{pmatrix}.\end{aligned}$$

Here U denotes a symmetry of the KdV equation, i.e., satisfies the linearized KdV equation (28). Then one easily finds that $U' = Q$ satisfies the same linearized KdV equation (28) as well, i.e., yields a recursion operator for the KdV equation. It is a matter of routine to check that this operator is the inverse of the Lenard operator. Moreover, it follows that $K : \mathcal{R} \to V\mathcal{E}$, originally given by $U = U'_{xx} + 4uU' + 2u_xW$, constitutes a three-dimensional covering (with nonlocal variables U, U_x and W).

According to Construction 26, to express the inverse recursion operator in terms of D_x^{-1}, all we need is to make the ZCR $\bar\alpha$ strictly lower triangular. As the first step we build up a covering $\mathcal{E}' \to \mathcal{E}$ with the quadratic pseudopotential $h = h_{11}$ defined by Eq. (14), i.e.,

$$\begin{aligned}h_x &= -h^2 - u,\\ h_t &= -2uh^2 + 2u_xh - u_{xx} - 2u^2.\end{aligned}$$

Then, using the gauge matrix

$$H = \begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix},$$

we get the lower triangular ZCR

$$\alpha' = \alpha^H = \begin{pmatrix}-h & 0\\ -1 & h\end{pmatrix} dx + \begin{pmatrix}u_x - 2uh & 0\\ -2u & -u_x + 2uh\end{pmatrix} dy$$

with $-h, h$ on the diagonal. As the second step, we construct the abelian covering $\mathcal{E}'' \to \mathcal{E}'$ with the potential z satisfying

$$z_x = -h, \qquad z_y = u_x - 2uh.$$

The gauge matrix

$$Z = \begin{pmatrix}\mathrm{e}^{-z} & 0\\ 0 & \mathrm{e}^{z}\end{pmatrix}$$

then gives the strictly lower triangular ZCR

$$\alpha'' = \alpha^{ZH} = \begin{pmatrix}0 & 0\\ -\mathrm{e}^{2z} & 0\end{pmatrix} dx + \begin{pmatrix}0 & 0\\ -2\mathrm{e}^{2z}u & 0\end{pmatrix} dy.$$

In the third step, we combine the above gauge matrices into one and compute its adjoint representation:

$$S = \begin{pmatrix} \mathrm{e}^{-z} & h\mathrm{e}^{-z} \\ 0 & \mathrm{e}^{z} \end{pmatrix}, \qquad \bar{S} = \begin{pmatrix} \mathrm{e}^{-2z} & -2h\mathrm{e}^{-2z} & -h^2 e^{-2z} \\ 0 & 1 & h \\ 0 & 0 & \mathrm{e}^{2z} \end{pmatrix}.$$

Acting by $\bar{S}$ on our operator, we get

$$\begin{aligned} \begin{pmatrix} P \\ Q \\ R \end{pmatrix}_x &= \begin{pmatrix} 0 & 0 & 0 \\ e^{2z} & 0 & 0 \\ 0 & -2e^{2z} & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + \begin{pmatrix} \mathrm{e}^{-2z} U' \\ 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} P \\ Q \\ R \end{pmatrix}_t &= \begin{pmatrix} 0 & 0 & 0 \\ 2ue^{2z} & 0 & 0 \\ 0 & -4ue^{2z} & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathrm{e}^{-2z} U'_{xx} - 2\mathrm{e}^{-2z} h U'_x + (2h^2 + 4u)\mathrm{e}^{-2z} U' \\ U'_x - 2hU' \\ -2\mathrm{e}^{2z} U' \end{pmatrix}, \\ U &= Q - h\mathrm{e}^{-2z} R. \end{aligned}$$

Rewritting the x-part in terms of inverse total derivatives D^{-1}, we get $P = D^{-1}(\mathrm{e}^{-2z} U)$, $Q = D^{-1}(\mathrm{e}^{2z} P)$, $R = -2D^{-1}(\mathrm{e}^{-2z} Q)$, hence

$$U = D^{-1}\mathrm{e}^{2z} D^{-1}\mathrm{e}^{-2z} U - h\mathrm{e}^{-2z} D^{-1}\mathrm{e}^{-2z} D^{-1}\mathrm{e}^{2z} D^{-1}\mathrm{e}^{-2z} U.$$

This is the well-known result [12, 20, 21], since $U' = Q - h\mathrm{e}^{-2z} R = -\frac{1}{2} D_x(R/h^2)$ and $z_{xx} = z_x^2 + u$.

The optional fourth step does not bring any significant improvement.

Example 29 Let us consider the Tzitzéica equation [38]

$$u_{xy} = \mathrm{e}^{u} - \mathrm{e}^{-2u},$$

later rediscovered as a member of the Zhiber–Shabat classification [45]. Its ZCR

$$\alpha = \begin{pmatrix} -u_x & 0 & \lambda \\ \lambda & u_x & 0 \\ 0 & \lambda & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & \mathrm{e}^{-2u}/\lambda & 0 \\ 0 & 0 & \mathrm{e}^{u}/\lambda \\ \mathrm{e}^{u}/\lambda & 0 & 0 \end{pmatrix} dy \tag{30}$$

as well as the Bäcklund transformation were essentially found by Tzitzéica himself.

One could invert the known recursion operator [36], but it is easier to compute the inverse recursion operator directly by the procedure outlined in Remark 27(2). Namely, we consider the eight-dimensional covering (29), where $\bar{A}$, $\bar{B}$, $A_\circ$ and $B_\circ$ are found from the formula (30) to be

$$\bar{A} = \begin{pmatrix} 0 & -\lambda & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & -2u_x & -\lambda & 0 & 0 & 0 & 0 & \lambda \\ -2\lambda & 0 & -u_x & 0 & -\lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 2u_x & -\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -\lambda & 0 & u_x & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & u_x & -\lambda \\ \lambda & 0 & 0 & 0 & 2\lambda & 0 & 0 & -u_x \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} 0 & 0 & -\mathrm{e}^{u}/\lambda & \mathrm{e}^{-2u}/\lambda & 0 & 0 & 0 & 0 \\ -\mathrm{e}^{-2u}\lambda & 0 & 0 & 0 & \mathrm{e}^{-2u}/\lambda & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{u}/\lambda & 0 & 0 & 0 & \mathrm{e}^{-2u}/\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{e}^{u}/\lambda & \mathrm{e}^{u}/\lambda & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{-2u}/\lambda & 0 & 0 & 0 & \mathrm{e}^{u}/\lambda \\ -\mathrm{e}^{u}/\lambda & 0 & 0 & 0 & -2\mathrm{e}^{u}/\lambda & 0 & 0 & 0 \\ 2\mathrm{e}^{u}/\lambda & 0 & 0 & 0 & \mathrm{e}^{u}/\lambda & 0 & 0 & 0 \\ 0 & \mathrm{e}^{u}/\lambda & 0 & 0 & 0 & 0 & -\mathrm{e}^{-2u}/\lambda & 0 \end{pmatrix},$$

$$A_\circ = \begin{pmatrix} -U_x \\ 0 \\ 0 \\ 0 \\ U_x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad B_\circ = \begin{pmatrix} 0 \\ -2\mathrm{e}^{-2u}U/\lambda \\ 0 \\ 0 \\ 0 \\ \mathrm{e}^{u}U/\lambda \\ \mathrm{e}^{u}U/\lambda \\ 0 \end{pmatrix},$$

W being a column $(W_{11}, W_{12}, W_{13}, W_{21}, W_{22}, W_{23}, W_{31}, W_{32})^\top$ of pseudopotentials. One easily finds that $W_{11} - W_{22}$ is a symmetry of the Tzitzéica equation if so is U. We have obtained the 'inverse' recursion operator of the Tzitzéica equation in the Guthrie form.

Let us express it in terms of D_x^{-1}. As the first step we introduce pseudopotentials p, q, r satisfying

$$\begin{aligned} p_x &= \lambda p^2 - 2pu_x - \lambda q, & p_y &= \frac{\mathrm{e}^u}{\lambda}pq - \frac{1}{\mathrm{e}^{2u}\lambda}, \\ q_x &= \lambda pq - qu_x - \lambda, & q_y &= \frac{\mathrm{e}^u}{\lambda}(q^2 - p), \\ r_x &= -\lambda pr + \lambda q + \lambda r^2 + u_x r, & r_y &= \frac{\mathrm{e}^u}{\lambda}(-pr^2 + qr - 1). \end{aligned}$$

to make the ZCR (30) lower triangular by providing a solution to Equations (17). Indeed, acting on α by the gauge matrix

$$H = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$$

we get

$$\begin{aligned} \alpha^H &= \begin{pmatrix} -u_x + \lambda p & 0 & 0 \\ \lambda & u_x - \lambda p + \lambda r & 0 \\ 0 & \lambda & -\lambda r \end{pmatrix} dx \\ &\quad + \begin{pmatrix} \mathrm{e}^u q/\lambda & 0 & 0 \\ \mathrm{e}^u r/\lambda & -\mathrm{e}^u pr/\lambda & 0 \\ \mathrm{e}^u/\lambda & -\mathrm{e}^u p/\lambda & \mathrm{e}^u(pr - q)/\lambda \end{pmatrix} dy. \end{aligned}$$

In the second step we remove the diagonal. To this end we introduce pseudopotentials s, t by

$$\begin{aligned} s_x &= -u_x + \lambda p, & s_y &= \frac{\mathrm{e}^u}{\lambda} q, \\ t_x &= u_x - \lambda p + \lambda r, & t_y &= -\frac{\mathrm{e}^u}{\lambda} pr. \end{aligned}$$

Acting on α^H by the gauge matrix

$$Z = \begin{pmatrix} \mathrm{e}^{-s} & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 \\ 0 & 0 & \mathrm{e}^{s+t} \end{pmatrix}$$

we finally get

$$\begin{aligned} \alpha^{ZH} &= \begin{pmatrix} 0 & 0 & 0 \\ \lambda \mathrm{e}^{s-t} & 0 & 0 \\ 0 & \lambda \mathrm{e}^{s+2t} & 0 \end{pmatrix} dx \\ &\quad + \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{e}^{u+s-t} r/\lambda & 0 & 0 \\ \mathrm{e}^{u+2s+t}/\lambda & -\mathrm{e}^{u+s+2t} p/\lambda & 0 \end{pmatrix} dy. \end{aligned}$$

Denoting $S = ZH$, we compute the adjoint representation $\bar{S}$ to be

$$\bar{S} = \begin{pmatrix} \mathrm{e}^{-2s-t} & -\mathrm{e}^{-2s-t}r & \mathrm{e}^{-2s-t}p & \mathrm{e}^{-2s-t}(pr-2q) & -\mathrm{e}^{-2s-t}(pr+q) & \mathrm{e}^{-2s-t}p(pr-q) & -\mathrm{e}^{-2s-t}qr & \mathrm{e}^{-2s-t}q(pr-q) \\ 0 & \mathrm{e}^{-s+t} & 0 & -\mathrm{e}^{-s+t}p & \mathrm{e}^{-s+t}p & -\mathrm{e}^{-s+t}p^2 & \mathrm{e}^{-s+t}q & -\mathrm{e}^{-s+t}pq \\ 0 & 0 & \mathrm{e}^{-s-2t} & -\mathrm{e}^{-s-2t}r & -2\mathrm{e}^{-s-2t}r & \mathrm{e}^{-s-2t}(pr-q) & -r^2\mathrm{e}^{-s-2t} & \mathrm{e}^{-s-2t}(pr-q)r \\ 0 & 0 & 0 & 1 & 0 & p & 0 & q \\ 0 & 0 & 0 & 0 & 1 & -p & r & -pr \\ 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{s-t} & 0 & \mathrm{e}^{s-t}r \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{s+2t} & -\mathrm{e}^{s+2t}p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{2s+t} \end{pmatrix}.$$

Acting by $\bar{S}$ on the above recursion operator we get

$$\bar{A}^{\bar{S}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda\mathrm{e}^{s+2t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda\mathrm{e}^{s-t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda\mathrm{e}^{s-t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda\mathrm{e}^{s-t} & -\lambda\mathrm{e}^{s+2t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda\mathrm{e}^{s-t} & -\lambda\mathrm{e}^{s-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda\mathrm{e}^{s+2t} & 2\lambda\mathrm{e}^{s+2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda\mathrm{e}^{s+2t} & -\lambda\mathrm{e}^{s-t} & 0 \end{pmatrix}$$

and

$$\bar{S}A_\circ = \begin{pmatrix} \mathrm{e}^{-2s-t}(-2pr+q)U_x \\ 2\mathrm{e}^{-s+t}pU_x \\ -\mathrm{e}^{-s-2t}rU_x \\ -U_x \\ U_x \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(we omit the matrices $\bar{B}^{\bar{S}}$ and $B_\circ$).

Thus, the inverse recursion operator for the Tzitzéica equation in terms of D^{-1} is

$$V = W_{21} - W_{22} - 2\mathrm{e}^{-s+t}pW_{23} + \mathrm{e}^{-s-2t}rW_{31} + \mathrm{e}^{-2s-t}(2pr - q)W_{32},$$

where

$$\begin{aligned}
W_{11} &= D^{-1}[\mathrm{e}^{-2s-t}(-2pr+q)U_x], \\
W_{12} &= D^{-1}[2\mathrm{e}^{-s+t}pU_x - \mathrm{e}^{s+2t}\lambda W_{11}], \\
W_{13} &= D^{-1}[-\mathrm{e}^{-s-2t}rU_x + \mathrm{e}^{s-t}\lambda W_{11}], \\
W_{22} &= D^{-1}[U_x + \mathrm{e}^{s-t}\lambda W_{12} - \mathrm{e}^{s+2t}\lambda W_{13}], \\
W_{21} &= D^{-1}[-U_x - \mathrm{e}^{s-t}\lambda W_{12}], \\
W_{31} &= D^{-1}[\mathrm{e}^{s+2t}\lambda(W_{21} + 2W_{22})], \\
W_{23} &= D^{-1}[-\mathrm{e}^{s-t}\lambda(-W_{21} + W_{22})], \\
W_{32} &= D^{-1}[\lambda(\mathrm{e}^{s+2t}W_{23} - \mathrm{e}^{s-t}W_{31})].
\end{aligned}$$

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