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by

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The $N = 1$ supersymmetric KdV equation: (non)local Hamiltonian and symplectic structures, recursions, and hierarchies

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ABSTRACT. Using methods of [4] and [5], we accomplish an extensive study of the $N = 1$ supersymmetric Korteweg-de Vries equation. Our results generalize the ones obtained previously and include: a description of local and nonlocal Hamiltonian and symplectic structures, five hierarchies of symmetries (including a new one), the corresponding hierarchies of conservation laws, recursion operators for symmetries and generating functions of conservation laws.

INTRODUCTION

There exists a number of super extensions of the classical KdV equation

$$u_t = -u_{xxx} + 6uu_x$$

(see [9] and the references therein). One of them, the so-called $N = 1$ supersymmetric extension, is

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x + \varphi_{xx}\varphi, \\ \varphi_t &= -\varphi_{xxx} + 3u\varphi_x + 3u_x\varphi, \end{aligned} \tag{1}$$

where φ is an odd (fermionic) variable. To deal with this system, it is convenient to introduce a new independent odd variable θ such that $D_\theta^2 = D_x$ (here D denotes the total derivative operator; see below) and a new odd function

$$\Phi = \varphi + \theta u.$$

Then (1) will acquire the form

$$\Phi_t = -\Phi_{xxx} + 3\Phi_\theta\Phi_x + 3\Phi_{x\theta}\Phi. \tag{2}$$

This equation is linear in θ and reduces to (1) if we equal to each other the corresponding coefficients at the left- and right-hand sides. System (1) (or equation (2)) was studied before (see, e.g., [8]) and a number of results related to its integrability were obtained. The aim of our paper is twofold: (1) to fill in a number of gaps in the existing picture (for example, we describe local and nonlocal Hamiltonian and symplectic structures, construct recursion operators for symmetries and generating functions of conservation laws, obtain a new hierarchy of symmetries) and to represent the known results in a more convenient form; (2) to demonstrate the efficiency of new methods of analysis of integrable systems described in [4, 5] and based on a general geometric approach to nonlinear PDE [1, 7].

This paper is organized as follows. In Section 1, we present the essential definitions and results needed for applications paying main attention to the computational aspects rather than to theoretical ones. All the proofs can be found in [1, 7, 4, 5]. In Section 2, the results for the $N = 1$ supersymmetric KdV equation are described. Finally, in the last section we briefly discuss the results and perspectives.

Key words and phrases. Super KdV equation, symmetry, conservation law, Hamiltonian structure, symplectic structure.

1. Description of the computational scheme

Here we deal with evolution systems $\mathcal{E}$ of the form

$$v_t = F(y, t, v_1, \dots, v_k), \tag{3}$$

where both the unknown variable $v = (v^1, \dots, v^m)$ and the right-hand side $F = (F^1, \dots, F^m)$ are vector-functions and $v_i = \partial^i v/\partial y^i$, y and t being the independent variables.

Remark 1. In applications, some of the variables v^j, as well as y, may be *odd*. In particular, in equation (2) θ and Φ are odd and x is even. Nevertheless, for the sake of simplicity, we expose the general theory for purely *even* equations. Necessary corrections needed for the super case the reader will find in Subsection 1.10.

Two basic operators related to (3),

$$D_y = \frac{\partial}{\partial y} + \sum_{i,j} v^j_{i+1} \frac{\partial}{\partial v^j_i},$$
$$D_t = \frac{\partial}{\partial t} + \sum_{i,j} D^i_y(F^j) \frac{\partial}{\partial v^j_i},$$

are called the *total derivatives*.

Remark 2. Note that the above expressions for total derivatives contain infinite number of terms. To make the action of these operators (as well as of similar operators introduced below) well defined, we introduce the space $\mathcal{F}(\mathcal{E})$ of functions smoothly depending on y, t and a *finite number* of variables v^j_i, and assume D_y and D_t to act in this space. Similarly, we shall consider the spaces $\mathcal{F}^m(\mathcal{E})$ of vector-functions of length m that depend on y, t and v^j_i in the same way.

1.1. **Symmetries.** A *symmetry* of equation (3) is a vector field

$$S = \sum_{i,j} S^j_i \frac{\partial}{\partial v^j_i}, \quad S^j_i \in \mathcal{F}(\mathcal{E}),$$

such that

$$[S, D_y] = [S, D_t] = 0.$$

Any symmetry is of the form

$$\partial_f = \sum_{i,j} D^i_y(f^j) \frac{\partial}{\partial v^j_i}, \tag{4}$$

where the vector-function $f = (f^1, \dots, f^m) \in \mathcal{F}^m(\mathcal{E})$ satisfies the system of equations

$$D_t(f^l) = \sum_{i,j} \frac{\partial F^l}{\partial v^j_i} D^i_y(f^j), \quad l = 1, \dots, m. \tag{5}$$

The operator at the right-hand side of (5) is called the *linearization* of F and is denoted by ℓ_F. Thus, equation (5) acquires the form

$$D_t(f) = \ell_F(f). \tag{6}$$

There exists a one-to-one correspondence between symmetries (4) and the corresponding functions $f \in \mathcal{F}^m(\mathcal{E})$, hence we shall identify symmetries with such functions and use the term ‘symmetry’ for any function that satisfy (6).

1.2. **Conservation laws and generating functions.** A *conservation law* of system (3) is a pair $\Omega = (Y, T)$, Y, $T \in \mathcal{F}(\mathcal{E})$, such that

$$D_t(Y) = D_y(T). \tag{7}$$

The function Y is called the *density* of Ω. A conservation law is called *trivial* if $Y = D_y(P)$, $T = D_t(P)$ for some function $P \in \mathcal{F}(\mathcal{E})$.

To any conservation law there corresponds its *generating function* defined by

$$g_\Omega = \frac{\delta Y}{\delta v} = \left(\frac{\delta Y}{\delta v^1}, \dots, \frac{\delta Y}{\delta v^m}\right),$$

where

$$\frac{\delta}{\delta v^j} = \sum_{i\ge 0}(-D_y)^i \circ \frac{\partial}{\partial v_i^j}$$

is the *variational derivative* with respect to v^j. Generating functions of conservation laws satisfy the system of equations

$$D_t(g) = -\ell_F^*(g), \tag{8}$$

or

$$D_t(g^l) = -\sum_{i,j}(-D_y)^i\left(\frac{\partial F^j}{\partial v_i^l} g^j\right), \quad l = 1, \dots, m, \tag{9}$$

where ℓ_F^* is *adjoint* to the operator ℓ_F.

Any conservation law is uniquely determined by its generating function and, in particular, Ω is trivial if and only if $g_\Omega = 0$. Stress that equation (9) may possess solutions that do not correspond to any conservation law of (3).

1.3. **Nonlocal variables.** Let us introduce a set of variables $w^1, \dots, w^j, \dots$ satisfying the equations

$$w_y^j = A^j(y, t, \dots, v_i^\alpha, \dots, w^\beta, \dots), \quad w_t^j = B^j(y, t, \dots, v_i^\alpha, \dots, w^\beta, \dots), \tag{10}$$

that are compatible modulo equation (3), where A^j, B^j are some smooth functions depending on a finite number of arguments. Consider the operators

$$\widetilde{D}_y = D_y + \sum_j A^j \frac{\partial}{\partial w^j}, \quad \widetilde{D}_t = D_t + \sum_j B^j \frac{\partial}{\partial w^j}.$$

Due to the compatibility conditions, one has

$$[\widetilde{D}_y, \widetilde{D}_t] = 0$$

modulo (3). The variables w^j are called *nonlocal*.

Using the operators $\widetilde{D}_y$, $\widetilde{D}_t$ instead of D_y and D_t in formulas (5), (7), and (9), we can introduce the notions of *nonlocal symmetries*, *nonlocal conservation laws*, and *nonlocal generating functions* depending on the new variables w^j. We shall denote the spaces of such symmetries and generating functions by $\mathbf{sym}(\mathcal{E})$ and $\mathbf{gf}(\mathcal{E})$, respectively.

Remark 3. An invariant geometric way to introduce nonlocal variables is based on the notion of *covering*, see [6, 1, 5, 7].

1.4. **ℓ- and ℓ^*-extensions.** There are two canonical ways to extend the initial system (3). The first one is related to the operator ℓ_F and is called the *ℓ-extension*. Namely, let us introduce the nonlocal variables ω_i^j (we shall also denote ω_0^j by ω^j), $j = 1, \dots, m$, $i = 0, 1, \dots$, satisfying the relations

$$(\omega_i^j)_y = \omega_{i+1}^j, \quad (\omega_i^j)_t = \widetilde{D}_y^i\left(\sum_{s,l}\frac{\partial F^j}{\partial v_s^l}\omega_s^l\right).$$

Clearly, these equations are consistent modulo (3) and are the consequences of the following ones

$$\omega_t^j = \sum_{i,l}\frac{\partial F^j}{\partial v_i^l}\omega_i^l. \tag{11}$$

In a similar way we construct the ℓ^*-*extension*: the nonlocal variables are p_i^j (p_0^j will also be denoted by p^j) and the defining relations are

$$(p_i^j)_y = p_{i+1}^j, \quad (p_i^j)_t = -\widetilde{D}_y^i\Big(\sum_{s,l}(-\widetilde{D}_y)^s\Big(\frac{\partial F^l}{\partial v_s^j}p^l\Big)\Big),$$

that reduce to the equations

$$p_t^j = -\sum_{s,l}(-\widetilde{D}_y)^s\Big(\frac{\partial F^l}{\partial v_s^j}p^l\Big) \tag{12}$$

and their differential consequences.

Remark 4. The parities of the variables ω^j and p^j are opposite to that of v^j: if v^j is *even*, then ω^j and p^j are *odd* and vice versa.

If the initial equation $\mathcal{E}$ was extended by nonlocal variables w^j, we can associate to these variables, in a canonical way, the corresponding ω's and p's whose 'behavior' is governed by linearization or, respectively, adjoint linearization of equations (10) in the corresponding nonlocal setting.

Associating operators to functions on the ℓ- and ℓ^*-extensions. Let $\mathcal{F}^m(\mathcal{E})$ be the space of vector-valued functions of length m (see Remark 2). Consider the case when $\mathcal{E}$ is not extended by nonlocal variables first. Let $a = (a_1, \dots, a_m)$, $a_i = \sum_{jl} a_l^{ij}\omega_l^j$, $a_l^{ij} \in \mathcal{F}(\mathcal{E})$, be a linear in ω vector-function. Then we put into correspondence to this function a differential operator $\Delta_a = \|\Delta_a^{ij}\|\colon \mathcal{F}^m(\mathcal{E}) \to \mathcal{F}^m(\mathcal{E})$, where

$$\Delta_a^{ij} = \sum_l a_l^{ij} D_y^l.$$

If $\mathcal{F}(\mathcal{E})$ contains nonlocal variables, the situation becomes more complicated. We shall consider here the simplest case when the functions A^j in (10) are independent of ω^β. Let $\bar{\omega}^\beta$ be the variable in the ℓ-extension associated to the nonlocal variable w^β and $b = (b^1, \dots, b^m)$, $b^i = \sum_\beta b^{i\beta}\bar{\omega}^\beta$, be a linear in $\bar{\omega}$ vector-function. Then the corresponding operator $\Delta_b = \|\Delta_b^{ij}\|\colon \mathcal{F}^m(\mathcal{E}) \to \mathcal{F}^m(\mathcal{E})$ is of the form

$$\Delta_b^{ij} = \sum_\alpha b^{i\alpha} D_y^{-1} \circ \sum_l \frac{\partial A^\alpha}{\partial v_l^j} D_y^l. \tag{13}$$

For the ℓ^*-extension the construction is completely similar.

Below we shall use the notation $\mathcal{L}^m(\ell_{\mathcal{E}})$ and $\mathcal{L}^m(\ell_{\mathcal{E}}^*)$ for the spaces of vector-functions linear in ω, $\bar{\omega}$ and p, $\bar{p}$, respectively.

1.5. **Recursion operators for symmetries.** Let $\Omega \in \mathcal{L}^m(\ell_{\mathcal{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(\Omega) = \widetilde{\ell}_F(\Omega).$$

Then the corresponding operator Δ_Ω maps $\mathbf{sym}(\mathcal{E})$ to $\mathbf{sym}(\mathcal{E})$ and thus is a recursion operator for (nonlocal) symmetries of $\mathcal{E}$.

1.6. **Recursion operators for generating functions.** Let $G \in \mathcal{L}^m(\ell_{\mathcal{E}}^*)$ be a function that satisfies the equation

$$\widetilde{D}_t(G) = -\widetilde{\ell}_F^*(G).$$

Then the corresponding operator Δ_G maps $\mathbf{gf}(\mathcal{E})$ to $\mathbf{gf}(\mathcal{E})$ and thus is a recursion operator for (nonlocal) generating functions of $\mathcal{E}$.

1.7. **Hamiltonian structures.** Let $A \in \mathcal{L}^m(\ell^*_{\mathcal{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(A) = \widetilde{\ell}_F(A).$$

Then the corresponding operator Δ_A maps $\mathbf{gf}(\mathcal{E})$ to $\mathbf{sym}(\mathcal{E})$. We call such maps *pre-Hamiltonian structures*. In order Δ_A to be a true *Hamiltonian structure*, it has to satisfy two conditions: skew-symmetry ($\Delta^*_A = -\Delta_A$) and the Jacobi identity for the corresponding Poisson bracket (that amounts to $[\![\Delta_A, \Delta_A]\!] = 0$, where $[\![\,,]\!]$ is the variational Schouten bracket, see [2, 4]). Both these conditions are easily checked in terms of the function A.

Namely, if $A = \big\| \sum_{jl} a^{ij}_l p^j_l \big\|$ then we consider the function $W_A = \sum_{ijl} a^{ij}_l p^j_l p^i$ and in terms of W_A the first condition reads

$$\sum_i \frac{\delta W_A}{\delta p^i} p^i = -2W_A, \tag{14}$$

while the second one is

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta p}\right) \sum_i \left(\frac{\delta W_A}{\delta v^i}\frac{\delta W_A}{\delta p^i}\right) = 0, \tag{15}$$

$(\delta/\delta v, \delta/\delta p) = (\delta/\delta v^1, \dots, \delta/\delta v^m, \delta/\delta p^1, \dots, \delta/\delta p^m)$ Note also that the compatibility condition for two Hamiltonian structures A and B amounts to

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta p}\right) \sum_i \left(\frac{\delta W_A}{\delta v^i}\frac{\delta W_B}{\delta p^i} + \frac{\delta W_B}{\delta v^i}\frac{\delta W_A}{\delta p^i}\right) = 0. \tag{16}$$

The equation $\mathcal{E}$ itself is in the Hamiltonian form if it possesses a Hamiltonian structure A and may be presented as

$$v_t = \Delta_A \frac{\delta Y}{\delta v} \tag{17}$$

for some $Y = (Y^1, \dots, Y^m)$.

1.8. **Symplectic structures.** Let $S \in \mathcal{L}^m(\ell_{\mathcal{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(S) = -\widetilde{\ell}^*_F(S).$$

Then the corresponding operator Δ_S maps $\mathbf{sym}(\mathcal{E})$ to $\mathbf{gf}(\mathcal{E})$ and may be called a *presymplectic structure* on $\mathcal{E}$. A presymplectic structure is called *symplectic* if it enjoys in addition the following properties. Let $S = \big\| \sum_{jl} b^{ij}_l \omega^j_l \big\|$. Similar to Subsection 1.7, we consider the function $W_S = \sum_{ijl} b^{ij}_l \omega^j_l \omega^i$ and impose the conditions

$$\sum_i \frac{\delta W_S}{\delta \omega^i} \omega^i = -2W_S, \tag{18}$$

i.e., the operator Δ_S is skew-adjoint, and

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta \omega}\right) \sum_i \frac{\delta W_S}{\delta v^i} \omega^i = 0 \tag{19}$$

that means that the 'form' W_S s is closed.

1.9. **Canonical representation.** As it will be seen below, all the operators constructed in our study are presented in the form

$$\sum_{\alpha \geq 0} c^\alpha_{ij} D^\alpha_y + \sum_\beta d^\beta_j D^{-1}_y \circ e^\beta_i,$$

where $\|c^\alpha_{ij}\|$ is an $m \times m$-matrix, $\|d^\beta_j\|$ is an $m \times l$-matrix, and $\|e^\beta_i\|$ is an $l \times m$-matrix for some $l > 0$ (matrix-valued functions, to be more precise). In the table it is shown how the matrices d and e look for different types of operators.

Type of operator	Lines of matrix d	Columns of matrix e
Recursions for symmetries	Symmetry	Generating function
Recursions for generating funct.	Generating function	Symmetry
Hamiltonian structures	Symmetry	Symmetry
Symplectic structures	Generating function	Generating function

1.10. **Super case.** We shall now assume that all objects under consideration belong to the super setting, i.e., may be either even or odd, which means that they obey the rule

$$AB = (-1)^{AB} BA.$$

Here and below, symbols used at the exponents of (-1) stand for the corresponding parity. Generalization of the above exposed theory to the super case is carried out along the line of [10, 7].

Then the basic formulas to be used in the calculus described above are:

(1) for evolutionary derivations

$$\partial_\varphi = \sum_{ij} (-1)^{\varphi v_i^j} D_y^i(\varphi^j) \frac{\partial}{\partial v_i^j}$$

(naturally, the parity of v_i^j equals that of v^j plus parity of y times i);

(2) for the linearization one has $\ell_f(\varphi) = (-1)^{f\varphi} \partial_\varphi(f)$ that amounts to

$$(\ell_f)_\alpha^\beta = \sum_i (-1)^{(f^\alpha+1)v_i^\beta} \frac{\partial f^\alpha}{\partial v_i^\beta} D_y^i;$$

(3) for the operator adjoint to $\Delta = \sum_i a_i D_y^i$ one has

$$\Delta^* = \sum_i (-1)^{i+ia_iy+\frac{i(i-1)}{2}y} D_y^i \circ a_i.$$

2. Main results for the $N = 1$ supersymmetric KdV equation

Here we apply the theory described above to equation (2)

$$\Phi_t = -\Phi_{xxx} + 3\Phi_\theta\Phi_x + 3\Phi_{x\theta}\Phi.$$

We use the notation

$$\Phi_k \quad \text{for} \quad \frac{\partial^{2k}\Phi}{\partial\theta^{2k}} = \frac{\partial^k\Phi}{\partial x^k}$$

and

$$\Phi_{k\frac{1}{2}} \quad \text{for} \quad \frac{\partial^{2k+1}\Phi}{\partial\theta^{2k+1}} = \frac{\partial^{k+1}\Phi}{\partial x^k\partial\theta}.$$

The functions Φ_k are *odd* while $\Phi_{k\frac{1}{2}}$ are *even*, the function $\Phi = \Phi_0$ itself being odd.

Gradings. We assign the following gradings $[\cdot]$ to the variables on our equation:

$$[\theta] = -1/2, \quad [x] = -1, \quad [t] = -3, \quad [\Phi] = 3/2.$$

Respectively, we have

$$[\Phi_k] = (2k+3)/2, \quad [\Phi_{k\frac{1}{2}}] = k+2.$$

With these gradings, equation (2) becomes homogeneous (of grading 9/2) and all constructions below can be considered to be homogeneous as well.

2.1. **Nonlocal functions.** Here we extend the equation $\mathcal{E}$ by four groups of nonlocal variables. We present here their θ-components only; the x- and t-components are given in [3] (they are found from the compatibility conditions).

2.1.1. *Group 1.* This group includes the even variables q_1, q_3, q_5, defined by

$$\begin{aligned}
(q_1)_\theta &= \Phi_0, \\
(q_3)_\theta &= \Phi_0\Phi_{\frac{1}{2}}, \\
(q_5)_\theta &= \Phi_{\frac{1}{2}}(-\Phi_2 + 2\Phi_0\Phi_{\frac{1}{2}})/2.
\end{aligned}$$

Gradings: $[q_1] = 1$, $[q_3] = 3$, $[q_5] = 5$.

2.1.2. *Group 2.* This group includes the odd variables $Q_{\frac{1}{2}}$, $Q_{\frac{5}{2}}$, $Q_{\frac{9}{2}}$ defined by

$$
\begin{aligned}
&(Q_{\frac{1}{2}})_\theta = q_1,\\
&(Q_{\frac{5}{2}})_\theta = q_1^3 - 6q_3,\\
&(Q_{\frac{9}{2}})_\theta = -60\Phi_0\Phi_1 q_1 + q_1^5 - 60q_1^2 q_3 + 240q_5.
\end{aligned}
$$

Gradings: $[Q_{\frac{1}{2}}] = 1/2$, $[Q_{\frac{5}{2}}] = 5/2$, $[Q_{\frac{9}{2}}] = 9/2$.

2.1.3. *Group 3.* This group includes the odd variables $Q_{\frac{3}{2}}$, $Q_{\frac{7}{2}}$, $Q_{\frac{11}{2}}$ defined by

$$
\begin{aligned}
&(Q_{\frac{3}{2}})_\theta = \Phi_0 Q_{\frac{1}{2}},\\
&(Q_{\frac{7}{2}})_\theta = (12\Phi_2 Q_{\frac{1}{2}} + 18\Phi_1 Q_{\frac{1}{2}} q_1 + \Phi_0 Q_{\frac{5}{2}})/3,\\
&(Q_{\frac{11}{2}})_\theta = (360\Phi_4 Q_{\frac{1}{2}} + 5280\Phi_3 Q_{\frac{1}{2}} q_1 - 760\Phi_2 Q_{\frac{5}{2}} + 4680\Phi_2 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 1200\Phi_2 Q_{\frac{1}{2}} q_1^2\\
&\qquad + 60\Phi_1 Q_{\frac{5}{2}} q_1 + \Phi_0 Q_{\frac{9}{2}})/60.
\end{aligned}
$$

Gradings: $[Q_{\frac{3}{2}}] = 3/2$, $[Q_{\frac{7}{2}}] = 7/2$, $[Q_{\frac{11}{2}}] = 11/2$.

2.1.4. *Group 4.* This group includes the even variables $\bar{q}_1$, $\bar{q}_3$, $\bar{q}_5$ defined by

$$
\begin{aligned}
&(\bar{q}_1)_\theta = Q_{\frac{3}{2}},\\
&(\bar{q}_3)_\theta = -(Q_{\frac{7}{2}} + Q_{\frac{3}{2}} q_1^2),\\
&(\bar{q}_5)_\theta = (12Q_{\frac{11}{2}} + 42Q_{\frac{7}{2}}\Phi_{\frac{1}{2}} + 6Q_{\frac{7}{2}} q_1^2 + 12Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}} q_1 + Q_{\frac{3}{2}} q_1^4 - 24Q_{\frac{3}{2}} q_1 q_3)/3.
\end{aligned}
$$

Gradings: $[\bar{q}_1] = 1$, $[\bar{q}_3] = 3$, $[\bar{q}_5] = 5$.

Remark 5. The last three variables are not used directly in the subsequent computations, but clarify the nonlocal picture and enter in the expressions for the higher terms of hierarchies of symmetries and generating functions.

2.2. **Seeding symmetries.** Solving equation (5), which in our case is of the form

$$
\widetilde{D}_t(f) = -\widetilde{D}_\theta^6(f) + 3\widetilde{D}_\theta(f)\Phi_1 + 3\Phi_{\frac{1}{2}}\widetilde{D}_\theta^2(f) + 3\widetilde{D}_\theta^3(f)\Phi + 3\Phi_{1\frac{1}{2}} f,
$$

where $\widetilde{D}_t$ and $\widetilde{D}_\theta$ are the total derivative operators extended to the nonlocal setting (see Subsection 2.1), we found a number of solutions that serve as seeding symmetries for constructing infinite hierarchies and are used to construct *nonlocal vectors* (see Subsection 2.4 below).

These symmetries are:

The Y_k series.

$$
\begin{aligned}
&Y_1 = \Phi_1,\\
&Y_3 = \Phi_3 - 3\Phi_1\Phi_{\frac{1}{2}} - 3\Phi_0\Phi_{1\frac{1}{2}},\\
&Y_5 = \Phi_5 - 5\Phi_3\Phi_{\frac{1}{2}} - 10\Phi_2\Phi_{1\frac{1}{2}} + 10\Phi_1\Phi_{\frac{1}{2}}^2 - 10\Phi_1\Phi_{2\frac{1}{2}} + 20\Phi_0\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\qquad - 5\Phi_0\Phi_{3\frac{1}{2}}.
\end{aligned}
$$

The $Y_{k\frac{1}{2}}$ series.

$$
\begin{aligned}
Y_{\frac{3}{2}} &= -2\Phi_1 Q_{\frac{1}{2}} - \Phi_{\frac{1}{2}} q_1 + \Phi_{1\frac{1}{2}},\\
Y_{\frac{7}{2}} &= -12\Phi_3 Q_{\frac{1}{2}} - 2\Phi_1 Q_{\frac{5}{2}} + 36\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 36\Phi_0 Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 12\Phi_0\Phi_2\\
&\quad - 6\Phi_0\Phi_1 q_1 + 12\Phi_{\frac{1}{2}}^2 q_1 - 36\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} - \Phi_{\frac{1}{2}} q_1^3 + 6\Phi_{\frac{1}{2}} q_3 + 3\Phi_{1\frac{1}{2}} q_1^2 - 6\Phi_{2\frac{1}{2}} q_1\\
&\quad + 6\Phi_{3\frac{1}{2}},\\
Y_{\frac{11}{2}} &= 240\Phi_5 Q_{\frac{1}{2}} + 40\Phi_3 Q_{\frac{5}{2}} - 1200\Phi_3 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} - 2400\Phi_2 Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\quad + 2\Phi_1 Q_{\frac{9}{2}} - 120\Phi_1 Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} + 2400\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 2400\Phi_1 Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} - 600\Phi_1\Phi_3\\
&\quad + 240\Phi_1\Phi_2 q_1 - 120\Phi_0 Q_{\frac{5}{2}}\Phi_{1\frac{1}{2}} + 4800\Phi_0 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} - 1200\Phi_0 Q_{\frac{1}{2}}\Phi_{3\frac{1}{2}}\\
&\quad - 480\Phi_0\Phi_4 + 360\Phi_0\Phi_3 q_1 + 1920\Phi_0\Phi_2\Phi_{\frac{1}{2}} - 120\Phi_0\Phi_2 q_1^2 - 720\Phi_0\Phi_1\Phi_{\frac{1}{2}} q_1\\
&\quad + 1680\Phi_0\Phi_1\Phi_{1\frac{1}{2}} + 20\Phi_0\Phi_1 q_1^3 - 120\Phi_0\Phi_1 q_3 + 660\Phi_{\frac{1}{2}}^3 q_1 - 3540\Phi_{\frac{1}{2}}^2\Phi_{1\frac{1}{2}}\\
&\quad - 40\Phi_{\frac{1}{2}}^2 q_1^3 + 240\Phi_{\frac{1}{2}}^2 q_3 + 360\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} q_1^2 - 960\Phi_{\frac{1}{2}}\Phi_{2\frac{1}{2}} q_1 + 1200\Phi_{\frac{1}{2}}\Phi_{3\frac{1}{2}}\\
&\quad + \Phi_{\frac{1}{2}} q_1^5 - 60\Phi_{\frac{1}{2}} q_1^2 q_3 + 240\Phi_{\frac{1}{2}} q_5 - 720\Phi_{1\frac{1}{2}}^2 q_1 + 2400\Phi_{1\frac{1}{2}}\Phi_{2\frac{1}{2}} - 5\Phi_{1\frac{1}{2}} q_1^4\\
&\quad + 120\Phi_{1\frac{1}{2}} q_1 q_3 + 20\Phi_{2\frac{1}{2}} q_1^3 - 120\Phi_{2\frac{1}{2}} q_3 - 60\Phi_{3\frac{1}{2}} q_1^2 + 120\Phi_{4\frac{1}{2}} q_1 - 120\Phi_{5\frac{1}{2}}.
\end{aligned}
$$

The Z_k series.

$$
\begin{aligned}
Z_1 &= Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + \theta(-2\Phi_1 Q_{\frac{1}{2}} - \Phi_{\frac{1}{2}} q_1 + \Phi_{1\frac{1}{2}}),\\
Z_3 &= (3Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} q_1 - 3Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}} + Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} - 12Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 3Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} q_1 + 6Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}}\\
&\quad + 6\Phi_1 Q_{\frac{1}{2}} Q_{\frac{3}{2}} + 6\Phi_0\Phi_1 Q_{\frac{1}{2}} + \theta(-12\Phi_3 Q_{\frac{1}{2}} - 2\Phi_1 Q_{\frac{5}{2}} + 36\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\\
&\quad + 36\Phi_0 Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 12\Phi_0\Phi_2 - 6\Phi_0\Phi_1 q_1 + 12\Phi_{\frac{1}{2}}^2 q_1 - 36\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\quad - \Phi_{\frac{1}{2}} q_1^3 + 6\Phi_{\frac{1}{2}} q_3 + 3\Phi_{1\frac{1}{2}} q_1^2 - 6\Phi_{2\frac{1}{2}} q_1 + 6\Phi_{3\frac{1}{2}}))/3,\\
Z_5 &= (-15Q_{\frac{7}{2}}\Phi_{\frac{1}{2}} q_1 + 15Q_{\frac{7}{2}}\Phi_{1\frac{1}{2}} + 120Q_{\frac{3}{2}}\Phi_{\frac{1}{2}}^2 q_1 - 360Q_{\frac{3}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\quad - 10Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} q_1^3 + 60Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} q_3 + 30Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}} q_1^2 - 60Q_{\frac{3}{2}}\Phi_{2\frac{1}{2}} q_1 + 60Q_{\frac{3}{2}}\Phi_{3\frac{1}{2}}\\
&\quad - Q_{\frac{9}{2}}\Phi_{\frac{1}{2}} + 40Q_{\frac{5}{2}}\Phi_{\frac{1}{2}}^2 - 5Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} q_1^2 + 15Q_{\frac{5}{2}}\Phi_{1\frac{1}{2}} q_1 - 20Q_{\frac{5}{2}}\Phi_{2\frac{1}{2}} - 660Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^3\\
&\quad + 90Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 q_1^2 - 390Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} q_1 + 960Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{2\frac{1}{2}} + 5Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} q_1^4\\
&\quad - 30Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} q_1 q_3 + 660Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}}^2 - 10Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} q_1^3 - 30Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} q_3 + 60Q_{\frac{1}{2}}\Phi_{3\frac{1}{2}} q_1\\
&\quad - 120Q_{\frac{1}{2}}\Phi_{4\frac{1}{2}} + 12\Phi_5 - 120\Phi_3 Q_{\frac{1}{2}} Q_{\frac{3}{2}} - 60\Phi_3\Phi_{\frac{1}{2}} - 120\Phi_2\Phi_{1\frac{1}{2}}\\
&\quad - 20\Phi_1 Q_{\frac{5}{2}} Q_{\frac{3}{2}} - 30\Phi_1 Q_{\frac{1}{2}} Q_{\frac{7}{2}} + 360\Phi_1 Q_{\frac{1}{2}} Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - 10\Phi_1 Q_{\frac{1}{2}} Q_{\frac{5}{2}} q_1\\
&\quad - 240\Phi_1\Phi_2 Q_{\frac{1}{2}} - 60\Phi_1\Phi_{\frac{1}{2}} q_1^2 + 60\Phi_1\Phi_{1\frac{1}{2}} q_1 - 120\Phi_1\Phi_{2\frac{1}{2}} + 360\Phi_0 Q_{\frac{1}{2}} Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}}\\
&\quad - 360\Phi_0\Phi_3 Q_{\frac{1}{2}} + 120\Phi_0\Phi_2 Q_{\frac{3}{2}} + 120\Phi_0\Phi_2 Q_{\frac{1}{2}} q_1 + 60\Phi_0\Phi_1 Q_{\frac{3}{2}} q_1\\
&\quad - 20\Phi_0\Phi_1 Q_{\frac{5}{2}} + 720\Phi_0\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} - 180\Phi_0\Phi_1 Q_{\frac{1}{2}} q_1^2 + 300\Phi_0\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\quad - 90\Phi_0\Phi_{\frac{1}{2}} q_1^3 + 90\Phi_0\Phi_{1\frac{1}{2}} q_1^2 - 60\Phi_0\Phi_{3\frac{1}{2}} + \theta(240\Phi_5 Q_{\frac{1}{2}} + 40\Phi_3 Q_{\frac{5}{2}}\\
&\quad - 1200\Phi_3 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} - 2400\Phi_2 Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 2\Phi_1 Q_{\frac{9}{2}} - 120\Phi_1 Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} + 2400\Phi_1 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2\\
&\quad - 2400\Phi_1 Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} - 600\Phi_1\Phi_3 + 240\Phi_1\Phi_2 q_1 - 120\Phi_0 Q_{\frac{5}{2}}\Phi_{1\frac{1}{2}} + 4800\Phi_0 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\quad - 1200\Phi_0 Q_{\frac{1}{2}}\Phi_{3\frac{1}{2}} - 480\Phi_0\Phi_4 + 360\Phi_0\Phi_3 q_1 + 1920\Phi_0\Phi_2\Phi_{\frac{1}{2}} - 120\Phi_0\Phi_2 q_1^2\\
&\quad - 720\Phi_0\Phi_1\Phi_{\frac{1}{2}} q_1 + 1680\Phi_0\Phi_1\Phi_{1\frac{1}{2}} + 20\Phi_0\Phi_1 q_1^3 - 120\Phi_0\Phi_1 q_3 + 660\Phi_{\frac{1}{2}}^3 q_1\\
&\quad - 3540\Phi_{\frac{1}{2}}^2\Phi_{1\frac{1}{2}} - 40\Phi_{\frac{1}{2}}^2 q_1^3 + 240\Phi_{\frac{1}{2}}^2 q_3 + 360\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} q_1^2 - 960\Phi_{\frac{1}{2}}\Phi_{2\frac{1}{2}} q_1\\
&\quad + 1200\Phi_{\frac{1}{2}}\Phi_{3\frac{1}{2}} + \Phi_{\frac{1}{2}} q_1^5 - 60\Phi_{\frac{1}{2}} q_1^2 q_3 + 240\Phi_{\frac{1}{2}} q_5 - 720\Phi_{1\frac{1}{2}}^2 q_1 + 2400\Phi_{1\frac{1}{2}}\Phi_{2\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&- 5\Phi_{1\frac12}q_1^4 + 120\Phi_{1\frac12}q_1q_3 + 20\Phi_{2\frac12}q_1^3 - 120\Phi_{2\frac12}q_3 - 60\Phi_{3\frac12}q_1^2 + 120\Phi_{4\frac12}q_1 \\
&- 120\Phi_{5\frac12}))/5.
\end{aligned}
$$

The $Z_{k\frac12}$ series.

$$
\begin{aligned}
Z_{\frac12} &= -2\theta\Phi_1 + \Phi_{\frac12}, \\
Z_{\frac52} &= -2\Phi_1Q_{\frac32} + \Phi_1Q_{\frac12}q_1 + 2\Phi_0\Phi_1 - 4\Phi_{\frac12}^2 + \Phi_{\frac12}q_1^2 - 2\Phi_{1\frac12}q_1 + 2\Phi_{2\frac12} \\
&\quad + \theta(-4\Phi_3 + 12\Phi_1\Phi_{\frac12} + 12\Phi_0\Phi_{1\frac12}), \\
Z_{\frac92} &= -24\Phi_3Q_{\frac32} + 24\Phi_3Q_{\frac12}q_1 - 6\Phi_1Q_{\frac72} + 72\Phi_1Q_{\frac32}\Phi_{\frac12} + 2\Phi_1Q_{\frac52}q_1 \\
&\quad - 36\Phi_1Q_{\frac12}\Phi_{\frac12}q_1 + 24\Phi_1Q_{\frac12}\Phi_{1\frac12} - 36\Phi_1Q_{\frac12}q_3 + 48\Phi_1\Phi_2 + 72\Phi_0Q_{\frac32}\Phi_{1\frac12} \\
&\quad - 72\Phi_0Q_{\frac12}\Phi_{1\frac12}q_1 + 72\Phi_0\Phi_3 - 48\Phi_0\Phi_2q_1 - 144\Phi_0\Phi_1\Phi_{\frac12} + 48\Phi_0\Phi_1q_1^2 \\
&\quad + \theta(-48\Phi_5 + 240\Phi_3\Phi_{\frac12} + 480\Phi_2\Phi_{1\frac12} - 480\Phi_1\Phi_{\frac12}^2 + 480\Phi_1\Phi_{2\frac12} \\
&\quad - 960\Phi_0\Phi_{\frac12}\Phi_{1\frac12} + 240\Phi_0\Phi_{3\frac12}) + 132\Phi_{\frac12}^3 - 24\Phi_{\frac12}^2q_1^2 + 144\Phi_{\frac12}\Phi_{1\frac12}q_1 \\
&\quad - 192\Phi_{\frac12}\Phi_{2\frac12} + \Phi_{\frac12}q_1^4 - 24\Phi_{\frac12}q_1q_3 - 144\Phi_{1\frac12}^2 - 4\Phi_{1\frac12}q_1^3 + 24\Phi_{1\frac12}q_3 \\
&\quad + 12\Phi_{2\frac12}q_1^2 - 24\Phi_{3\frac12}q_1 + 24\Phi_{4\frac12}.
\end{aligned}
$$

Gradings. There are two points of view on symmetries: as on functions and as on vector fields $\mathcal{Э}_f$ (see Subsection 1.1). For functions we have:

$$
\begin{array}{llll}
[Y_1] = 5/2, & [Y_3] = 9/2, & [Y_5] = 13/2, & \text{odd;} \\
[Y_{\frac32}] = 3, & [Y_{\frac72}] = 5, & [Y_{\frac{11}{2}}] = 7, & \text{even;} \\
[Z_1] = 5/2, & [Z_3] = 7/2, & [Z_5] = 13/2, & \text{odd;} \\
[Z_{\frac12}] = 2, & [Z_{\frac52}] = 4, & [Z_{\frac92}] = 6, & \text{even.}
\end{array}
$$

For vector fields we have:

$$
\begin{array}{llll}
[\mathcal{Э}_{Y_1}] = 1, & [\mathcal{Э}_{Y_3}] = 3, & [\mathcal{Э}_{Y_5}] = 5, & \text{even;} \\
[\mathcal{Э}_{Y_{\frac32}}] = 3/2, & [\mathcal{Э}_{Y_{\frac72}}] = 7/2, & [\mathcal{Э}_{Y_{\frac{11}{2}}}] = 11/2, & \text{odd;} \\
[\mathcal{Э}_{Z_1}] = 1, & [\mathcal{Э}_{Z_3}] = 3, & [\mathcal{Э}_{Z_5}] = 5, & \text{even;} \\
[\mathcal{Э}_{Z_{\frac12}}] = 1/2, & [\mathcal{Э}_{Z_{\frac52}}] = 5/2, & [\mathcal{Э}_{Z_{\frac92}}] = 9/2, & \text{odd.}
\end{array}
$$

Note also that the symmetries Y_α do not depend on θ, while Z_α are linear functions with respect to θ.

2.3. **Seeding generating functions.** Solving equation (5), which in our case is of the form

$$
\widetilde{D}_t(f) = -\widetilde{D}_\theta^6(f) + 6\Phi_{\frac12}\widetilde{D}_\theta^2(f) - 3\Phi_0\widetilde{D}_\theta^3(f),
$$

we found a number of solutions that serve as seeding generating functions for constructing infinite hierarchies and used to construct *nonlocal forms* (see Subsection 2.5 below). These generating functions are:

The F_k series.

$$
\begin{aligned}
F_0 &= 1, \\
F_2 &= \Phi_{\frac12}, \\
F_4 &= (-2\Phi_0\Phi_1 + 3\Phi_{\frac12}^2 - \Phi_{2\frac12})/3 \qquad .
\end{aligned}
$$

The $F_{k\frac{1}{2}}$ series.

$$
\begin{aligned}
F_{\frac{1}{2}} &= Q_{\frac{1}{2}},\\
F_{\frac{5}{2}} &= (Q_{\frac{5}{2}} - 12Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 6\Phi_1 + 6\Phi_0 q_1)/6,\\
F_{\frac{9}{2}} &= (Q_{\frac{9}{2}} - 40Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} + 720Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 240Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} + 120\Phi_3 + 120\Phi_2 q_1\\
&\quad - 480\Phi_1\Phi_{\frac{1}{2}} + 60\Phi_1 q_1^2 - 480\Phi_0\Phi_1 Q_{\frac{1}{2}} - 420\Phi_0\Phi_{\frac{1}{2}} q_1 - 240\Phi_0\Phi_{1\frac{1}{2}} + 20\Phi_0 q_1^3\\
&\quad - 120\Phi_0 q_3)/20.
\end{aligned}
$$

The G_k series.

$$
\begin{aligned}
G_0 &= \theta Q_{\frac{1}{2}},\\
G_2 &= (3Q_{\frac{1}{2}}Q_{\frac{3}{2}} + 6\Phi_0 Q_{\frac{1}{2}} + \theta Q_{\frac{5}{2}} - 12\theta Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 6\theta\Phi_1 + 6\theta\Phi_0 q_1)/3,\\
G_4 &= (-10Q_{\frac{5}{2}}Q_{\frac{3}{2}} + 15Q_{\frac{1}{2}}Q_{\frac{7}{2}} + 120Q_{\frac{1}{2}}Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - 5Q_{\frac{1}{2}}Q_{\frac{5}{2}}q_1 - 120\Phi_2 Q_{\frac{1}{2}}\\
&\quad - 60\Phi_1 Q_{\frac{3}{2}} - 60\Phi_0 Q_{\frac{3}{2}} q_1 - 20\Phi_0 Q_{\frac{5}{2}} + 420\Phi_0 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 90\Phi_0 Q_{\frac{1}{2}} q_1^2\\
&\quad - 120\Phi_0\Phi_1 - \theta Q_{\frac{9}{2}} + 40\theta Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} - 720\theta Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 + 240\theta Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} - 120\theta\Phi_3\\
&\quad - 120\theta\Phi_2 q_1 + 480\theta\Phi_1\Phi_{\frac{1}{2}} - 60\theta\Phi_1 q_1^2 + 480\theta\Phi_0\Phi_1 Q_{\frac{1}{2}} + 420\theta\Phi_0\Phi_{\frac{1}{2}} q_1\\
&\quad + 240\theta\Phi_0\Phi_{1\frac{1}{2}} - 20\theta\Phi_0 q_1^3 + 120\theta\Phi_0 q_3)/90.
\end{aligned}
$$

The $G_{k\frac{1}{2}}$ series.

$$
\begin{aligned}
G_{-\frac{1}{2}} &= \theta,\\
G_{\frac{3}{2}} &= -Q_{\frac{3}{2}} + Q_{\frac{1}{2}}q_1 + 2\Phi_0 - 4\theta\Phi_{\frac{1}{2}},\\
G_{\frac{7}{2}} &= (3Q_{\frac{7}{2}} - 24Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - Q_{\frac{5}{2}}q_1 + 6Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}q_1 - 12Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 18Q_{\frac{1}{2}}q_3 - 24\Phi_2\\
&\quad - 12\Phi_1 q_1 + 84\Phi_0\Phi_{\frac{1}{2}} + 6\Phi_0 q_1^2 + 96\theta\Phi_0\Phi_1 - 144\theta\Phi_{\frac{1}{2}}^2 + 48\theta\Phi_{2\frac{1}{2}})/6.
\end{aligned}
$$

Gradings. These generating functions have the following gradings and parities:

$$
\begin{array}{cccl}
[F_0] = 0, & [F_2] = 2, & [F_4] = 4, & \text{even;}\\
[F_{\frac{1}{2}}] = 1/2, & [F_{\frac{5}{2}}] = 5/2, & [F_{\frac{9}{2}}] = 9/2, & \text{odd;}\\
[G_0] = 0, & [G_2] = 2, & [G_4] = 4, & \text{even;}\\
[G_{-\frac{1}{2}}] = -1/2, & [G_{\frac{3}{2}}] = 3/2, & [G_{\frac{7}{2}}] = 7/2, & \text{odd.}
\end{array}
$$

Note again that the generating functions F_α do not depend on θ, while G_α are linear functions with respect to θ.

2.4. **Nonlocal vectors.** We pass now to the ℓ^*-extension of equation (2). The additional coordinates on this extension are denoted by $P = P_0$, $P_{\frac{1}{2}}$, P_1, etc.

Now we introduce nonlocal variables in the ℓ^*-extension that we call *nonlocal vectors* and which are defined by

$$
\begin{array}{lll}
(P_{Y_1})_\theta = Y_1 P_0, & (P_{Y_3})_\theta = Y_3 P_0, & (P_{Y_5})_\theta = Y_5 P_0;\\
(P_{Y_{\frac{3}{2}}})_\theta = Y_{\frac{3}{2}} P_0, & (P_{Y_{\frac{7}{2}}})_\theta = Y_{\frac{7}{2}} P_0, & (P_{Y_{\frac{11}{2}}})_\theta = Y_{\frac{11}{2}} P_0;\\
(P_{Z_1})_\theta = Z_1 P_0, & (P_{Z_3})_\theta = Z_3 P_0, & (P_{Z_5})_\theta = Z_5 P_0;\\
(P_{Z_{\frac{1}{2}}})_\theta = Z_{\frac{1}{2}} P_0, & (P_{Z_{\frac{5}{2}}})_\theta = Z_{\frac{5}{2}} P_0, & (P_{Z_{\frac{9}{2}}})_\theta = Z_{\frac{9}{2}} P_0,
\end{array}
$$

where the symmetries Y_α and Z_α were described in Subsection 2.2.

The x- and t-components of these variables are given in [3].

Gradings. The variable P_0 is even and we assign grading 0 to it. Then P_k are also even variables with $[P_k]=k$ while $P_{k\frac{1}{2}}$ are odd and $[P_{k\frac{1}{2}}]=(2k+1)/2$. Consequently,

$$\begin{array}{llll}
[P_{Y_1}]=2, & [P_{Y_3}]=4, & [P_{Y_5}]=6, & \text{even;}\\
[P_{Y_{\frac{3}{2}}}]=5/2, & [P_{Y_{\frac{7}{2}}}]=9/2, & [P_{Y_{\frac{11}{2}}}]=13/2, & \text{odd;}\\
[P_{Z_1}]=2, & [P_{Z_3}]=4, & [P_{Z_5}]=6, & \text{even;}\\
[P_{Z_{\frac{1}{2}}}]=3/2, & [P_{Z_{\frac{5}{2}}}]=7/2, & [P_{Z_{\frac{9}{2}}}]=11/2, & \text{odd.}
\end{array}$$

2.5. **Nonlocal forms.** Passing now to the ℓ-extension of equation (2), we introduce the additional coordinates on this extension that are denoted by $\Omega=\Omega_0$, $\Omega_{\frac{1}{2}}$, Ω_1, etc.

Now we introduce nonlocal variables in the ℓ-extension called *nonlocal forms* and described by

$$\begin{array}{lll}
(\Omega_{F_0})_\theta=\Omega_0F_0, & (\Omega_{F_2})_\theta=\Omega_0F_2, & (\Omega_{F_4})_\theta=\Omega_0F_4;\\
(\Omega_{F_{\frac{1}{2}}})_\theta=\Omega_0F_{\frac{1}{2}}, & (\Omega_{F_{\frac{5}{2}}})_\theta=\Omega_0F_{\frac{5}{2}}, & (\Omega_{F_{\frac{9}{2}}})_\theta=\Omega_0F_{\frac{9}{2}};\\
(\Omega_{G_0})_\theta=\Omega_0G_0, & (\Omega_{G_2})_\theta=\Omega_0G_2, & (\Omega_{G_4})_\theta=\Omega_0G_4;\\
(\Omega_{G_{-\frac{1}{2}}})_\theta=\Omega_0G_{-\frac{1}{2}}, & (\Omega_{G_{\frac{3}{2}}})_\theta=\Omega_0G_{\frac{3}{2}}, & (\Omega_{G_{\frac{7}{2}}})_\theta=\Omega_0G_{\frac{7}{2}},
\end{array}$$

where the generating functions F_α and G_α were described in Subsection 2.3.

The x- and t-components of these variables are given in [3].

Gradings. The variable Ω_0 is even and we assign grading 0 to it. Then Ω_k are also even variables with $[\Omega_k]=k$, while $\Omega_{k\frac{1}{2}}$ are odd and $[\Omega_{k\frac{1}{2}}]=(2k+1)/2$. Consequently,

$$\begin{array}{llll}
[\Omega_{F_0}]=-1/2, & [\Omega_{F_2}]=3/2, & [\Omega_{F_4}]=7/2, & \text{odd;}\\
[\Omega_{F_{\frac{1}{2}}}]=0, & [\Omega_{F_{\frac{5}{2}}}]=2, & [\Omega_{F_{\frac{9}{2}}}]=4, & \text{even;}\\
[\Omega_{G_0}]=-1/2, & [\Omega_{G_2}]=3/2, & [\Omega_{G_4}]=7/2, & \text{odd;}\\
[\Omega_{G_{-\frac{1}{2}}}]=-1, & [\Omega_{G_{\frac{3}{2}}}]=1, & [\Omega_{G_{\frac{7}{2}}}]=3, & \text{even.}
\end{array}$$

2.6. **Recursion operators for symmetries.** Using the method described in Subsection 1.5, we found two nontrivial solutions of the linearized equation in the ℓ-extension enriched with nonlocal variables. The first one is

$$\begin{aligned}
\Omega^1 = {} & -Q_{\frac{1}{2}}\Omega_{F_0}\Phi_{\frac{1}{2}} - 2\Phi_1\Omega_{G_0} - \Phi_1\Omega_{F_0} + 2\Phi_1Q_{\frac{1}{2}}\Omega_{G_{-\frac{1}{2}}}\\
& - 2\Phi_0\Omega_{\frac{1}{2}} + \theta\Omega_{F_0}\Phi_{\frac{1}{2}}q_1 - \theta\Omega_{F_0}\Phi_{1\frac{1}{2}} + 2\theta\Phi_1Q_{\frac{1}{2}}\Omega_{F_0}\\
& + 2\theta\Phi_1\Omega_{F_{\frac{1}{2}}} - \Omega_{F_{\frac{1}{2}}}\Phi_{\frac{1}{2}} + \Omega_{G_{-\frac{1}{2}}}\Phi_{\frac{1}{2}}q_1 - \Omega_{G_{-\frac{1}{2}}}\Phi_{1\frac{1}{2}}\\
& - 2\Omega_0\Phi_{\frac{1}{2}} + \Omega_2.
\end{aligned}$$

The operator corresponding to the first solution is

$$\begin{aligned}
\Delta_{\Omega^1} = {} & D_\theta^4 - 2\Phi_0D_\theta - 2\Phi_{\frac{1}{2}}\\
& - (Y_1+Z_1)D_\theta^{-1}\circ F_0 - Z_{\frac{1}{2}}D_\theta^{-1}\circ F_{\frac{1}{2}} - Y_{\frac{3}{2}}D_\theta^{-1}\circ G_{-\frac{1}{2}} - 2Y_1D_\theta^{-1}\circ G_0.
\end{aligned}$$

The second solution is given in [3].

Gradings. The operator Ω^1 is even and its grading is 2.

2.7. **Recursion operators for generating functions.** Using the method described in Subsection 1.6, we found three nontrivial solutions of the adjoint linearized equation in the ℓ^*-extension enriched with nonlocal variables. The first one is

$$P^1 = Q_{\frac{1}{2}}P_{Z_{\frac{1}{2}}} + 2\Phi_0P_{\frac{1}{2}} + \theta P_{Y_{\frac{3}{2}}} + 2\theta Q_{\frac{1}{2}}P_{Y_1} - 4\Phi_{\frac{1}{2}}P_0 + P_{Y_1} + P_{Z_1} + P_2.$$

The operator corresponding to the first solution is

$$\begin{aligned}\Delta_{P^1} &= D_\theta^4 + 2\Phi_0 D_\theta - 4\Phi_{\frac12}\\ &+ (F_0 + 2G_0)D_\theta^{-1}\circ Y_1 + G_{-\frac12}D_\theta^{-1}\circ Y_{\frac32} + F_0 D_\theta^{-1}\circ Z_1 + F_{\frac12}D_\theta^{-1}\circ Z_{\frac12}.\end{aligned}$$

The second and third solutions are given in [3].

Gradings. The operator P^1 is even and its grading is 2.

2.8. **Hamiltonian structures.** Using the method described in Subsection 1.7, we found three nontrivial solutions of the linearized equation in the ℓ^*-extension enriched with nonlocal variables. The first one is

$$H^1 = P_{2\frac12} - P_{\frac12}\Phi_{\frac12} - 2\Phi_1 P_0 - 3\Phi_0 P_1.$$

The operator corresponding to the first solution is

$$\Delta_{H^1} = D_\theta^5 - 3\Phi_0 D_\theta^2 - \Phi_{\frac12} D_\theta - 2\Phi_1.$$

This operator satisfies criteria (14) and (15) and thus is Hamiltonian. Moreover, there exists a conservation law (corresponding to the nonlocal variable q_3)

$$\begin{aligned}X &= \Phi_0\Phi_{\frac12},\\ T &= -2\Phi_1\Phi_2 + \Phi_0\Phi_3 - 9\Phi_0\Phi_1\Phi_{\frac12} + 4\Phi_{\frac12}^3 - 2\Phi_{\frac12}\Phi_{2\frac12} + \Phi_{1\frac12}^2\end{aligned}$$

such that our equation can be represented as

$$\Phi_t = \Delta_{H^1}\frac{\delta}{\delta\Phi}\Big(-\frac12 X\Big),$$

and so (17) is also satisfied.

The second Hamiltonian structure is of the form

$$\begin{aligned}H^2 &= -P_{Z_{\frac12}}\Phi_{\frac12}q_1 + P_{Z_{\frac12}}\Phi_{1\frac12} - P_{Y_{\frac32}}\Phi_{\frac12} + P_{4\frac12} - 3P_{2\frac12}\Phi_{\frac12} - 3P_{1\frac12}\Phi_{1\frac12}\\ &+ 3P_{\frac12}\Phi_{\frac12}^2 - P_{\frac12}\Phi_{2\frac12} - 2Q_{\frac12}\Phi_{\frac12}P_{Y_1} - 2\Phi_3 P_0 - 7\Phi_2 P_1 - 2\Phi_1 Q_{\frac12}P_{Z_{\frac12}}\\ &+ 9\Phi_1\Phi_{\frac12}P_0 - 2\Phi_1 P_{Z_1} - 9\Phi_1 P_2 - \Phi_0\Phi_1 P_{\frac12} + 13\Phi_0\Phi_{\frac12}P_1 + 7\Phi_0\Phi_{1\frac12}P_0\\ &- 5\Phi_0 P_3 + 2\theta\Phi_1 P_{Y_{\frac32}} + 4\theta\Phi_1 Q_{\frac12}P_{Y_1} + 2\theta\Phi_{\frac12}q_1 P_{Y_1} - 2\theta\Phi_{1\frac12}P_{Y_1}.\end{aligned}$$

The corresponding operator is

$$\begin{aligned}\Delta_{H^2} &= D_\theta^9 - 5\Phi_0 D_\theta^6 - 3\Phi_{\frac12}D_\theta^5 - 9\Phi_1 D_\theta^4 - 3\Phi_{1\frac12}D_\theta^3 + (13\Phi_0\Phi_{\frac12} - 7\Phi_2)D_\theta^2\\ &+ (3\Phi_{\frac12}^2 - \Phi_{2\frac12} - \Phi_0\Phi_1)D_\theta + (9\Phi_1\Phi_{\frac12} + 7\Phi_0\Phi_{1\frac12} - 2\Phi_3)\\ &+ Y_{\frac32}D_\theta^{-1}\circ Z_{\frac12} - Z_{\frac12}D_\theta^{-1}\circ Y_{\frac32} - 2Y_1 D_\theta^{-1}\circ Z_1 - 2Z_1 D_\theta^{-1}\circ Y_1.\end{aligned}$$

The third solution is given in [3].

Gradings. The operator Δ_{H^1} is odd and of grading $5/2$. The operator Δ_{H^2} is also odd and of grading $9/2$.

2.9. **Symplectic structures.** Using the method described in Subsection 1.8, we found three nontrivial solutions of the adjoint linearized equation in the ℓ-extension enriched with nonlocal variables. The first one is

$$S^1 = \Omega_{G_0} + \Omega_{F_0} - Q_{\frac12}\Omega_{G_{-\frac12}} + \theta Q_{\frac12}\Omega_{F_0} + \theta\Omega_{F_{\frac12}}.$$

The operator corresponding to the first solution is

$$\Delta_{S^1} = (F_0 + G_0)D_\theta^{-1}\circ F_0 + G_{-\frac12}D_\theta^{-1}\circ F_{\frac12} - F_{\frac12}D_\theta^{-1}\circ G_{-\frac12} + F_0 D_\theta^{-1}\circ G_0.$$

This operator satisfies criteria (18) and (19) and thus is symplectic.

The second solution is of the form

$$\begin{aligned} S^2 = {} & (3\Omega_{G_2} - 12\Omega_{G_0}\Phi_{\frac{1}{2}} - 12\Omega_{F_2} - 12\Omega_{F_0}\Phi_{\frac{1}{2}} + 6\Omega_{1\frac{1}{2}} - 3Q_{\frac{3}{2}}\Omega_{F_{\frac{1}{2}}} \\ & - Q_{\frac{5}{2}}\Omega_{G_{-\frac{1}{2}}} + 3Q_{\frac{1}{2}}Q_{\frac{3}{2}}\Omega_{F_0} + 3Q_{\frac{1}{2}}\Omega_{F_{\frac{1}{2}}}q_1 + 12Q_{\frac{1}{2}}\Omega_{G_{-\frac{1}{2}}}\Phi_{\frac{1}{2}} \\ & - 3Q_{\frac{1}{2}}\Omega_{G_{\frac{3}{2}}} - 6\Phi_1\Omega_{G_{-\frac{1}{2}}} + 6\Phi_0 Q_{\frac{1}{2}}\Omega_{F_0} + 6\Phi_0\Omega_{F_{\frac{1}{2}}} - 6\Phi_0\Omega_{G_{-\frac{1}{2}}}q_1 \\ & + 6\Phi_0\Omega_0 + \theta Q_{\frac{5}{2}}\Omega_{F_0} - 12\theta Q_{\frac{1}{2}}\Omega_{F_2} - 12\theta Q_{\frac{1}{2}}\Omega_{F_0}\Phi_{\frac{1}{2}} + 6\theta\Phi_1\Omega_{F_0} \\ & + 6\theta\Phi_0\Omega_{F_0}q_1 - 12\theta\Omega_{F_{\frac{1}{2}}}\Phi_{\frac{1}{2}} + 6\theta\Omega_{F_{\frac{5}{2}}})/6. \end{aligned}$$

The corresponding operator is

$$\begin{aligned} \Delta_{S^2} = {} & D_\theta^3 + \Phi_0 + (\frac{1}{2}G_2 - 2F_2)D_\theta^{-1}\circ F_0 \\ & - 2(F_0 + G_0)D_\theta^{-1}\circ F_2 + \frac{1}{2}G_{\frac{3}{2}}D_\theta^{-1}\circ F_{\frac{1}{2}} + G_{-\frac{1}{2}}D_\theta^{-1}\circ F_{\frac{5}{2}} \\ & - 2F_2 D_\theta^{-1}\circ G_0 + \frac{1}{2}F_0 D_\theta^{-1}\circ G_2 - 6F_{\frac{5}{2}}D_\theta^{-1}\circ G_{-\frac{1}{2}} - \frac{1}{2}F_{\frac{1}{2}}D_\theta^{-1}\circ G_{\frac{3}{2}}. \end{aligned}$$

The third solution is given in [3].

Gradings. The operator Δ_{S^1} is odd and of grading $-1/2$. The second operator is also odd and its grading equals $3/2$.

2.10. **Interrelations.** Using the symmetries computed in Subsection 2.2 and applying the recursion operator obtained in Subsection 2.6, we get four infinite series of (generally, nonlocal) symmetries

$$\begin{array}{lll} Y_{2k-1}, & [Y_{2k-1}] = (4k+1)/2, & \text{odd}, \\ Y_{\frac{4k-1}{2}}, & [Y_{\frac{4k-1}{2}}] = 2k+1, & \text{even}, \\ Z_{2k-1}, & [Z_{2k-1}] = (4k+1)/2, & \text{odd}, \\ Z_{\frac{4k-3}{2}}, & [Z_{\frac{4k-3}{2}}] = 2k, & \text{even}, \end{array}$$

$k = 1, 2, \ldots$

In a similar war, using the results of Subsections 2.3 and 2.7, we get four infinite series of generating functions

$$\begin{array}{lll} F_{2k-2}, & [F_{2k-2}] = 2k-2, & \text{even}, \\ F_{\frac{4k-3}{2}}, & [F_{\frac{4k-3}{2}}] = (4k-3)/2, & \text{odd}, \\ G_{2k}, & [G_{2k}] = 2k, & \text{even}, \\ G_{\frac{4k-5}{2}}, & [G_{\frac{4k-5}{2}}] = (4k-5)/2, & \text{odd}, \end{array}$$

$k = 1, 2, \ldots$

These series are related to each other (up to rational coefficients) by the operators of Subsections 2.6–2.9 in the following way:

$$\begin{tikzcd} & Y_{2k-1} \arrow[r, "\Delta_{\Omega^1}"] \arrow[d, "\Delta_{S^1}"] & Y_{2k+1} \arrow[d, "\Delta_{S^1}"] \\ F_{2k-2} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & F_{2k} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & F_{2k+2} \end{tikzcd} \qquad \begin{tikzcd} & Z_{2k-1} \arrow[r, "\Delta_{\Omega^1}"] \arrow[d, "\Delta_{S^1}"] & Z_{2k+1} \arrow[d, "\Delta_{S^1}"] \\ G_{2k} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & G_{2k+2} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & G_{2k+4} \end{tikzcd}$$

$$\begin{tikzcd} & Y_{\frac{4k-1}{2}} \arrow[r, "\Delta_{\Omega^1}"] \arrow[d, "\Delta_{S^1}"] & Y_{\frac{4k+3}{2}} \arrow[d, "\Delta_{S^1}"] \\ F_{\frac{4k-3}{2}} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & F_{\frac{4k+1}{2}} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & F_{\frac{4k+5}{2}} \end{tikzcd} \qquad \begin{tikzcd} & Z_{\frac{4k-3}{2}} \arrow[r, "\Delta_{\Omega^1}"] \arrow[d, "\Delta_{S^1}"] & Z_{\frac{4k+1}{2}} \arrow[d, "\Delta_{S^1}"] \\ G_{\frac{4k-5}{2}} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & G_{\frac{4k-1}{2}} \arrow[ur, "\Delta_{H^1}"] \arrow[r, "\Delta_{P^1}"'] & G_{\frac{4k+3}{2}} \end{tikzcd}$$

Remark 6. Actually, there exists another hierarchy of symmetries S_{2k}, $k = 0, 1, \dots$, with the seeding element

$$S_0 = 6(-\Phi_3 + 3\Phi_1\Phi_{\frac{1}{2}} + 3\Phi\Phi_{1\frac{1}{2}})t + 2\Phi_1 x + \theta\Phi_{\frac{1}{2}} + 3\Phi$$

(the scaling symmetry). All these symmetries are odd, linear with respect to x, t, and θ, and have grading $[S_{2k}] = (4k + 3)/2$.

3. Conclusion

The study of the $N = 1$ supersymmetric KdV equation exposed in this paper demonstrates the power and efficiency of the geometrical methods elaborated in [1] and [4]. In particular, we found recursion operators for symmetries and generating functions, Hamiltonian and symplectic structures, constructed five infinite series of symmetries, one of which was not known before.

Our experience shows that the methods applied are of a universal nature and may be used to analyze a lot of other equations, both classical and supersymmetric. In particular, from technical point of view, the canonical representation of nonlocal operators (see Subsection 1.9) seems to be quite efficient and convenient when dealing with such operators. Note that all nonlocal operators constructed in this paper are represented in the canonical form.

We strongly believe that the majority of the problems formulated in [9] can be solved by our methods. We plan to demonstrate this in forthcoming publications. Note in particular that the nonlocal Hamiltonian structure indicated in [9] is inverse to our symplectic structure S^1.

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References

[1] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor′kova, I. S. Krasil′shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov, *Symmetries and conservation laws for differential equations of mathematical physics*, Monograph, Amer. Math. Soc., 1999.

[2] S. Igonin, A. Verbovetsky, and R. Vitolo, *On the formalism of local variational differential operators*, Memorandum 1641, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2002, URL http://www.math.utwente.nl/publications/2002/1641abs.html.

[3] P. Kersten, I. Krasil′shchik, and A. Verbovetsky, *An extensive study of the $N = 1$ supersymmetric KdV equation*, Memorandum 1656, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2002, URL http://www.math.utwente.nl/publications/2002/1656abs.html.

[4] P. Kersten, I. Krasil′shchik, and A. Verbovetsky, *Hamiltonian operators and ℓ^*-coverings*, Memorandum 1640, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2002, URL http://www.math.utwente.nl/publications/2002/1640abs.html.

[5] I. S. Krasil′shchik and P. H. M. Kersten, *Symmetries and recursion operators for classical and supersymmetric differential equations*, Kluwer, 2000.

[6] I. S. Krasil′shchik and A. M. Vinogradov, *Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws, and Bäcklund transformations*, Acta Appl. Math. **15** (1989), 161–209.

[7] J. Krasil′shchik and A. M. Verbovetsky, *Homological methods in equations of mathematical physics*, Advanced Texts in Mathematics, Open Education & Sciences, Opava, 1998, arXiv:math.DG/9808130.

[8] Yu. I. Manin and A. O. Radul, *A supersymmetric extension of the Kadomtsev-Petviashvili hierarchy*, Comm. Math. Phys. **98** (1985), 65–77.

[9] P. Mathieu, *Open problems for the super KdV equations*, Bäcklund and Darboux Transformations. The Geometry of Solitons, CRM Proc. Lecture Notes, vol. 29, Amer. Math. Soc., 2001, pp. 325–334, arXiv:math-ph/0005007.

[10] A. M. Verbovetsky, *Lagrangian formalism over graded algebras*, J. Geom. Phys. **18** (1996), 195–214, arXiv:hep-th/9407037.

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