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Homogeneous spaces associated with Miura transformations

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Abstract. For scalar evolution equations, we prove that a zero-curvature representation with values in a Lie algebra $\mathfrak{g}$ and a certain type homogeneous space of $\mathfrak{g}$ determine a Miura type transformation. Using this result, we show how to construct and classify Miura type transformations for a given equation.

Recall that an *action* of a Lie algebra $\mathfrak{g}$ on a manifold W is a homomorphism $\rho\colon \mathfrak{g} \to D(W)$ to the Lie algebra $D(W)$ of vector fields on W. The action is said to be *transitive* if for each point $a \in W$ the mapping

$$\mathfrak{g} \to T_a W, \quad v \mapsto \rho(v)_a$$

is surjective. In this case W is called a *homogeneous space* of $\mathfrak{g}$. Two actions $\rho_i\colon \mathfrak{g} \to D(W_i)$, $i = 1, 2$, are said to be *isomorphic* if there is a diffeomorphism $\varphi\colon W_1 \to W_2$ such that $\rho_2 = \varphi_* \rho_1$.

Below our considerations are always local. The results are valid in both categories of smooth and analytic manifolds. Depending on the category considered, all functions are supposed to be smooth or analytic.

Consider two scalar evolution equations

$$u_t = P(u, u_1, \dots, u_p), \quad u_k = \frac{\partial^k u}{\partial x^k}, \tag{1}$$

$$v_t = R(v, v_1, \dots, v_r), \quad v_k = \frac{\partial^k v}{\partial x^k}. \tag{2}$$

and a transformation

$$u = S(v, v_1, \dots, v_n) \tag{3}$$

such that for any solution $v(x, t)$ of (2) function (3) satisfies (1). Such transformations are called *Miura type transformations* (MT in short) by analogy with the famous Miura transformation connecting the KdV and the modified KdV equations. The maximal integer n such that (3) depends nontrivially on v_n is called the *order* of the MT.

In this paper we develop a description of MTs in terms of homogeneous spaces of Lie algebras.

Introduce new variables

$$w^i = \partial^{i-1} v / \partial x^{i-1}, \quad i = 1, \dots, n, \tag{4}$$

and rewrite system (2), (3) as follows

$$\begin{aligned}\frac{\partial w^i}{\partial x} &= w^{i+1}, \quad i=1,\dots,n-1,\\ \frac{\partial w^n}{\partial x} &= a(w^1,\dots,w^n,u),\\ \frac{\partial w^i}{\partial t} &= b^i(w^1,\dots,w^n,u,\dots,u_{p-1}), \quad i=1,\dots,n.\end{aligned} \tag{5}$$

where p is the order of (1). And vice versa, it is easily seen that any consistent system of this form with

$$\frac{\partial}{\partial u}a(w^1,\dots,w^n,u)\neq 0 \tag{6}$$

determines a MT of order n for (1) as follows:

- substitute (4) to (5),
- from equation (5) express $u=S(v,v_1,\dots,v_n)$,
- let $D=\sum_{i\ge 0} v_{i+1}\partial/\partial v_i$, then equation (2) is given by

$$v_t = b^1(v,v_1,\dots,v_{n-1},S,D(S),\dots,D^{p-1}(S)).$$

Consider the *total derivative* operators

$$\begin{aligned}D_x &= \frac{\partial}{\partial x}+\sum_{j\ge 0} u_{j+1}\frac{\partial}{\partial u_j},\\ D_t &= \frac{\partial}{\partial t}+\sum_{j\ge 0} D_x^j\big(P(u,u_1,\dots,u_p)\big)\frac{\partial}{\partial u_j},\end{aligned}$$

and more general overdetermined systems

$$\begin{aligned}\frac{\partial w^i}{\partial x} &= a^i(w^1,\dots,w^n,u), \quad i=1,\dots,n,\\ \frac{\partial w^i}{\partial t} &= b^i(w^1,\dots,w^n,u,\dots,u_{p-1}), \quad i=1,\dots,n.\end{aligned} \tag{7}$$

consistent modulo (1). Clearly, an invertible change of variables

$$w^i \mapsto f^i(w^1,\dots,w^n)$$

leads to a new system of the form (7). Two systems related by such a change of variables are said to be *equivalent*.

System (7) is completely determined by the vector fields

$$A=\sum_{i=1}^n a^i(w^1,\dots,w^n,u)\frac{\partial}{\partial w^i}, \quad B=\sum_{i=1}^n b^i(w^1,\dots,w^n,u,\dots,u_{p-1})\frac{\partial}{\partial w^i}.$$

Consistency of (7) modulo (1) is equivalent to the equation

$$[D_x+A,\, D_t+B]=0. \tag{8}$$

Remark 1. Coordinate-independent study of such objects is performed in [1, 6] in the framework of the theory of coverings of PDEs. Our results are considerably inspired by this theory. Another source of the present ideas is the paper [3], which announces some remarkable results on MTs for the KdV equation.

Recall that two functions

$$M(u), \quad N(u, \dots, u_{p-1}) \tag{9}$$

with values in a Lie algebra $\mathfrak{g}$ constitute a *zero-curvature representation* (ZCR in short) for (1) if

$$[D_x + M,\, D_t + N] = D_x N - D_t M + [M, N] = 0 \tag{10}$$

Then each action $\rho\colon \mathfrak{g} \to D(W)$ and a choice of local coordinates $w^1, \dots, w^n$ in W determine a consistent system of the form (7) with $A = \rho(M)$ and $B = \rho(N)$, since equation (8) follows from (10). Clearly, different choices of coordinates in W or isomorphic actions determine equivalent systems (7).

For each $k \in \mathbb{N}$ we define the subalgebra $\mathfrak{g}_k$ of $\mathfrak{g}$ by induction on k as follows:

- $\mathfrak{g}_0 = 0$,
- $\mathfrak{g}_1$ is the subalgebra generated by all the elements
$$M(u) - M(u') \in \mathfrak{g},$$
- $\mathfrak{g}_{k+1}$ is generated by the subspaces $\mathfrak{g}_k$ and $[\mathfrak{g}_k, M(u)]$.

Set also $\tilde{\mathfrak{g}} = \cup_{k\ge 0}\mathfrak{g}_k$.

Let us present the main result of this paper.

Theorem 1. *Consider ZCR (9) with values in $\mathfrak{g}$ and an action $\rho\colon \mathfrak{g} \to D(W)$, where $\dim W = n$. The corresponding system (7) is equivalent to a system of the form (5) with (6) (i.e., determines a MT for (1)) if and only if the following conditions hold.*

1. *The subalgebra $\rho(\tilde{\mathfrak{g}})$ acts on W transitively.*
2. *The subalgebra $\rho(\mathfrak{g}_{n-1})$ acts on W nontransitively.*

In this case a nonconstant function w on W invariant under $\rho(\mathfrak{g}_{n-1})$ is unique up to a change $w \mapsto g(w)$. The functions

$$w^i = \rho\big(M(u)\big)^{i-1}(w), \quad i = 1, \dots, n, \tag{11}$$

do not depend on u and are local coordinates in which system (7) constructed by ρ takes the desired form (5), (6).

Proof. If the system constructed by ρ is of the form (5), (6) then by the definition of $\mathfrak{g}_k$ we obtain that the image of $\mathfrak{g}_k$ in each tangent space is spanned by $\partial/\partial w^{n-k+1}, \dots, \partial/\partial w^n$. This obviously implies Conditions 1 and 2.

Conversely, let ρ satisfy Conditions 1, 2. Consider a generic point $a \in W$ and the local orbits $O_k \subset W$ of this point under the action of $\rho(\mathfrak{g}_k)$. Conditions 1, 2 say that

$$\dim O_{n-1} < n, \quad \exists\, c: \;\; O_k \text{ is open in } W \;\; \forall\, k \ge c. \tag{12}$$

On the other hand, from the definition of $\mathfrak{g}_k$ it follows easily that if $U \cap O_m = U \cap O_{m+1}$ for some $m \ge 0$ and some neighbourhood U of a then $U \cap O_k = U \cap O_m$ for all $k \ge m$.

Combining this with (12), we obtain

$$a = O_0 \subsetneq O_1 \subsetneq \cdots \subsetneq O_n, \quad \dim O_k = k, \ k = 0, 1, \ldots, n. \tag{13}$$

Therefore, there is a nonconstant function w invariant under $\rho(\mathfrak{g}_{n-1})$, and it is unique up to a change $w \mapsto g(w)$. Consider now functions (11). Using (13) and the definition of $\mathfrak{g}_k$, by induction on k one proves

$$\rho(\mathfrak{g}_{n-k})(w^k) = 0, \quad \rho(\mathfrak{g}_{n-k+1})(w^k) \neq 0, \quad k = 1, \ldots, n. \tag{14}$$

In particular, $w^1, \ldots, w^{n-1}$ are invariant under $\rho(\mathfrak{g}_1)$, which implies that $w^1, \ldots, w^n$ do not depend on u. Combining (13) and (14), we obtain that $w^1, \ldots, w^n$ form a system of local coordinates in W. It is easily seen that system (7) constructed by ρ is of the form (5), (6) in these coordinates. □

Corollary 1. *If $\mathfrak{g}_m = \mathfrak{g}_{m+1}$ for some $m \geq 0$ then ZCR (9) cannot produce MTs of order greater than m.*

Proof. By Theorem 1, a MT of order n is determined by a transitive action ρ of $\tilde{\mathfrak{g}}$ such that $\rho(\mathfrak{g}_{n-1})$ is not transitive. Since in our case $\tilde{\mathfrak{g}} = \mathfrak{g}_k$ for any $k \geq m$, there are no such actions for $n > m$. □

It can be shown that for any equation (1) there is a *universal* ZCR with values in the (possibly infinite-dimensional) *Wahlquist-Estabrook Lie algebra* $\mathfrak{we}$ of (1) such that any consistent system (7) arises from an action of $\mathfrak{we}$ (see [2, 8, 6] and references therein). In particular, by Theorem 1, all MTs of (1) are determined by some transitive actions of $\mathfrak{we}$.

There is an algorithmic procedure to find the universal ZCR for a given equation (1), see [2, 6]. However, in this way the Lie algebra $\mathfrak{we}$ is described in terms of generators and relations, which makes it difficult to study transitive actions of $\mathfrak{we}$. Using heavy computations, for a number of equations the explicit structure of $\mathfrak{we}$ in terms of well-known Lie algebras was described (see [8] and references therein). It is shown in [5] how these computations can be simplified.

Example 1. According to [8], for the KdV equation $u_t = u_3 + u_1 u$ we have

$$\mathfrak{we} \cong H \oplus \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$$

where H is the 5-dimensional nilpotent Heisenberg algebra with the basis r_i, $i = -2, -1, 0, 1, 2$ and the commutator table

$$[r_{-1}, r_1] = [r_2, r_{-2}] = r_0, \quad [r_i, r_j] = 0 \ \forall\, i + j \neq 0.$$

The universal ZCR is $M(u) = X_1 + \frac{1}{3}uX_2 + \frac{1}{6}u^2 X_3$, where

$$X_1 = r_1 - \frac{1}{2}y + \frac{1}{2}\lambda z, \quad X_2 = r_{-1} + z, \quad X_3 = r_{-2},$$

and h, y, z is a basis of $\mathfrak{sl}_2$ with the relations $[h, y] = 2y$, $[h, z] = -2z$, $[y, z] = h$. Here the form of $N(u, u_1, u_2)$ in (9) is not important for us.

We have

$$\mathfrak{g}_1 = \langle X_2,\ X_3 \rangle, \quad \mathfrak{g}_2 = \langle r_{-2},\ r_{-1},\ z,\ 2r_0 - h \rangle,$$

and

$$\mathfrak{g}_3 = \mathfrak{g}_k = \tilde{\mathfrak{g}} = \langle \mathfrak{sl}_2 \otimes \mathbb{C}[\lambda],\ r_{-2},\ r_{-1},\ r_0,\ r_1 \rangle \quad \forall\, k \geq 3.$$

By Corollary 1, any MT of the KdV equation is of order not greater than 3. A weaker fact that each MT is reduced to a MT of order not greater than 3 'by introduction of a potential' was announced in [3].

Let us explain how our method of constructing MTs includes the one of [3].

Consider the part of this ZCR with values in $\mathfrak{f} = \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$. We have

$$\mathfrak{f}_1 = \langle z \rangle, \quad \mathfrak{f}_2 = \langle z, h \rangle, \quad \mathfrak{f}_3 = \tilde{\mathfrak{f}} = \mathfrak{f}.$$

By Theorem 1, each homogeneous space of $\mathfrak{f}$ of dimension ≤ 3 determines a MT for the KdV equation. Conditions 2 is satisfied, since $\dim \mathfrak{f}_k = k$ for $k = 0, 1, 2$.

According to [5], for a transitive action $\rho\colon \mathfrak{f} \to D(W)$ the image $\rho(\mathfrak{f})$ is finite-dimensional and is of the form

$$\mathfrak{sl}_2 \otimes \mathbb{C}[\lambda]/(f(\lambda)), \quad f(\lambda) \in \mathbb{C}[\lambda]. \tag{15}$$

Let

$$f(\lambda) = a \prod_{s=1}^{k} (\lambda - e_s)^{k_s}, \quad a,\, e_s \in \mathbb{C}, \quad a \neq 0,\; e_i \neq e_j \;\forall\, i \neq j.$$

Then Lie algebra (15) is isomorphic to

$$\bigoplus_{s=1}^{k} \mathfrak{sl}_2 \otimes \mathbb{C}[\lambda]/(\lambda^{k_s}). \tag{16}$$

The Lie groups

$$\prod_{s} \mathrm{SL}_2\big(\mathbb{C}[\lambda]/(\lambda^{k_s})\big).$$

that appear in [3] have (16) as their Lie algebras. Thus construction of MTs is reduced to local description of homogeneous spaces of dim ≤ 3 of these Lie groups. This description and the corresponding MTs are presented in [3].

Example 2. The equation

$$u_t = u_5 + 10uu_3 + 25u_1u_2 + 20u^2u_1 \tag{17}$$

admits the following Lax pair [4]

$$L_t = [L, A], \tag{18}$$

$$L = D_x^3 + 2uD_x + u_1,$$

$$A = 9D_x^5 + 30uD_x^3 + 45u_1D_x^2 + (20u^2 + 35u_2)D_x + (10u_3 + 20uu_1).$$

Let us show how this Lax pair leads to a ZCR and MTs for (17). As usual, equation (18) is equivalent to consistency of the following system

$$L\psi = \lambda\psi, \quad \psi_t = -A\psi, \tag{19}$$

where λ is a parameter. Introduce new variables

$$q^1 = \psi, \quad q^2 = \psi_x, \quad q^3 = \psi_{xx} + u\psi \tag{20}$$

and rewrite (19) as follows

$$Q_x = -MQ, \quad Q_t = -NQ, \tag{21}$$

where

$$Q = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -1 & 0 \\ -u & 0 & -1 \\ \lambda & -u & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} u_3 + 8uu_1 + 3u\lambda & -u_2 - 4u^2 & 9\lambda \\ f + 3u_1\lambda & -6u\lambda & -u_2 - 4u^2 \\ 5u^2\lambda + 2u_2\lambda & f - 3u_1\lambda & 3u\lambda - u_3 - 8uu_1 \end{pmatrix},$$

$$f = u_4 + 8u_1^2 + 9uu_2 + 9\lambda^2 + 4u^3.$$

Let us treat λ as a formal variable. Then the matrices M, N constitute a ZCR (9) with values in $\mathfrak{sl}_3(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$. More precisely, M and N take values in the subalgebra $\mathfrak{g} \subset \mathfrak{sl}_3 \otimes \mathbb{C}[\lambda]$ generated by the two matrices

$$U = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda & 0 & 0 \end{pmatrix}. \tag{22}$$

The subalgebra $\mathfrak{g}_1$ is spanned by the matrix U, the subalgebra $\mathfrak{g}_2$ has the basis

$$U, \quad [U, V] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and $\mathfrak{g}_3 = \mathfrak{g}$. By Corollary 1, this ZCR leads to MTs of order not greater than 3. By Theorem 1, every homogeneous space W of $\mathfrak{g}$ with $\dim W \leq 3$ indeed determines a MT of order $\dim W$. Conditions 2 is satisfied, since $\dim \mathfrak{g}_k = k$ for $k = 0, 1, 2$.

For example, substituting $\lambda = 0$ we obtain an epimorphism $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_2(\mathbb{C})$. The canonical transitive action of $\mathfrak{sl}_2$ on the projective line determines the MT [4]

$$u = -v_1 - \frac{1}{2}v^2, \quad v_t = v_5 - 5(v_1v_3 + v_2^2 + v_1^3 + 4vv_1v_2 + v^2v_3 - v^4v_1).$$

Remark 2. It is easily seen that for any Lax pair $L_t = [L, A]$ such that L depends on u, u_1 only and is linear in u_1 there is a similar to (20) change of variables that transforms (19) to a ZCR (21) of the form (9). Therefore, each Lax pair of the described type leads to MTs.

Remark 3. A ZCR of the form (9) with values in a Lie algebra $\mathfrak{g}$ is equivalent to a homomorphism from the Wahlquist-Estabrook algebra to $\mathfrak{g}$. Therefore, we have a homomorphism from the Wahlquist-Estabrook algebra of (17) onto the algebra generated by (22). We conjecture that the kernel of this homomorphism is nilpotent. Then any MT of (17) is obtained from the MTs described here by introduction of a potential in the sense of [3].

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