The Diffiety Institute Preprint Series

Preprint DIPS–5 April 13, 2002

Bäcklund transformations for the Liouville equation provide nonlocal variables and nonlocal structures

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Bäcklund transformations for the Liouville equation provide nonlocal variables and nonlocal structures

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Abstract. Bäcklund autotransformation for the hyperbolic Liouville equation, Bäcklund transformation between the latter equation and the wave equation, and Bäcklund transformation between the Liouville equation and the scal$^+$-Liouville equation are integrated in nonlocal variables. For the Liouville equation, shadows of nonlocal symmetries are obtained, and these shadows are extended up to true nonlocal symmetries in non-abelian coverings. Nonlocal classical conservation laws are reconstructed for the latter equation.
Mathematics Subject Classification (2000): 35Q53, 58J72, 32G08,

Introduction. In this paper, following [1, 2] we study nonlocal aspects of integrating Bäcklund transformations between PDE, regarding these transformations as non-abelian one-dimensional coverings over infinitely prolonged equations. The aim of this paper is to illustrate a natural scheme that provides nonlocal variables associated with a certain PDE and to obtain nonlocal structures in the case of the hyperbolic Liouville equation. Notation and all definitions follow [3]. The paper is the extended version of the report [4].

Definition 1 ([3])**.** A *Bäcklund transformation* $\mathcal{B}(\tilde{\mathcal{E}}, \tau_1, \tau_2, \mathcal{E}_1, \mathcal{E}_2)$ between equations $\mathcal{E}_1$ and $\mathcal{E}_2$ is the diagram

$$\mathcal{E}_1^\infty \xleftarrow{\tau_1} \tilde{\mathcal{E}} \xrightarrow{\tau_2} \mathcal{E}_2^\infty \tag{1}$$

such that τ_1 and τ_2 are coverings with the same covering equation $\tilde{\mathcal{E}}$ over the infinitely prolonged equations $\mathcal{E}_1^\infty$ and $\mathcal{E}_2^\infty$, respectively.

The following three Bäcklund transformations for Eq. (2) are well-known, see [5]: The equations $\mathcal{B}_u(u, \tilde{u}; t)$ of Bäcklund autotransformation for the hyperbolic Liouville equation

$$\mathcal{E}_u = \{u_{xy} - \exp(2u) = 0\} \tag{2}$$

are

$$(\tilde{u} - u)_x = \exp(-t) \cdot \exp(\tilde{u} + u), \qquad (\tilde{u} + u)_y = 2\exp(t) \cdot \sinh(\tilde{u} - u) \quad t \in \mathbb{R}, \tag{3}$$

Bäcklund transformation $\mathcal{B}_{uv}(u, v; t)$ between Liouville's equation (2) and the wave equation $\mathcal{E}_v = \{v_{xy} = 0\}$ is

$$(v - u)_x = \exp(-t)\exp(u + v), \qquad (v + u)_y = -\exp(t)\exp(u - v), \quad t \in \mathbb{R}, \tag{4}$$

and Bäcklund transformation $\mathcal{B}_{u\Upsilon}(u, \Upsilon; t)$ between the Liouville equation and the scal$^+$-Liouville equation $\mathcal{E}_\Upsilon = \{\Upsilon_{xy} = \exp(-2\Upsilon)\}$ is

$$(\Upsilon - u)_x = 2\exp(-t)\cosh(\Upsilon + u), \quad (\Upsilon + u)_y = -\exp(t)\exp(u - \Upsilon), \quad t \in \mathbb{R}. \tag{5}$$

Remark 1. Note that to each solution $u(x,y)$ to the Liouville equation $\mathcal{E}_u$ we can assign the function $\Upsilon(x,y) = -u(-x,y)$ such that $\Upsilon(x,y)$ is a solution to the scal$^+$-Liouville equation $\mathcal{E}_\Upsilon$ and vice versa, i.e. these solutions are in bijective correspondence and Eq. (5) provides nonlocal (invoking $x \mapsto -x$) Bäcklund autotransformation for Eq. (2).

In what follows, we reconstruct the coverings τ_j for Eq. (3)–(5) and demonstrate the nonlocal variables to be potentials for the fiber variables u, v, and Υ.

1. In this section, we introduce the one-dimensional non-abelian coverings such that we can integrate Bäcklund transformations (3)–(5) in the corresponding nonlocal variables.

Further on, we choose an arbitrary $t \in \mathbb{R}$ and consider the extended total derivatives

$$\tilde{D}_x^{\mathcal{E}_u} = \bar{D}_x^{\mathcal{E}_u} - \exp(2u)\frac{\partial}{\partial \Xi_t} \quad \text{and} \quad \tilde{D}_y^{\mathcal{E}_u} = \bar{D}_y^{\mathcal{E}_u} + (\Xi_t^2 + 2u_y\Xi_t - \exp(2t))\frac{\partial}{\partial \Xi_t},$$

$$\tilde{D}_x^{\mathcal{E}_u} = \bar{D}_x^{\mathcal{E}_u} - \exp(2u)\frac{\partial}{\partial \Xi_\infty} \quad \text{and} \quad \tilde{D}_y^{\mathcal{E}_u} = \bar{D}_y^{\mathcal{E}_u} + (\Xi_\infty^2 + 2u_y\Xi_\infty)\frac{\partial}{\partial \Xi_\infty},$$

$$\tilde{D}_y^{\mathcal{E}_u} = \bar{D}_y^{\mathcal{E}_u} - \exp(2u)\frac{\partial}{\partial \Xi'_t} \quad \text{and} \quad \tilde{D}_x^{\mathcal{E}_u} = \bar{D}_x^{\mathcal{E}_u} + ({\Xi'_t}^2 + 2u_x\Xi'_t + \exp(-2t))\frac{\partial}{\partial \Xi'_t},$$

$$\tilde{D}_x^{\mathcal{E}_v} = \bar{D}_x^{\mathcal{E}_v} + \exp(2v)\frac{\partial}{\partial \Xi_t^v} \quad \text{and} \quad \tilde{D}_y^{\mathcal{E}_v} = \bar{D}_y^{\mathcal{E}_v} + (2v_y\Xi_t^v + \exp(2t))\frac{\partial}{\partial \Xi_t^v},$$

and finally,

(6)
$$\tilde{D}_y^{\mathcal{E}_\Upsilon} = \bar{D}_y^{\mathcal{E}_\Upsilon} + \exp(-2\Upsilon)\frac{\partial}{\partial \Xi_t^\Upsilon} \quad \text{and} \quad \tilde{D}_x^{\mathcal{E}_\Upsilon} = \bar{D}_x^{\mathcal{E}_\Upsilon} + ((\Xi_t^\Upsilon)^2 - 2\Upsilon_x\Xi_t^\Upsilon + \exp(-2t))\frac{\partial}{\partial \Xi_t^\Upsilon}.$$

We see that in all cases the extended derivatives commute: $[\tilde{D}_x, \tilde{D}_y] = 0$, and thus the coverings

(7)
$$\tau_t\colon \tilde{\mathcal{E}}_t \to \mathcal{E}_u^\infty, \quad \tau_\infty\colon \tilde{\mathcal{E}}_\infty \to \mathcal{E}_u^\infty, \quad \tau'_t\colon \tilde{\mathcal{E}}'_t \to \mathcal{E}_u^\infty, \quad \tau_t^v\colon \tilde{\mathcal{E}}_t^v \to \mathcal{E}_v^\infty, \quad \text{and } \tau_t^\Upsilon\colon \tilde{\mathcal{E}}_t^\Upsilon \to \mathcal{E}_\Upsilon^\infty$$

are defined. Explicit form of the covering equations $\tilde{\mathcal{E}}_t$, $\tilde{\mathcal{E}}_\infty$, $\tilde{\mathcal{E}}'_t$, $\tilde{\mathcal{E}}_t^v$, and $\tilde{\mathcal{E}}_t^\Upsilon$, is given in (9).

Remark 2. The coverings (7) with nonlocal variables (6) are non-abelian and thus cannot be reduced to local conservation laws for the underlying equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$. We also claim that the t-parameterized coverings, e.g. τ_t at the points t_1 and t_2, are equivalent: $\tau_{t_1} \simeq \tau_{t_2}$, i.e., there is the functional dependence between the nonlocal variables, Ξ_{t_1} and Ξ_{t_2} in the case under consideration. Really,

$$\Xi_{t_1} + y\cdot\exp(2t_1) = \Xi_{t_2} + y\cdot\exp(2t_2) = \Xi_{t=-\infty} \qquad \forall t_1, t_2 \in \mathbb{R}.$$

According to [1], the deformations of coverings (7) are such that their structural elements U_t evolve by $dU_t/dt = [\![\hat{X}_t, U_t]\!]^{\text{FN}}$, where $\hat{X}_t$ is a shadow field and $[\![\cdot,\cdot]\!]^{\text{FN}}$ is the Frölicher–Nijenhuis bracket.

Conversely, to each local symmetry of the equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$, we can assign the one-parametric family of one-dimensional non-abelian coverings, thus generalizing (7). If such deformations provided by the symmetries of equations $\mathcal{E}_f$ and $\mathcal{E}_g$ coincide for some f and g, then we obtain the one-parametric family of Bäcklund transformations between the equations $\mathcal{E}_f$ and $\mathcal{E}_g$.

Remark 3. The covering equations can be obtained explicitly since the nonlocal variables in (6) are easily seen to be potentials for at least one of the dependent variables u, v, and Υ, in every case, e.g., $u = \frac{1}{2}\ln(-\partial\Xi_\infty/\partial x)$.

For Ξ_t and its limit Ξ_∞ at the point $t = -\infty$, we have

$$\tilde{\mathcal{E}}_t = \left\{ \frac{\partial \Xi_t}{\partial y} = \Xi_t^2 + \frac{\Xi_t \cdot \frac{\partial^2 \Xi_t}{\partial x \partial y}}{\frac{\partial \Xi_t}{\partial x}} - \exp(2t) \right\} \text{ and } \tilde{\mathcal{E}}_\infty = \left\{ \frac{\partial \Xi_\infty}{\partial y} = \Xi_\infty^2 + \frac{\Xi_\infty \cdot \frac{\partial^2 \Xi_\infty}{\partial x \partial y}}{\frac{\partial \Xi_\infty}{\partial x}} \right\}.$$

Besides, we obtain the equations $\tilde{\mathcal{E}}_t'$ and $\tilde{\mathcal{E}}_t^v$:

$$\tilde{\mathcal{E}}_t' = \left\{ \frac{\partial \Xi_t'}{\partial y} = (\Xi_t')^2 + \frac{\Xi_t' \cdot \frac{\partial^2 \Xi_t'}{\partial x \partial y}}{\frac{\partial \Xi_t'}{\partial x}} + \exp(-2t) \right\}, \tag{8}$$

$$\tilde{\mathcal{E}}_t^v = \left\{ \frac{\partial \Xi_t^v}{\partial y} = (\Xi_t^v)^2 + \frac{\Xi_t^v \cdot \frac{\partial^2 \Xi_t^v}{\partial x \partial y}}{\frac{\partial \Xi_t^v}{\partial x}} + \exp(2t) \right\},$$

and the equation

$$\tilde{\mathcal{E}}_t^\Upsilon = \left\{ \frac{\partial \Xi_t^\Upsilon}{\partial y} = (\Xi_t^\Upsilon)^2 + \frac{\Xi_t^\Upsilon \cdot \frac{\partial^2 \Xi_t^\Upsilon}{\partial x \partial y}}{\frac{\partial \Xi_t^\Upsilon}{\partial x}} + \exp(-2t) \right\}. \tag{9}$$

Remark 4. The initial equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$ are of the Liouville type, i.e., they admit finite sequences of the Laplace invariants [6]. Not so it is for covering equations (9).

For any covering τ in (7), the nonlocal τ-structures for the underlying equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$, are just *local* structures for covering equations (9). Due to Remark 3, we obtain the natural splitting of these nonlocal τ-symmetries and τ-conservation laws, and the local theory is embedded into the nonlocal one. Namely, there are two distinct types of local conserved currents for covering equations (9): true nonlocal conserved currents for equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$, and local conserved currents for the latter equations such that the dependent variables u, v, and Υ are expressed in terms the nonlocal variables and their derivatives.

Example 1. For the equation $\tilde{\mathcal{E}}_\infty$, the generating sections $\psi \in \ker \ell^*_{\tilde{\mathcal{E}}_\infty}$ of the classical conserved current $\tilde{h}$ are

$$\psi = \frac{\chi'(x)}{\partial\Xi_\infty/\partial x} + \frac{\chi(x)}{\Xi_\infty} + \frac{1}{\Xi_\infty}\Phi\left(\frac{\partial\Xi_\infty/\partial y}{\Xi_\infty} + \Xi_\infty\right),$$

where $\chi(x)$ and $\Phi(z)$ are arbitrary smooth functions of their arguments. Leaving apart $\chi(x)$ that corresponds to a local conservation law for Eq. (2) as pointed above, consider the true nonlocal component $\tilde{\psi} = \Xi_\infty^{-1}\cdot\Phi(\Xi_\infty + \Xi_\infty^{-1}\cdot\partial\Xi_\infty/\partial y)$ and represent the equation $\tilde{\mathcal{E}}_\infty$ in the form

$$\tilde{\mathcal{E}}_\infty \simeq \left\{ F(\tilde{\mathcal{E}}_\infty) = \frac{\partial^2\Xi_\infty}{\partial x\partial y} + \Xi_\infty \cdot \frac{\partial\Xi_\infty}{\partial x} - \frac{\frac{\partial\Xi_\infty}{\partial x}\cdot\frac{\partial\Xi_\infty}{\partial y}}{\Xi_\infty} = 0 \right\}.$$

Now we recall some basic facts [3] on the correlation between conservation laws and their generating sections. Suppose h is a conservation law for the equation $\{F = 0\}$.

Then $dh = \square(F)\, dx^1 \wedge \ldots \wedge dx^n$ for some $\square \in \mathcal{C}\mathrm{Diff}(C^\infty(\pi), C^\infty(\pi))$. The fundamental theorem states that $\psi = \square^*(1)$, where the generating section $\psi \in \ker \bar{\ell}_F^*$.

Consider the current $h = \Psi(\Xi_\infty + \partial\Xi_\infty/\partial y \cdot \Xi_\infty^{-1})\, dy$. We have

$$dh = \left(\frac{\partial^2\Xi_\infty/\partial x\partial y}{\Xi_\infty} - \frac{\partial\Xi_\infty/\partial x \cdot \partial\Xi_\infty/\partial y}{\Xi_\infty^2} + \frac{\partial\Xi_\infty}{\partial x}\right) \cdot \Psi'\left(\frac{\partial\Xi_\infty/\partial y}{\Xi_\infty} + \Xi_\infty\right) dx \wedge dy = \\ = \underbrace{\frac{1}{\Xi_\infty} \cdot \Psi'\left(\frac{\partial\Xi_\infty/\partial y}{\Xi_\infty} + \Xi_\infty\right)}_{\square,\quad \operatorname{ord}\square = 0} \cdot F(\tilde{\mathcal{E}}_\infty)\, dx \wedge dy,$$

whence h is $\bar{d}$-closed and

$$\tilde{\psi} = \square^*(1) = \frac{1}{\Xi_\infty} \cdot \Psi'\left(\frac{\partial\Xi_\infty/\partial y}{\Xi_\infty} + \Xi_\infty\right).$$

We see that the initial $\Phi(z) = \Psi'(z)$ and the conservation laws are in bijective correspondence with their generating functions.

For the equation $\tilde{\mathcal{E}}_\infty$, the generating functions φ of the point symmetries are

$$\varphi = \Phi(x) \cdot \frac{\partial\Xi_\infty}{\partial x} + \Psi(y) \cdot \frac{\partial\Xi_\infty}{\partial y} + \Psi'(y) \cdot \Xi_\infty$$

for any smooth functions Φ and Ψ. The corresponding vector field is nothing more than a point symmetry of the Liouville equation lifted onto $\tilde{\mathcal{E}}_\infty$.

In Section 2, we apply the lifting technique to the point symmetries of the base diffiety $\mathcal{E}_u^\infty$ and show that the low-order symmetries of the Liouville equation $\mathcal{E}_u$ can be successfully extended up to true local symmetries of the covering equation $\tilde{\mathcal{E}}_\infty$, still demonstrating that there appear no new nonlocal τ_t-symmetries for the initial Eq. (2).

Integrating in nonlocal variables. We stress that transformations (3)–(5) cannot be integrated in local variables; considering one-dimensional non-abelian coverings (7) and extending the sets of local variables with new nonlocal variables, see (6), we can integrate the transformations successfully. These results can be summarized as follows.

Proposition 1. *For equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$, Bäcklund (auto)transformations (3)–(5) are integrated in nonlocal variables explicitly:*

(1) *Bäcklund autotransformation (3) for Eq. (2):*

$$\tilde{u} = u + t - \ln \Xi_t \text{ and } u = t + \tilde{u} - \ln \Xi_t[\tilde{u}](-x, -y),$$

i.e., to inverse the transformation and obtain $u[\tilde{u}]$, the inversion $x \mapsto -x$ and $y \mapsto -y$ is required.

(2) *Bäcklund transformation (4) between Eq. (2) and the wave equation $v_{xy} = 0$:*

$$v = u + t - \ln \Xi_\infty \text{ and, conversely, } u = v + t - \ln \Xi_t^v.$$

(3) *Bäcklund transformation (5) between Eq. (2) and the* scal$^+$*-Liouville equation $\Upsilon_{xy} = \exp(-2\Upsilon)$:*

$$\Upsilon = -u + t + \ln \Xi_t' \text{ and, conversely, } u = -\Upsilon - t - \ln \Xi_t^\Upsilon.$$

Proof. We consider the case $\tilde{u}[u](x,y)$ in Bäcklund autotransformation (3). By definition, put $\mathcal{U} = \exp(\tilde{u})$ and $\mathcal{T} = \exp(-\tilde{u})$. From Eq. (3) we obtain the Bernoulli equation

$$\mathcal{U}_x = u_x \cdot \mathcal{U} + \exp(u-t)\,\mathcal{U}^2,$$

whence $\mathcal{U}^{-1} = \mathcal{T} = \exp(-u-t)\cdot\Xi$, where the nonlocal variable Ξ is such that $\tilde{D}_x(\Xi) = -\exp(2u)$, and the Riccati equation

$$\mathcal{T}_y = u_y \cdot \mathcal{T} + \exp(u+t)\,\mathcal{T}^2 - \exp(t-u). \tag{10}$$

Substituting $\exp(-u-t)\cdot\Xi$ for $\mathcal{T}$ in (10), we get $\tilde{D}_y(\Xi) = \Xi^2 + 2u_y\Xi - \exp(2t)$; refer (6) and compare the result with the derivatives $\tilde{D}_x(\Xi_t)$ and $\tilde{D}_y(\Xi_t)$.

The proof of other 5 cases is quite analogous: Assuming $f(x,y) \in \{u,\tilde{u},v,\Upsilon\}$ to be a known solution to the PDE $\mathcal{E}_f$, we obtain either two Bernoulli equations for Eq. (4) or one Bernoulli equation and one Riccati equation for Eq. (3) and Eq. (5) after a proper change of variables. Solving these ordinary differential equations for the solution $g(x,y) \in \{u,\tilde{u},v,\Upsilon\}$ to the PDE $\mathcal{E}_g$, related with $\mathcal{E}_f$ by one of Bäcklund (auto)transformations (3)–(5), we finally obtain the rules to differentiate the nonlocal variable in one of the coverings (7). □

Remark 5. We see that the one-dimensional coverings τ in (7) are sufficient for the transformations (3)–(5) to be integrated. In [7], pairs of the nonlocal variables $f \pm g$, where $f,\ g \in \{u,v,\Upsilon\}$, were demonstrated to provide the pairs f and g, related by Bäcklund transformations. Nevertheless, for any Eq. (3)–(5), at least one of these nonlocals was treated as the potential for the other one, so that the resulting covering was one-dimensional effectively.

Consider the diagrams (1) arising in the definition of Bäcklund transformation and apply them to Proposition 1. Bearing in mind that in all cases in (6) one of the projectors τ_1 and τ_2 is the first order differential operator depending on the nonlocal variable only, while another covering is the zero order morphism, we have

Proposition 2. *Consider the equations* (9). *The relations*

$$\begin{aligned}
u &= \frac{1}{2}\ln(-\Xi_t)_x & \qquad \tilde{u} &= t + \ln\frac{\sqrt{(-\Xi_t)_x}}{\Xi_t}\\
u &= \frac{1}{2}\ln(-\Xi_\infty)_x & \qquad v &= t + \ln\frac{\sqrt{(-\Xi_\infty)_x}}{\Xi_\infty}\\
u &= \frac{1}{2}\ln(-\Xi'_t)_y & \qquad \text{and} \qquad \Upsilon &= t + \ln\frac{\Xi'_t}{\sqrt{(-\Xi'_t)_y}}\\
v &= \frac{1}{2}\ln(\Xi^v_t)_x & \qquad u &= t + \ln\frac{\sqrt{(\Xi^v_t)_x}}{\Xi^v_t}\\
\Upsilon &= \frac{-1}{2}\ln(\Xi^\Upsilon_t)_y & \qquad u &= -t + \ln\frac{\sqrt{(\Xi^\Upsilon_t)_y}}{\Xi^\Upsilon_t}
\end{aligned}$$

hold.

In other words, the nonlocal variables satisfying (9) are potentials for *both* solutions to the equations $\mathcal{E}_u$, $\mathcal{E}_v$, and $\mathcal{E}_\Upsilon$, arising in diagrams (1). The property of all the coverings in (7) to be the nonlinear differential operators of order not grater than 1 is a really remarkable fact.

2. By definition, for an equation $\mathcal{E}$ and a covering $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}^\infty$, the τ-*shadows* φ are the solutions of the linearized equation $\tilde{\ell}_{\mathcal{E}}(\varphi) = 0$. The shadow fields $\tilde{\mathfrak{I}}_\varphi \in \bar{\mathrm{D}}_{\mathcal{C}}(\tilde{\mathcal{E}})$ are not true nonlocal symmetries since they do not describe the evolution of the nonlocal variable, and, as we shall see for the Liouville equation, not all of them can be extended up to true nonlocal symmetries.

Now we introduce new nonlocal variables such that the symmetries we are in search of depend on them.

Let $\overleftrightarrow{\Sigma}_t = \Xi_t + u_y$ be the new nonlocal variable such that $\tilde{D}_x^{\mathcal{E}_u}(\overleftrightarrow{\Sigma}_t) = 0$ and $\tilde{D}_y^{\mathcal{E}_u}(\overleftrightarrow{\Sigma}_t) = (\overleftrightarrow{\Sigma}_t)^2 + u_{yy} - u_y^2 - \exp(2t)$. Consider the limit of $\overleftrightarrow{\Sigma}_t$ as $t \to -\infty$. We stress that at the point $t = -\infty$ there appeares new automodel variable $\Sigma_\infty = u_x + \exp(2u)/\Xi_\infty$ such that $\tilde{D}_y^{\mathcal{E}_u}(\Sigma_\infty) = 0$. We claim that $\overleftrightarrow{\Sigma}_\infty = \lim_{t\to-\infty} \overleftrightarrow{\Sigma}_t$ and Σ_∞ differ by the discrete symmetry $x \leftrightarrow y$. Really, expressing the derivatives $\tilde{D}_x$ and $\tilde{D}_y$ of Σ_∞ and $\overleftrightarrow{\Sigma}_\infty$, we get $\tilde{D}_x(\Sigma_\infty) = \Sigma_\infty^2 + u_{xx} - u_x^2$ and $\tilde{D}_y(\Sigma_\infty) = 0$ and also $\tilde{D}_x(\overleftrightarrow{\Sigma}_\infty) = 0$ and $\tilde{D}_y(\overleftrightarrow{\Sigma}_t) = (\overleftrightarrow{\Sigma}_\infty)^2 + u_{yy} - u_y^2$. Thence, we use the nonlocal variable Σ_∞ only and treat all relations mod the symmetry $x \leftrightarrow y$ of the Liouville equation. By definition, put $\Sigma_t = (x \leftrightarrow y) \cdot (\overleftrightarrow{\Sigma}_t)$: $\tilde{D}_x(\Sigma_t) = \Sigma_t^2 + u_{xx} - u_x^2 - \exp(2t)$ and $\tilde{D}_y(\Sigma_t) = 0$.

Remark 6. Now, we can generalize the nonlocal conservation laws from Example 1 in Section 2 for the Liouville equation as

$$h_\Sigma = \Phi(x, \Sigma_t, u_{xx} - u_x^2, \ldots, \bar{D}_x^l(u_{xx} - u_x^2), \ldots)\, dx, \quad l \in \mathbb{N}: \qquad \bar{d}h_\Sigma = 0,$$

where Φ is an arbitrary smooth function and $u_{xx} - u_x^2$ is the minimal y-integral for the Liouville equation.

Nonlocal variables enable us to find shadows of nonlocal symmetries for Eq. (2) and then to extend these shadows up to true nonlocal symmetries of the Liouville equation.

Proposition 3. (1) *Let $f(t, x, \Sigma_t) \in C^\infty(\mathbb{R}^3)$ be a smooth function. Then the function*

$$\varphi = ({\Sigma_t}^2 + u_{xx} - u_x^2 - \exp(2t)) \cdot \frac{\partial f}{\partial \Sigma_t} + \frac{\partial f}{\partial x} + 2u_x \cdot f = (\tilde{D}_x + 2u_x) \cdot f(t, x, \Sigma_t) \tag{11}$$

is a τ_t-shadow of a nonlocal symmetry of the Liouville equation.

(2) *Let $f(x, \Sigma_\infty) \in C^\infty(\mathbb{R}^2)$ be a smooth function. Then, for the Liouville equation, the second order τ-shadow $\varphi(x, \Sigma_\infty, u, u_x, u_{xx})$ is*

$$\varphi = (\Sigma_\infty^2 + u_{xx} - u_x^2) \cdot \frac{\partial f}{\partial \Sigma_\infty} + \frac{\partial f}{\partial x} + 2u_x f = (\tilde{D}_x + 2u_x) \cdot f(x, \Sigma_\infty). \tag{12}$$

Remark 7. We stress that the generating functions φ depend on the second order derivative u_{xx} of the fiber variable u, in contrast with the local case: if the sets of base variables $\{x, y\}$ and fiber variables $\{u, u_x, u_y, \ldots\}$ are purely local, then the second order generating functions $\varphi(u_{xx}, u_{yy}, u_x, u_y, u, x, y)$ of local symmetries of the Liouville equation are missing at all.

Reconstruction of the nonlocal symmetries. In order to reconstruct the τ_t-shadows $\tilde{\ni}_\varphi$ up to true nonlocal symmetries $\tilde{\ni}_{\varphi,a} = \tilde{\ni}_\varphi + a \cdot \partial/\partial\Sigma_t$, $a \in C^\infty(\tilde{\mathcal{E}})$, $t \in \mathbb{R} \cup \{-\infty\}$, we solve the equations

$$\tilde{D}_x(a) = \tilde{\ni}_{\varphi,a}(\tilde{D}_x(\Sigma_t)) \text{ and } \tilde{D}_y(a) = \tilde{\ni}_{\varphi,a}(\tilde{D}_y(\Sigma_t))$$

for the function a.

Remark 8. In [8], the general scheme of symmetries of the Liouville equation is obtained: $\varphi = (\bar{D}_x + 2u_x) \cdot f(x, w, \bar{D}_x^l w)$, $l \geq 0$, where the minimal y-integral is $w = u_{xx} - u_x^2$. We see that nonlocal shadows (11) and (12) belong to the same class.

We also note that for all variables concerned we can define the weight such that the symmetries become homogeneous w.r.t. this weight. Really, by definition, put the weight $\mathrm{wt}(u_k) = k$, where $k \geq 0$, $\mathrm{wt}(u_k \cdot u_l) = k + l$, and $\mathrm{wt}(u_{k_1} + u_{k_2}) = k_1$, if $k_1 = k_2$. Note that the minimal y-integral $w = u_{xx} - u_x^2$ and the operator $\bar{D}_x + u_x$ both are homogeneous w.r.t the weight wt with its values 2 and 1, respectively. We assume the weight of a arbitrary function f to be trivial: $\mathrm{wt}(f) = 0$, if f is a free functional parameter in a relation we deal with.

By definition, put $\mathrm{wt}(\Sigma_t) = 1$ $\forall t \in \mathbb{R} \cup \{-\infty\}$ and $\mathrm{wt}(k \equiv \exp(t)) = 1$. Then, the weight $\mathrm{wt}(\tilde{D}_x(\Sigma_t))$ is well defined and equals 2. Obviously, $\mathrm{wt}(a) = \mathrm{wt}(\varphi) + 1$, since $\mathrm{wt}(\partial/\partial\Sigma_t) = -1$ and nonlocal symmetries are homogeneous w.r.t. the weight. Note that for shadows (11) and (12), $\mathrm{wt}(\varphi) = 1$, although φ depends on $\partial\Sigma_t/\partial x$ (resp., $\partial\Sigma_\infty/\partial x$).

Proposition 4. (1) *Let $f(t)$ be a smooth function and the functions φ and $a(t, \Sigma_t, u_x, u_{xx})$ be*

$$\varphi = 2u_x \cdot f(t) \text{ and } a = 2(\Sigma_t^2 + u_{xx} - u_x^2 - \exp(2t)) \cdot f(t). \tag{13}$$

Then, for Eq. (2), the field $\tilde{\ni}_\varphi + a \cdot \partial/\partial\Sigma_t$ is a true nonlocal symmetry of the weight 1.

(2) *Let $f(x)$ be a smooth function and the functions φ and $a(\Sigma_\infty, u_x, u_{xx})$ be*

$$\varphi = 2u_x \cdot f(x) + \frac{df}{dx} \text{ and } a = 2(\Sigma_\infty^2 + u_{xx} - u_x^2) \cdot f(x) + 2\Sigma_\infty \cdot \frac{df}{dx} + \frac{d^2 f}{dx^2}. \tag{14}$$

Then, for Eq. (2), the field $\tilde{\ni}_\varphi + a \cdot \partial/\partial\Sigma_\infty$ is a true nonlocal symmetry of the weight 1.

Nonlocal symmetry (13) is defined mod $\mathrm{CD}(\tilde{\mathcal{E}}) \ni g \cdot \tilde{D}_x$, $g \in C^\infty(\tilde{\mathcal{E}})$. Thus, the nonlocal symmetry class is $[\tilde{\ni}_{\varphi,a}] = [-f(t) \cdot \partial/\partial x]$, where $f(t) \cdot \partial/\partial x$ is the translation. Symmetry (14) is the lifting of a classical point symmetry.

As usual, there are nonlocal symmetries $\tilde{\ni}_{\varphi,a}$ with $x \leftrightarrow y$.

Remark 9. If any shadow (11) or (12) depends explicitly on Σ_t, $t \in \mathbb{R} \cup \{-\infty\}$, then its shadow field $\tilde{\ni}_\varphi$ cannot be extended up to a true nonlocal symmetry $\tilde{\ni}_{\varphi,a}$.

Remark 10. We see that for any smooth function ϕ the rules $\tilde{D}_x(\Sigma_t) = \phi(\Sigma_t) + \bar{D}_y$-closed summand or even $\tilde{D}_x(\Sigma_t) = \phi(t, \Sigma_t, \bar{D}_y\text{-closed arguments})$ and $\tilde{D}_y(\Sigma_t) = 0$ to differentiate the variable Σ_t provide a one-dimensional non-abelian covering over the Liouville equation by means of (6). Still, we note that the coverings (7) provided by Bäcklund autotransformation (3) respect the weight $\mathrm{wt}(\Sigma_t) = 1$, so that the weights

$\mathrm{wt}(\varphi)$ and $\mathrm{wt}(a)$ are well defined and $\mathrm{wt}(\tilde{\partial}_\varphi + a\cdot\partial/\partial\Sigma_t) = 1$ for any φ and a in Proposition 4.

The reasoning for other possible couplings w.r.t. the coveings in (7) of the base diffieties $\mathcal{E}_u^\infty$, $\mathcal{E}_v^\infty$, and $\mathcal{E}_\Upsilon^\infty$, and total diffieties (9) is quite analogous.

3. Finally, we observe the permutability property of Bäcklund (auto)transformations (3)–(5).

Proposition 5. (1) *Let u^j, $j=\mathrm{i},\mathrm{ii}$, be solutions of Eq. (2) such that $\mathcal{B}_u(u,u^j;t_j)=0$, $t_j\in\mathbb{R}$. Then there exists the unique solution $u'''(x,y)$ to the system*

$$\left\{\begin{array}{rcl}\mathcal{B}_u(u',u''';t_2) &=& 0,\\ \mathcal{B}_u(u'',u''';t_1) &=& 0,\end{array}\right.\tag{15}$$

namely, the solution u''' is such that the relation

$$\exp(u''')=\exp(u)\cdot\frac{k_2\exp(u')-k_1\exp(u'')}{k_2\exp(u'')-k_1\exp(u')}\tag{16}$$

holds, where $k_j\equiv\exp(t_j)$.

(2) *Let $j=\mathrm{i},\mathrm{ii}$, and $t_j\in\mathbb{R}$. Let v^j be solutions of the wave equation $v_{xy}=0$ such that $\mathcal{B}_{uv}(u,v^j;t_j)=0$ and u^j be solutions of the Liouville equation such that $\mathcal{B}_{uv}(u^j,v;t_j)=0$. Then there exist the unique solutions u''' and v''' to the systems*

$$\left\{\begin{array}{rcl}\mathcal{B}_{uv}(u''',v';t_2) &=& 0\\ \mathcal{B}_{uv}(u''',v'';t_1) &=& 0\end{array}\right.\quad\text{and}\quad\left\{\begin{array}{rcl}\mathcal{B}_{uv}(u',v''';t_2) &=& 0\\ \mathcal{B}_{uv}(u'',v''';t_1) &=& 0,\end{array}\right.$$

respectively. Denote $k_j\equiv\exp(t_j)$, then we have

$$\exp(u''')=\exp(u)\cdot\frac{k_2\exp(v')-k_1\exp(v'')}{k_2\exp(v'')-k_1\exp(v')},$$
$$\exp(v''')=\exp(v)\cdot\frac{k_1\exp(u'')-k_2\exp(u')}{k_2\exp(u'')-k_1\exp(u')}.$$

(3) *Let $j=\mathrm{i},\mathrm{ii}$, and $t_j\in\mathbb{R}$. Let Υ^j be solutions of the scal^+-equation $\mathcal{E}_\Upsilon$ such that $\mathcal{B}_{u\Upsilon}(u,\Upsilon^j;t_j)=0$ and u^j be solutions of the Liouville equation such that $\mathcal{B}_{u\Upsilon}(u^j,\Upsilon;t_j)=0$. Then there exist the unique solutions u''' and Υ''' to the systems*

$$\left\{\begin{array}{rcl}\mathcal{B}_{u\Upsilon}(u''',\Upsilon';t_2) &=& 0\\ \mathcal{B}_{u\Upsilon}(u''',\Upsilon'';t_1) &=& 0\end{array}\right.\quad\text{and}\quad\left\{\begin{array}{rcl}\mathcal{B}_{u\Upsilon}(u',\Upsilon''';t_2) &=& 0\\ \mathcal{B}_{u\Upsilon}(u'',\Upsilon''';t_1) &=& 0,\end{array}\right.$$

respectively. We also have

$$\exp(u''')=\exp(u)\cdot\frac{k_2\exp(\Upsilon')-k_1\exp(\Upsilon'')}{k_2\exp(\Upsilon'')-k_1\exp(\Upsilon')},$$
$$\exp(\Upsilon''')=\exp(\Upsilon)\cdot\frac{k_1\exp(u'')-k_2\exp(u')}{k_2\exp(u'')-k_1\exp(u')},$$

where $k_j\equiv\exp(t_j)$.

Proof. We consider Bäcklund autotransformation (3) only, dealing with cases 2 and 3 quite analogously. Consider the subsystem of (15) consisting of the relations (3) with derivatives w.r.t. x only. Then u''' defined in (16) is a unique solution to this subsystem demonstrating the linear dependence of the l. h. s. expressions. One easily checks that the other subsystem composed by the relations in (3) containing the derivatives w.r.t. y has two solutions, u''' and $\bar{u}'''$: u''' is defined in Eq. (16) and $\bar{u}'''$ is such that

$$\exp(\bar{u}''') = \exp(-u) \cdot \frac{k_1 \exp(u') - k_2 \exp(u'')}{k_2 \exp(-u') - k_1 \exp(-u'')},$$

the latter being an irrelevant solution. Thus, the function u''' is the unique solution to the whole system (15). □

Locally, the image of any Bäcklund transformation (3)–(5) is unique in the vicinity of a generic point $\mu = (x, y)$ in the base M. Thus, in terms of the Lamb diagrams, we have

Proposition 6. *All transformations (3)–(5) are permutable, i.e., for any t_1, t_2, and $t_3 \in \mathbb{R}$ the diagrams*

$$\begin{array}{ccc} u & \xrightarrow{t_1} & u' \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_2} \\ u'' & \xrightarrow[t_1]{} & u''' \end{array}, \qquad \begin{array}{ccccc} u & \xrightarrow{t_1} & v' & \xrightarrow{t_3} & u'' \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_2} \\ v'' & \xrightarrow[t_1]{} & u' & \xrightarrow[t_3]{} & v''' \end{array}, \quad \text{and} \quad \begin{array}{ccccc} u & \xrightarrow{t_1} & \Upsilon' & \xrightarrow{t_3} & u'' \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_2} \\ \Upsilon'' & \xrightarrow[t_1]{} & u' & \xrightarrow[t_3]{} & \Upsilon''' \end{array}$$

are commutative.

To finish with, we conclude that Bäcklund transformation (3)–(5) themselves do contain certain information about nonlocal variables such that these transformations can be integrated successfully. We see that the corresponding non-abelian coverings in (7) provide true nonlocal conservation laws for the underlying diffieties. Still, the structures on the covering equations are "too close" to the initial ones in some sence, so that the point symmetries of the base and the total diffieties are in bijective correspondence, and there are no nonlocal symmetries except the liftings.

The permutability property of Bäcklund transformations in terms of the structural elements deformations by means of the Frölicher–Nijenhuis bracket will be discussed elsewhere.

Acknowledgement. The author thanks M. Marvan for stimulating remarks and for the Jet software [9]. The work was partially supported by INTAS Grant YS 2001/2-33.

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