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possessing an irreducible \mathfrak{sl}_2 -valued zero
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Scalar second order evolution equations possessing an irreducible \mathfrak{sl}_2 -valued zero curvature representation

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Abstract

We find all scalar second order evolution equations possessing an \mathfrak{sl}_2 -valued zero curvature representation that is not reducible to a proper subalgebra of \mathfrak{sl}_2 . None of these zero-curvature representations admits a parameter.

For more than twenty years, researchers are being attracted by the problem of classification of nonlinear systems possessing a zero-curvature representation (ZCR). Efforts are focused on ZCR's taking values in a non-solvable Lie algebra \mathfrak{g} and depending on a nonremovable parameter, in expectation that they will be suitable for the Zakharov and Shabat [16] formulation of integrability (*S-integrability*) and hence admit soliton solutions. But even without parameter, a ZCR may be useful for construction of Bäcklund transformations, nonlocal symmetries and nonlocal conservation laws. The problem of finding a ZCR is then equivalent to that of computing finite-dimensional linear coverings in the sense of Krasil'shchik and Vinogradov [3], which are very often just realizations of the Wahlquist–Estabrook prolongation structures [15]. However, existing computational procedures are insufficient for solving general classification problems, unless in combination with methods based on different criteria of integrability. The most complete lists of integrable systems obtained so far resulted from the formal symmetry approach [6, 7, 10].

In this paper we apply the method of [4, 5] and find all second order scalar evolution equations

$$u_t = F(t, x, u, u_x, u_{xx}) \tag{1}$$

possessing an \mathfrak{sl}_2 -valued zero curvature representation that is irreducible in the sense of being not reducible to a proper subalgebra of \mathfrak{sl}_2 . We arrive at a single previously unnoticed class of equations parametrized by a single function of the coordinates t, x . We also distinguish a particular subclass of equations that admit a single conservation law. None of the corresponding ZCR's admits a substantial parameter, which is in

accordance with the general belief that no second order scalar evolution equation is S-integrable. All the previously known ZCR's [1, 8] for second order scalar evolution equations turn out to be reducible to a solvable subalgebra and as such fall outside our classification.

1 Preliminaries

Let E be a nonlinear partial differential equation on a number of functions in two independent variables t, x . Let \mathfrak{g} be a non-solvable matrix Lie algebra. By a \mathfrak{g} -valued zero-curvature representation (ZCR) for E we mean a \mathfrak{g} -valued one-form $\alpha = A dx + B dt$ such that

$$d\alpha = \frac{1}{2} [\alpha, \alpha] \quad (2)$$

holds as a consequence of E .

Let G be the connected and simply connected matrix Lie group associated with \mathfrak{g} . Then for an arbitrary G -valued function S , the form $\alpha^S = dSS^{-1} + S\alpha S^{-1}$ is another ZCR, which is said to be *gauge equivalent* to the former. Gauge equivalent ZCR's may be viewed as identical geometric objects (connections). A \mathfrak{g} -valued ZCR is said to be *reducible* if it is gauge equivalent to a ZCR taking values in a proper subalgebra of \mathfrak{g} ; otherwise it is said to be *irreducible*.

Let us proceed to a description of the general algorithm of computing ZCR's [4, 5] as we use it here. For simplicity we restrict ourselves to a single non-linear n th order evolution equation

$$u_t = F(t, x, u, u_1, \dots, u_n) = 0. \quad (3)$$

Here t, x are coordinates, u is a single field variable, and $u_1 = u_x$, $u_2 = u_{xx}$, etc., represent the derivatives. Let us consider the corresponding equation manifold E , that is, the infinite-dimensional space \mathbb{R}^∞ endowed with coordinates t, x, u and u_k , $1 \leq k$. The vector fields $D_x = \partial/\partial x + u_1\partial/\partial u + \dots + u_{k+1}\partial/\partial u_k + \dots$, $D_t = \partial/\partial t + F\partial/\partial u + \dots + D_x^k F \partial/\partial u_k + \dots$ defined on E generate a diffiety structure in the sense of [3] and encode all essential geometric information about the equation. In these terms, a ZCR for eq. (3) is a pair of \mathfrak{g} -valued functions A, B on E satisfying eq. (2), which may be written as

$$D_t A - D_x B + [A, B] = 0. \quad (4)$$

Let us introduce operators \widehat{D}_I acting on an arbitrary \mathfrak{g} -valued function C on E as follows:

$$\widehat{D}_x C = D_x C - [A, C], \quad \widehat{D}_t C = D_t C - [B, C]. \quad (5)$$

Operators $\widehat{D}_x, \widehat{D}_t$ commute whenever (A, B) is a ZCR. We also set $\widehat{D}_i = \widehat{D}_x \circ \dots \circ \widehat{D}_x$ (i times).

By [4] for every ZCR there exists a *characteristic matrix* R , which is a \mathfrak{g} -valued function defined on E (see also the independent work by Sakovich [11]). The following proposition is proved in [4, Prop. 2.7 and Prop. 3.9], see also [5, Prop. 2]

Proposition 1 (1) *The characteristic matrix R for a ZCR of the evolution equation (3) satisfies*

$$-\widehat{D}_t R = \sum_i (-\widehat{D})_i \left(\frac{\partial F}{\partial u_i} R \right). \quad (6)$$

(2) *Gauge-equivalent ZCR's have conjugate characteristic matrices.*

Using Proposition 1(2), one can restrict the gauge freedom by requiring that the characteristic matrix be in the Jordan normal form. The remaining gauge freedom then may be fixed by transforming A into its normal form with respect to the gauge action by the stabilizer $S \subset G$ of R .

In the sequel we consider a ZCR $A dx + B dt$ taking values in \mathfrak{sl}_2 . We shall write the two \mathfrak{sl}_2 -matrices as

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}. \quad (7)$$

Reducible ZCR's will be excluded from our classification. A subalgebra of \mathfrak{sl}_2 to which the ZCR may be reduced is either an abelian algebra or the two-dimensional solvable subalgebra representable by lower triangular matrices. Obviously, a ZCR taking values in an abelian algebra is equivalent to one or more conservation laws. What concerns solvable algebras, the situation is not much different.

Definition 2 An \mathfrak{sl}_2 -valued ZCR satisfying the condition $a_2 = b_2 = 0$ is said to be *lower triangular*.

For a lower triangular ZCR, it follows from eq. (4) that $\phi = a_1 dx + b_1 dt$ is a conservation law. Let h be the potential of ϕ ; then, by the same eq. (4), $\phi' = (a_3 dx + b_3 dt)e^{2h}$ is a conservation law nonlocal over the potential h . This situation will be referred to as a *chain* of conservation laws. Clearly, one can reconstruct the reducible ZCR from the corresponding chain (ϕ, ϕ') . In this way, reducible \mathfrak{sl}_2 -valued ZCR's are equivalent to certain chains of conservation laws. It also follows that methods to find them are to be sought among methods to compute nonlocal conservation laws.

Proposition 3 *Let the matrices (7) form a ZCR for for the evolution equation (3). Suppose that $a_2 = 0$. Then also $b_2 = 0$ or the ZCR is gauge equivalent to zero.*

Proof Let us denote by C the matrix (4) evaluated at $a_2 = 0$. By assumption, C is zero on the equation manifold E . If $b_2 \neq 0$, then from the condition $0 = c_2 = -D_x b_2 + 2a_1 b_2$ we compute that $a_1 = \frac{1}{2} D_x b_2 / b_2$ on E , and then from the condition $0 = c_1 = D_t a_1 - D_x b_1 - a_3 b_2$ we compute that $a_3 = \frac{1}{2} D_{tx} b_2 / b_2^2 - \frac{1}{2} D_t b_2 D_x b_2 / b_2^3 - D_x b_1 / b_2$ on E . Let us introduce a function G by the requirement $b_3 = -b_1^2 / b_2 - D_t b_1 / b_2 + b_1 D_t b_2 / b_2^2 + \frac{1}{2} D_{tt} b_2 / b_2^2 - \frac{3}{4} (D_t b_2)^2 / b_2^3 + G / b_2$. Then $0 = c_3 = -D_x G / b_2$, which in the case of an evolution equation implies that G is a function of t only. Then, under the above substitutions for a_1, a_3, b_3 , the gauge matrix

$$\begin{pmatrix} b_2^{-1/2} & 0 \\ -\frac{1}{2} D_t b_2 b_2^{-3/2} + b_1 b_2^{-1/2} & b_2^{1/2} \end{pmatrix}$$

sends A to zero and B to

$$\begin{pmatrix} 0 & 1 \\ G & 0 \end{pmatrix},$$

which depends on t at most. The last matrix is sent to zero by gauge transformation with the gauge matrix composed of independent solutions of the equation $s_{tt} = Gs$.

2 The classification

Let us consider a second order evolution equation (1) along with the \mathfrak{sl}_2 -matrices A, B satisfying eq. (4) but not reducible to a solvable subalgebra. We also assume that $\partial F / \partial u_{xx} \neq 0$. Following [5], we consider the two cases distinguished by their Segre characteristics separately.

2.1 The nilpotent case

Under the notation (7), the Jordan form for R corresponds to $r_1 = 0, r_2 = 0, r_3 = 1$. The normal form for A , obtained in [5], is given by the single requirement $a_1 = 0$. Indeed, one can set $a_1 = 0$ in a general matrix A by means of the gauge matrix

$$\begin{pmatrix} 1 & 0 \\ a_1/a_2 & 1 \end{pmatrix}$$

whenever $a_2 \neq 0$; otherwise the ZCR is either lower triangular or trivial by Proposition 3.

The equation (6) then reduces to the system $T_i = 0, i = 1, 2, 3$, where

$$T_1 := 2D_x \left(\frac{\partial F}{\partial u_{xx}} \right) a_2 + \frac{\partial F}{\partial u_{xx}} D_x a_2 - \frac{\partial F}{\partial u_x} a_2 + b_2,$$

$$T_2 := 2 \frac{\partial F}{\partial u_{xx}} a_2^2,$$

$$T_3 := -D_{xx} \frac{\partial F}{\partial u_{xx}} + D_x \frac{\partial F}{\partial u_x} - 2 \frac{\partial F}{\partial u_{xx}} a_2 a_3 - \frac{\partial F}{\partial u} - 2b_1.$$

Then $a_2 = 0$ by the second equation and $b_2 = 0$ by the first equation, whence the ZCR is lower triangular. Consequently, this case is void in our classification.

2.2 The semisimple case

It will be convenient to change the notation for \mathfrak{sl}_2 -matrices to

$$A = \begin{pmatrix} a_1 & a_2 + a_3 \\ a_2 - a_3 & -a_1 \end{pmatrix}.$$

The Jordan form for R has $r_2 = r_3 = 0$ with $r := r_1$ arbitrary. Unlike in [5], we choose the normal form for A characterized by the single requirement $a_3 = 0$. And indeed, whenever $a_2 + a_3 \neq 0$, which is irrestrictive by Proposition 3, one can set a_3 to zero by gauge transformation. The relevant gauge matrix is diagonal with the diagonal entries h and $1/h$, where $h = ((a_2 - a_3)/(a_2 + a_3))^{1/4}$.

Eq. (4) and (6) then assume the form $S_i = 0 = T_i$, $i = 1, 2, 3$, with

$$\begin{aligned} S_1 &= -D_t a_1 + D_x b_1 + 2a_2 b_3, \\ S_2 &= -D_t a_2 + D_x b_2 - 2a_1 b_3, \\ S_3 &= D_x b_3 + 2a_2 b_1 - 2a_1 b_2, \\ T_1 &= -D_t r - \frac{\partial F}{\partial u} r + D_x \left(\frac{\partial F}{\partial u_x} r \right) - D_{xx} \left(\frac{\partial F}{\partial u_{xx}} r \right) - 4 \frac{\partial F}{\partial u_{xx}} r a_2^2, \\ T_2 &= -b_3 + 2 \frac{\partial F}{\partial u_{xx}} a_1 a_2, \\ T_3 &= -b_2 + \frac{\partial F}{\partial u_x} a_2 - 2 D_x \left(\frac{\partial F}{\partial u_{xx}} \right) a_2 - 2 \frac{D_x r}{r} \frac{\partial F}{\partial u_{xx}} a_2 - \frac{\partial F}{\partial u_{xx}} D_x a_2. \end{aligned} \tag{8}$$

If $a_2 = 0$, then we have $b_2 = b_3 = 0$ by the last two equations and the ZCR reduces to a single conservation law. Therefore, we assume that $a_2 \neq 0$ in the sequel.

Proposition 4 *As solutions to eq. (8), functions r, a_1, a_2, b_3 cannot depend on jet coordinates other than t, x, u, u_x, u_{xx} , whereas functions b_1, b_2 cannot depend on jet coordinates other than $t, x, u, u_x, u_{xx}, u_{xxx}$.*

Proof We may assume, without loss of generality, that the functions r, a_1, a_2, b_3 depend on t, x, u, \dots, u_k and the functions b_1, b_2 depend on $t, x, u, \dots, u_k, u_{k+1}$ for some $k \geq 2$. We perform a downward induction, each step of which consists in deriving appropriate differential consequences of the system (8). Thus, let $k > 2$. Then we have

$$0 = \frac{\partial T_1}{\partial u_{k+2}} = -2 \frac{\partial F}{\partial u_{xx}} \frac{\partial r}{\partial u_k},$$

but $\partial F / \partial u_{xx} \neq 0$, whence r does not depend on u_k . Then, similarly,

$$\begin{aligned} 0 &= \frac{\partial S_2}{\partial u_{k+2}} - \frac{\partial T_3}{\partial u_{k+1}} = 2 \frac{\partial b_2}{\partial u_{k+1}} \\ 0 &= \frac{\partial S_2}{\partial u_{k+2}} + \frac{\partial T_3}{\partial u_{k+1}} = -2 \frac{\partial F}{\partial u_{xx}} \frac{\partial a_2}{\partial u_k}, \end{aligned}$$

whence b_2 does not depend on u_{k+1} and a_2 does not depend on u_k . Finally,

$$\begin{aligned} 0 &= -2a_2 \frac{\partial S_1}{\partial u_{k+2}} + \frac{\partial S_3}{\partial u_{k+1}} + \frac{\partial T_2}{\partial u_k} = 4a_2 \frac{\partial F}{\partial u_{xx}} \frac{\partial a_1}{\partial u_k}, \\ 0 &= 2a_2 \frac{\partial S_1}{\partial u_{k+2}} + \frac{\partial S_3}{\partial u_{k+1}} + \frac{\partial T_2}{\partial u_k} = 4a_2 \frac{\partial b_1}{\partial u_{k+1}}, \\ 0 &= -2a_2 \frac{\partial S_1}{\partial u_{k+2}} + \frac{\partial S_3}{\partial u_{k+1}} - \frac{\partial T_2}{\partial u_k} = 2 \frac{\partial b_3}{\partial u_k}, \end{aligned}$$

whence a_1, b_3 do not depend on u_k and b_1 does not depend on u_{k+1} (recall that $a_2 \neq 0$). This completes the induction step.

Under the restrictions established in Proposition 4, the determining system (8) becomes an overdetermined system of partial differential equations. As such, it can be solved routinely, but its solution is troublesome even with the employment of software capable of automated deriving of differential consequences. The reason is that the class of second order evolution equations is invariant with respect to a large group of contact transformations $\bar{x} = \bar{x}(t, x, u, u_x)$, $\bar{u} = \bar{u}(t, x, u, u_x)$, $\bar{t} = \bar{t}(t)$. Below we shall apply a series of suitably chosen contact transformations to achieve substantial reduction of the matrix A .

Proposition 5 *For every second order evolution equation (1) possessing an irreducible \mathfrak{sl}_2 -valued ZCR there exists a contact transformation such that the transformed a_2 depends on t, x, u, u_x at most.*

Proof Let functions r, a_i, b_i depend on $t, x, u, u_x, u_{xx}, u_{xxx}$ as in Proposition 4. Taking successively the derivatives $\partial S_1/\partial u_{xxxx}, \partial S_2/\partial u_{xxxx}, T_2, \partial T_3/\partial u_{xxx}, \partial T_1/\partial u_{xxxx}, \partial S_3/\partial u_{xxx}, \partial T_3/\partial u_{xxx}, \partial^2 S_1/\partial u_{xxx}^2, \partial^2 S_2/\partial u_{xxx}^2$ one may check routinely that

$$\frac{\partial^2 a_2}{\partial u_{xx}^2} = 0 \quad \text{and} \quad a_1 \frac{\partial a_2}{\partial u_{xx}} - \frac{\partial a_1}{\partial u_{xx}} a_2 = 0,$$

are among differential consequences of the system (8). Hence,

- (a) a_2 is linear in u_{xx} , i.e., $a_2 = a_{21}(t, x, u, u_x)u_{xx} + a_{20}(t, x, u, u_x)$;
- (b) the ratio a_1/a_2 does not depend on u_{xx} .

Now, if $a_{21} = 0$, then the statement is proved. Otherwise, let f_1, f_2 be two functionally independent solutions of the linear equation

$$-\frac{a_{20}}{a_{21}} \frac{\partial f}{\partial u_x} + u_x \frac{\partial f}{\partial u} + \frac{\partial f}{\partial x} = 0. \tag{9}$$

In particular, both f_1 and f_2 do depend on u_x . Then $\bar{t} = t$, $\bar{x} = f_1$, $\bar{u} = f_2$ and $\bar{u}_{\bar{x}} = (\partial f_2 / \partial u_x) / (\partial f_1 / \partial u_x)$ satisfy the well-known necessary conditions of being a contact transformation:

$$\frac{\frac{\partial \bar{u}}{\partial u_x}}{\frac{\partial \bar{x}}{\partial u_x}} = \bar{u}_{\bar{x}} = \frac{\frac{\partial \bar{u}}{\partial x} + u_x \frac{\partial \bar{u}}{\partial u}}{\frac{\partial \bar{x}}{\partial x} + u_x \frac{\partial \bar{x}}{\partial u}}.$$

Under this transformation, $A dx + B dt$ becomes $\bar{A} d\bar{x} + \bar{B} d\bar{t}$ with $d\bar{x} = D_x \bar{x} dx + D_t \bar{x} dt$, $d\bar{t} = dt$, so that

$$A = \bar{A} D_x \bar{x} = \left(\frac{\partial f_2}{\partial x} + u_x \frac{\partial f_2}{\partial u} + u_{xx} \frac{\partial f_2}{\partial u_x} \right) \bar{A} = \frac{a_2}{a_{21}} \frac{\partial f_2}{\partial u_x} \bar{A},$$

where we have used eq. (9). Hence

$$\bar{A} = \frac{a_{21}}{\partial f_2 / \partial u_x} \begin{pmatrix} a_1/a_2 & 1 \\ 1 & -a_1/a_2 \end{pmatrix},$$

which is independent of u_{xx} , hence of \bar{u}_{xx} , by virtue of statement (b) above.

Theorem 6 *Every second order scalar evolution equation (1) possessing an irreducible \mathfrak{sl}_2 -valued ZCR is transformable to an equation of the form*

$$u_t = \frac{\partial \beta}{\partial x} u^2 u_{xx} + 2 \frac{\partial^2 \beta}{\partial x^2} u^2 u_x + 4\beta u_x + \left(\frac{\partial^3 \beta}{\partial x^3} - 4 \frac{\partial \beta}{\partial x} \right) u^3 - 4 \frac{\partial \beta}{\partial x} u \quad (10)$$

through a contact transformation. Here β is an arbitrary function of t, x with $\partial \beta / \partial x \neq 0$. The ZCR is then $A dx + B dt$ with

$$A = \begin{pmatrix} \frac{1}{u} & 1 \\ 1 & -\frac{1}{u} \end{pmatrix}, \quad (11)$$

$$B = \begin{pmatrix} -\frac{\partial \beta}{\partial x} u_x + 4 \frac{\beta}{u} - \frac{\partial^2 \beta}{\partial x^2} u & 4\beta + 2 \frac{\partial \beta}{\partial x} u \\ 4\beta - 2 \frac{\partial \beta}{\partial x} u & \frac{\partial \beta}{\partial x} u_x - 4 \frac{\beta}{u} + \frac{\partial^2 \beta}{\partial x^2} u \end{pmatrix}.$$

Proof Following Proposition 5, we assume that the matrix A depends on t, x, u, u_x at most. One may check routinely that

$$\frac{\partial^2 a_2}{\partial u_x^2} = 0 \quad \text{and} \quad \frac{\partial^2 a_2}{\partial x \partial u_x} + u_x \frac{\partial^2 a_2}{\partial u \partial u_x} - \frac{\partial a_2}{\partial u} = 0$$

are among differential consequences of the system (8). The general solution is $a_2 = \partial h / \partial x + u_x \partial h / \partial u = D_x h$ for a suitable function $h(t, x, u)$. If a_2 does depend on u_x ,

then $\partial h/\partial u \neq 0$, whence $\bar{t} = t$, $\bar{x} = h$, $\bar{u} = x$ is a point transformation. If a_2 does not depend on u_x , then h does not depend on u , but does depend on x (otherwise $a_2 = 0$), and $\bar{t} = t$, $\bar{x} = h$, $\bar{u} = u$ is a point transformation. In both cases $A = \bar{A} D_x h = \bar{A} a_2$, whence $\bar{a}_2 = 1$ in the transformed matrix \bar{A} .

With $a_2 = 1$, one can check routinely that $\partial a_1/\partial u_x = 0$ is among differential consequences of system (8). If moreover $\partial a_1/\partial u = 0$, then A is completely independent of u and its derivatives, and then so is B , whence the ZCR is gauge equivalent to zero. Therefore, we shall continue with $\partial a_1/\partial u \neq 0$. Then we can apply a point transformation $\bar{x} = \bar{x}$, $\bar{u} = 1/a_1$, which sends a_1 to $1/\bar{u}$ (this choice prevents terms quadratic in u_x from appearing on the right-hand side of eq. (1)). It is then a matter of routine to compute all possible forms of the right-hand side F of eq. (1) and also the corresponding matrices B .

There seem to be no earlier appearance of the class (10) in the literature, let alone its ‘simplest’ member $u_t = u^2 u_{xx} + 4x u_x - 4u^3 - 4u$.

The results would be incomplete if we do not establish irreducibility of the ZCR (11). Since reducibility implies existence of at least one local conservation law, we shall start the following result.

Proposition 7 *Within the class (10), the only equations to possess a conservation law are those with*

$$\beta = \frac{1}{8} \frac{p_t e^{2x} + q_t e^{-2x}}{p e^{2x} - q e^{-2x}}, \quad (12)$$

where p, q are arbitrary functions of t such that $(pq)_t \neq 0$. In all these cases the equation has a single conservation law

$$\begin{aligned} D_t \frac{p e^{2x} + q e^{-2x}}{u} &= D_x \left(\frac{1}{2} \frac{(pq)_t (p e^{2x} + q e^{-2x})}{(p e^{2x} - q e^{-2x})^2} u_x \right. \\ &\quad + \frac{1}{2} \frac{(p_t e^{2x} + q_t e^{-2x})(p e^{2x} + q e^{-2x})}{(p e^{2x} - q e^{-2x})} \frac{1}{u} \\ &\quad \left. - \frac{(pq)_t (3p^2 e^{4x} + 2pq + 3q^2 e^{-4x})}{(p e^{2x} - q e^{-2x})^3} u \right). \end{aligned} \quad (13)$$

Proof A routine computation shows that any characteristics ψ of a conservation law depends on t, x, u at most and satisfies the equations

$$\frac{\partial^2 \psi}{\partial x^2} - 4\psi = 0, \quad \frac{\partial \psi}{\partial u} + 2\frac{\psi}{u} = 0, \quad \frac{\partial \psi}{\partial t} - 4\beta \frac{\partial \psi}{\partial x} = 0.$$

The rest is easy.

Another computation shows that for none of the equations of the class (12) the corresponding ZCR (11) can be reduced to the lower triangular form with multiples of (13) on the diagonal. Thus, the ZCR’s (11) are indeed irreducible.

Finally, a remark on equations determining pseudospherical surfaces (PSS equations) is due. In anticipation of finding new S-integrable nonlinear systems, a number of attempts have been made to classify equations describing pseudospherical surfaces (PSS equations), see [14] and references therein. Even though being a PSS equation is equivalent to possessing an \mathfrak{sl}_2 -valued ZCR, the classification of second order scalar evolution PSS equations as obtained by Reyes [8] (see also [9, 2]) has no intersection with ours. This seeming paradox is easily resolved. In fact, each of the ZCR's found by Reyes is reducible to the lower triangular form (the generalized Burgers equation) or even to a single conservation law (the other equations), which are disregarded in our classification. On the other side, equations (10) are not integrable, hence do not enter the classification of integrable equations by Svinolupov and Sokolov [12, 13], which was the starting point of the Reyes work.

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