The Diffiety Institute Preprint Series

Preprint DIPS–03 **March 21, 2002**

Conservation laws for multidimensional systems and related linear algebra problems

by

Sergei Igonin

Available via INTERNET:
http://diffiety.ac.ru/

The Diffiety Institute
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

Conservation laws for multidimensional systems and related linear algebra problems

SERGEI IGONIN

Abstract. We consider multidimensional systems of PDEs of generalized evolution form with t-derivatives of arbitrary order on the left-hand side and with the right-hand side dependent on lower order t-derivatives and arbitrary space derivatives. For such systems we find an explicit necessary condition for existence of higher conservation laws in terms of the system's symbol. For systems that violate this condition we give an effective upper bound on the order of conservation laws. Using this result, we completely describe conservation laws for viscous transonic equations, for the Brusselator model, and the Belousov-Zhabotinskii system. To achieve this, we determine the conditions for a quadratic matrix A with entries from an arbitrary field to be similar (conjugate) to its transpose A^t or to the matrix $-A^t$, which is of independent interest.

Mathematics Subject Classification (2000): Primary 37K05, 37K10; Secondary 15A24, 76H05, 35K57.

1. Introduction

It is well known that conservation laws are of fundamental importance for clarifying the structure of PDEs. In particular, a common feature of soliton equations is to have conservation laws of arbitrarily high order. Existence of higher order conservation laws imposes very strong conditions on a system of PDEs. Explicit formulation of these conditions would help to classify integrable systems of a given type.

The straightforward study of the conserved current condition is hampered by the fact that one is interested in equivalence classes of conserved currents modulo trivial ones. Therefore, it is convenient to switch from a conserved current to its characteristic, which is the same for equivalent currents and satisfies the equation adjoint to the linearization of the initial system [1, 2, 8, 12].

Thus a part of the problem is to determine conditions for the adjoint linearized equation to have higher order solutions χ. In the present article we perform the first natural step in this direction. For determined, possibly multidimensional, systems of PDEs we find the conditions imposed on the symbol of the system by the fact that some higher order vector-functions satisfy the adjoint linearized equation modulo lower order terms.

Key words and phrases. Multidimensional systems, conservation laws, adjoint equation, characteristic polynomial, invariant factors, viscous transonic equations, Brusselator.

Revised version, minor mistakes corrected.

More precisely, we consider systems of generalized evolution form

$$\frac{\partial^h u^i}{\partial t^h} = F^i(x_j, t, \frac{\partial^s u^a}{\partial t^s}, \frac{\partial^{i_1+\cdots+i_n} u^b}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}), \tag{1}$$
$$i,\, a,\, b = 1, \ldots, m, \quad j = 1, \ldots, n, \quad s = 0, \ldots, h-1, \quad i_1 + \cdots + i_n \leq N,$$

with t-derivatives of fixed order $h > 0$ on the left-hand side and with the right-hand side dependent on lower order t-derivatives and arbitrary space derivatives.

To any m-component vector-function χ of the variables x_j, t, u^i and their derivatives we associate its symbol with respect to the space variables x_j, which is an $m \times m$ matrix, whose entries are homogeneous polynomials in n variables of degree equal to the order $\mathrm{o}(\chi)$ of χ with respect to x_j.

Let A be the symbol of the right-hand side of (1) and S be the symbol of the characteristic χ of a conservation law for (1). It turns out that if $\mathrm{o}(\chi) > O$, where $O \leq N$ is some constant associated to (1), then the adjoint linearized equation implies the matrix equation

$$SA = (-1)^{N+h} A^t S. \tag{2}$$

Here and below A^t is the transpose of A.

A linear algebra problem arises naturally: for what matrices A does there exist a nonzero matrix S such that (2) holds? In addition, since for known integrable systems there are normally higher conservation laws with nonsingular S, one is also interested for which A the matrix S can be taken nonsingular, i.e., when the matrices A and $\pm A^t$ are similar (conjugate).

In solving these problems there is a difference for the cases $n = 1$ and $n > 1$. If $n = 1$, one can switch from homogeneous polynomial in one variable matrices A and S to the corresponding matrices of coefficients and, allowing the coefficients to be complex, make use of the Jordan normal form [3]. While if $n > 1$ then the entries of the matrices belong to the field of rational functions in several variables, which is essentially not algebraicly closed, hence the Jordan normal form is not generally applicable. Using more sophisticated algebraic technique, we prove the following effective criteria.

Theorem. *For any $m \times m$ matrix A with the entries from an arbitrary field F and the characteristic polynomial $d(\lambda) \in F[\lambda]$ we have the following.*

1. *The matrices A and A^t are always similar.*
2. *A nonzero $m \times m$ matrix S such that $SA = -A^t S$ exists if and only if the polynomials $d(\lambda)$ and $d(-\lambda)$ have a common divisor of positive degree.*
3. *The matrices A and $-A^t$ are similar if and only if all the invariant factors $d_i(\lambda)$ of A (certain divisors of the characteristic polynomial [4]) satisfy $d_i(-\lambda) = (-1)^{\deg d_i(\lambda)} d_i(\lambda)$. In particular, $d(\lambda) = (-1)^{\deg d(\lambda)} d(\lambda)$, which in the case $\mathrm{char}\, F \neq 2$ implies $\mathrm{tr}\, A = 0$ and, if m is odd, $\det A = 0$.*

Statements 2 and 3 of the theorem give a necessary condition for existence of higher conservation laws for systems (1) with odd $N + h$. In particular, a scalar equation ($m = 1$) of the form (1) with odd $N + h$ can not have conservation laws of order greater than O. For different ways to write system (1) in the generalized evolution form the symbols A and the resulting conditions are generally different. In order to have higher conservation laws a system of PDEs must satisfy all conditions obtained from various ways to write it in the generalized evolution form.

Let us discuss the previous research on this theme. It seems that only evolution systems ($h = 1$) in one space variable ($n = 1$) were studied in this respect. For such systems equation (2) was obtained by a similar technique in [6] and rediscovered in [3]. In [6] it is noticed that $SA = -A^tS$ for nonsingular S implies $\det(A + \lambda\mathbf{I}) = (-1)^m \det(A - \lambda\mathbf{I})$, which is a weaker version of our Statement 3. Here and below $\mathbf{I}$ is the unity matrix. In [3] Statement 2 is proved for complex matrices, and the corresponding necessary condition for existence of conservation laws of order greater than the order of the evolution system is formulated. Even in this simplest case our result is stronger, since the upper bound O is normally much smaller than the order of the system (see the examples in Section 7).

The paper is organized as follows. In Section 2 the method of characteristics of conservation laws is recalled. We specify the method for systems of generalized evolution form in Section 3 and derive equation (2) in Section 4. The above theorem on quadratic matrices is proved in Section 5. In Section 6 we explicitly formulate the obtained necessary conditions for existence of higher conservation laws. Finally, Section 7 contains some mathematical physics equations of the form (1), which violate these conditions and, therefore, do not have conservation laws of order greater than O. This result allows us to describe all conservation laws for two basic equations in the theory of viscous transonic gas flows (see, for example, [5, 7, 9, 11] and references therein) and for two popular reaction-diffusion systems: the Brusselator model and the Belousov-Zhabotinskii system [10, Section 15.4].

2. Characteristics of conservation laws

This is a brief review of the method of characteristic for computation of conservation laws. We refer to [1, 2, 8, 12] for more details.

Consider a system $\mathcal{E}$ of differential equations

$$F_s(y_i, v^j, \dots, v^k_I, \dots) = 0, \;\; s = 1, \dots, p, \tag{3}$$

with independent variables $y_1, \dots, y_a$, unknown functions $v^1, \dots, v^b$, and

$$v^k_I = \frac{\partial^{|I|} v^k}{\partial y_1^{i_1} \dots \partial y_a^{i_a}}, \quad I = (i_1, \dots, i_a) \in \mathbb{Z}^a_+,$$

being their derivatives. Here and below $\mathbb{Z}_+$ is the set of nonnegative integers and $|I| = i_1 + \dots + i_a$.

Let $\mathcal{F}$ be the algebra of smooth functions of the variables y_i, u^j, and u^j_I. Although the whole set of the variables is infinite, each function is supposed to depend only on a finite subset. Denote by $\mathcal{F}_{\mathcal{E}}$ the quotient algebra with respect to the ideal $\mathcal{I}$ generated by the left-hand sides of equations (3) and their differential consequences $D_{y_{i_1}} \dots D_{y_{i_k}}(F_s) \in \mathcal{F}$. Here

$$D_{y_i} = \frac{\partial}{\partial y_i} + \sum_{j,I} u^j_{I+1_i} \frac{\partial}{\partial u^j_I}$$

is the *total derivative with respect to* y_i, where 1_i is the multi-index with 1 at the i-th place, the other indices of 1_i being zero. For two equivalent functions $f_1, f_2 \in \mathcal{F}$, $f_1 - f_2 \in \mathcal{I}$, one has $f_1(y_i, v^j(y_i)) = f_2(y_i, v^j(y_i))$ for any local solution $v^j(y_i)$ to (3).

By definition, the ideal $\mathcal{I}$ is invariant under the action of D_{y_i}, which, therefore, defines a derivation $\bar{D}_{y_i}$ of $\mathcal{F}_\mathcal{E}$. A *conserved current for* (3) is an a-tuple $J = (J_1, \dots, J_a)$, where $J_k \in \mathcal{F}_\mathcal{E}$, that satisfies the equation

$$\sum_{i=1}^{a} \bar{D}_{y_i}(J_i) = 0. \tag{4}$$

A conserved current is called *trivial*, if it has the form

$$J_k = \sum_{l<k} \bar{D}_l(\mathcal{L}_{lk}) - \sum_{k<l} \bar{D}_l(\mathcal{L}_{kl})$$

for some functions $\mathcal{L}_{kl} \in \mathcal{F}_\mathcal{E}$, $1 \le k < l \le a$. Two conserved currents are said to be *equivalent* if they differ by a trivial one. *Conservation laws* are defined to be the equivalent classes of conserved currents.

Let $\tilde{J}_k \in \mathcal{F}$ be such that $J_k = \tilde{J}_k + \mathcal{I}$. Identity (4) means that

$$\sum_{i=1}^{a} D_{y_i}(\tilde{J}_i) = \sum_{s,I} g_I^s D_y^I(F_s) \tag{5}$$

for some functions $g_I^s \in \mathcal{F}$, only a finite number of which is nonzero. Here and in what follows for each multi-index $I = (i_1, \dots, i_a)$ we denote $D_y^I = D_{y_1}^{i_1} \dots D_{y_a}^{i_a}$. Consider the functions

$$\tilde{\chi}_s = \sum_I (-1)^{|I|} D_y^I(g_I^s), \quad s = 1, \dots, p. \tag{6}$$

Generally speaking, representation (5) and functions $\tilde{\chi}_s$ are not uniquely defined by the conserved current J. Assume that system (3) is non-overdetermined and nondegenerated [1, 8], then the corresponding elements $\chi_s = \tilde{\chi}_s + \mathcal{I}$ of $\mathcal{F}_\mathcal{E}$ are well-defined by J and are all zero if and only if J is trivial. The p-tuple $\chi = (\chi_1, \dots, \chi_p)$ is the same for equivalent conserved currents and is called the *characteristic* (or *generating function* [1, 12]) of the corresponding conservation law.

In addition, χ satisfies the *adjoint linearized equation*

$$K(\chi) = 0, \tag{7}$$

where K is the $b \times p$ matrix differential operator with the entries

$$[K]_{ij} = \sum_I (-1)^{|I|} \bar{D}_y^I \circ \frac{\partial F_j}{\partial v_I^i}. \tag{8}$$

The homological interpretation of these concepts can be found in [1, 2, 12].

3. Formulas for systems of generalized evolution form

Consider a system $\mathcal{E}$ of m partial differential equations in $n+1$ independent variables $t, x_1, \dots, x_n$ and m unknown functions $u^1, \dots, u^m$ of the form

$$\frac{\partial^h u^i}{\partial t^h} = F^i, \quad i = 1, \dots, m, \tag{9}$$

where F^i are smooth functions of the variables x_j, t, u^a and the following derivatives

$$\frac{\partial^{i_1+\dots+i_n} u^b}{\partial x_1^{i_1}\dots\partial x_n^{i_n}},\ i_1,\dots,i_n \in \mathbb{Z}_+,\quad \frac{\partial^s u^a}{\partial t^s},\ s<h. \tag{10}$$

For each multi-index $I=(i_1,\dots,i_n)\in\mathbb{Z}_+^n$ and integer $s\geq 0$ we denote

$$u^i_{s,I} = \frac{\partial^{|I|+s} u^i}{\partial t^s \partial x_1^{i_1}\dots\partial x_n^{i_n}}. \tag{11}$$

Let us describe the algebra $\mathcal{F}_\mathcal{E}$ for this system. By induction on s, it follows from (9) that the derivatives $u^i_{s,I}$ with $s\geq h$ are expressed in terms of

$$t,\quad x_j,\quad u^i_{s,I},\ s<h. \tag{12}$$

Therefore, for each function of t, x_j, u^i and arbitrary derivatives (11) there is a unique equivalent modulo (9) function of variables (12). Thus we can identify $\mathcal{F}_\mathcal{E}$ with the algebra of smooth functions of variables (12). Below all functions are supposed to be from $\mathcal{F}_\mathcal{E}$.

The restrictions $\bar{D}_{x_i}, \bar{D}_t\colon \mathcal{F}_\mathcal{E}\to\mathcal{F}_\mathcal{E}$ of the total derivatives are written in coordinates (12) as follows

$$\bar{D}_{x_i} = \frac{\partial}{\partial x_i} + \sum_{a,I,s} u^a_{s,I+1_i}\frac{\partial}{\partial u^a_{s,I}},$$
$$\bar{D}_t = \frac{\partial}{\partial t} + \sum_{s<h-1} u^a_{s+1,I}\frac{\partial}{\partial u^a_{s,I}} + \sum D^I_x(F^a)\frac{\partial}{\partial u^a_{h-1,I}}. \tag{13}$$

The equation $\bar{D}_t J_0 + \sum_{i=1}^n \bar{D}_{x_i} J_i = 0$ for a conserved current $J=(J_0,J_1,\dots,J_n)$ implies the identity

$$D_t J_0 + \sum_{i=1}^n D_{x_i} J_i = \sum_{i,I} \frac{\partial J_0}{\partial u^i_{h-1,I}} D^I_x(u^i_{h,0} - F^i),$$

which is the specification of (5) for system (9). According to general formula (6), the characteristic $\chi=(\chi_1,\dots,\chi_m)$ is computed as follows

$$\chi_i = \sum_I (-1)^{|I|}\bar{D}^I_x\Big(\frac{\partial J_0}{\partial u^i_{h-1,I}}\Big). \tag{14}$$

From (7) and (8) we see that the characteristic regarded as a column vector satisfies the equation

$$(-)^h \bar{D}^h_t(\chi) = L(\chi), \tag{15}$$

where L is the $m\times m$ matrix differential operator with the entries

$$[L]_{ij} = \sum_I (-1)^{|I|}\bar{D}^I_x \circ \frac{\partial F^j}{\partial u^i_{0,I}} + \sum_{s=0}^{h-1}(-1)^s \bar{D}^s_t \circ \frac{\partial F^j}{\partial u^i_{s,0}}. \tag{16}$$

4. Solving the adjoint equation for the highest order terms

For a (vector-)function f the maximal integer k such that $\partial f/\partial u^i_{s,I} \neq 0$ for some $0 \le s < h$, $1 \le i \le m$, $|I| = k$ is called the *order* of f and denoted by $\mathrm{o}(f)$. If $\partial f/\partial u^i_{s,I} = 0$ for all s, i, I then we set $\mathrm{o}(f) = -1$. The maximal integer $s < h$ such that $\partial f/\partial u^i_{s,I} \neq 0$ for some $1 \le i \le m$, $|I| = \mathrm{o}(f)$ is denoted by $\mathrm{t}(f)$. The order of the characteristic of a conservation law for (9) is called the *order* of the conservation law.

Consider the ring $\mathcal{F}[q_1, \dots, q_n]$ of polynomials in n variables with $\mathcal{F}$ as the ring of coefficients. For each multi-index $I = (i_1, \dots, i_n)$ denote by q^I the monomial $q_1^{i_1} \dots q_n^{i_n} \in \mathcal{F}[q_1, \dots, q_n]$. For any k-component vector-function $\chi = (\chi_1, \dots, \chi_k)$ and two integers a, $b \ge 0$ let $\mathbf{S}^{a,b}_\chi$ be the $k \times m$ matrix with the entries

$$[\mathbf{S}^{a,b}_\chi]_{ij} = \sum_{|I|=a} \frac{\partial \chi_i}{\partial u^j_{b,I}} q^I \in \mathcal{F}[q_1, \dots, q_n].$$

We call the nonzero matrix $\mathbf{S}^{\mathrm{o}(\chi),\mathrm{t}(\chi)}_\chi$ the *symbol* of χ and denote it by $\mathbf{S}_\chi$.

Let A be the symbol of the right-hand side $(F^1, \dots, F^m)$ of (9) and set $N = \mathrm{o}(F^1, \dots, F^m) > 0$. By assumption (10), one has $\mathrm{t}(F^1, \dots, F^m) = 0$. Therefore, by definition,

$$[A]_{ij} = \sum_{|I|=N} \frac{\partial F^i}{\partial u^j_{0,I}} q^I. \tag{17}$$

Applying the Leibniz rule, differential operators (16) can be uniquely rewritten in the usual form

$$[L]_{ij} = \sum_{|I| \le N} f^{ij}_I \bar{D}^I_x + \sum_{s<h} g^{ij}_s \bar{D}^s_t.$$

In particular, from definition (16) one has

$$f^{ij}_I = (-1)^N \frac{\partial F^j}{\partial u^i_{0,I}} \quad \forall\, I : |I| = N. \tag{18}$$

We set

$$O = \max_{i,j,I,s} \{-1,\, \mathrm{o}(f^{ij}_I) - N,\, \mathrm{o}(g^{ij}_s) - N\}. \tag{19}$$

From definition (16) it follows that $O \le N$.

Theorem 1. *Let χ be the characteristic of a conservation law for (9). If $\mathrm{o}(\chi) > O$ then we have*

$$\mathbf{S}_\chi A = (-1)^{N+h} A^t \mathbf{S}_\chi. \tag{20}$$

Proof. Set $a = N + \mathrm{o}(\chi)$, $b = \mathrm{t}(\chi)$. Equation (15) implies, in particular,

$$(-1)^h \mathbf{S}^{a,b}_{\bar{D}^h_t(\chi)} = \mathbf{S}^{a,b}_{L(\chi)}. \tag{21}$$

By formula (13) and assumption (10), for any vector-function χ with $\mathrm{o}(\chi) \ge 0$ the vector-function $\bar{D}^h_t(\chi)$ does not depend on the coordinates $u^j_{s,I}$ with $|I| > a$ or $|I| = a$, $s > b$. Moreover, by formula (17), it is easily seen that

$$\mathbf{S}^{a,b}_{\bar{D}^h_t(\chi)} = \mathbf{S}_\chi A. \tag{22}$$

In the case $\mathrm{o}(\chi) > O$ only the part $\sum_{|I|=N} f_I^{ij} \bar{D}_x^I$ of L contributes to $\mathbf{S}^{a,b}_{L(\chi)}$, since $\bar{D}_t^s(\chi)$ for $s < h$ does not depend on $u^j_{p,I}$ with $|I| = a$, $p \geq b$ and the number a is greater than the order of any coefficient f_I^{ij} or g_s^{ij}. Therefore, taking into account formulas (18) and (17), we obtain

$$\mathbf{S}^{a,b}_{L(\chi)} = (-1)^N A^t \mathbf{S}_\chi.$$

Combining this with (22) and (21), one gets (20). □

In the next section we study the conditions imposed on A by equation (20).

5. Linear algebra problems

For a ring R we denote by $M_k(R)$ the ring of $k \times k$ matrices with entries from R. Consider an arbitrary field F and denote by $\tilde{F}$ its algebraic closure, i.e., the minimal algebraicly closed extention of F.

Theorem 2. *Let $A \in M_k(F)$ and let $d(\lambda) = \det(A - \lambda \mathbf{I})$ be the characteristic polynomial of A. A nonzero matrix $S \in M_k(F)$ such that*

$$SA = -A^t S \tag{23}$$

exists if and only if the polynomials $d(\lambda)$ and $d(-\lambda)$ have a common divisor of positive degree. Equivalently, there are two roots λ_1, $\lambda_2 \in \tilde{F}$ of $d(\lambda)$ such that $\lambda_1 + \lambda_2 = 0$.

Proof. The polynomials $d(\lambda)$ and $d(-\lambda)$ have a common divisor of positive degree if and only if they have a common root λ in $\tilde{F}$, i.e., both λ and $-\lambda$ are roots of $d(\lambda)$.

If (23) holds then for any similar to A matrix $A' = CAC^{-1}$, $C \in M_k(F)$, one has $S'A' = -A'^t S'$ with $S' = C^{-1t} S C^{-1}$. Let us regard A as a matrix from $M_k(\tilde{F}) \supset M_k(F)$. Then we can assume A to be in the Jordan normal form. For such A one can easily show that the linear map

$$M_k(\tilde{F}) \to M_k(\tilde{F}), \ S \mapsto SA + A^t S,$$

has a nontrivial kernel if and only if there are two eigenvalues λ_1, $\lambda_2 \in \tilde{F}$ of A such that $\lambda_1 + \lambda_2 = 0$ (see [3] for $F = \mathbb{C}$).

It remains to prove that if (23) holds for some $S \in M_k(\tilde{F})$ then there is nonzero $S' \in M_k(F)$ such that $S'A = -A^t S'$. Consider a (possibly infinite) basis $\{a_i\}$ of $\tilde{F}$ regarded as a vector space over F. One has $S = \sum_i a_i S_i$, where $S_i \in M_k(F)$. Since $A \in M_k(F)$, equation (23) implies $S_i A = -A^t S_i$. □

Recall some criteria for two matrices A, $B \in M_k(F)$ to be similar (see, for example, [4, Chapter 13]). Consider the ring $F[\lambda]$ of polynomials in one variable. A matrix $C \in M_k(F[\lambda])$ is said to be *unimodular* if $\det C$ is nonzero and belongs to F. For any matrix $A \in M_k(F)$ the matrix $A - \lambda \mathbf{I} \in M_k(F[\lambda])$ admits a *canonical decomposition*

$$A - \lambda \mathbf{I} = C_1 D C_2, \quad C_1,\ D,\ C_2 \in M_k(F[\lambda]), \tag{24}$$

such that C_1, C_2 are unimodular, while D is diagonal. Moreover, the polynomials $d_i = [D]_{ii}$ have the leading coefficient 1, and d_{i+1} is divisible by d_i for each $i = 1, \ldots, k-1$. Then the k polynomials d_i are defined uniquely by A and are called the *invariant factors* of A. Note that there is a simple procedure to compute the invariant factors [4, Chapter 13].

Proposition 1 ([4, Chapter 13]). *Two matrices A, $B \in M_k(F)$ are similar if and only if they have the same invariant factors.*

We call a polynomial $d(\lambda) = \sum_i a_i\lambda^i \in F[\lambda]$ *skew* if $d(-\lambda) = (-1)^{\deg d(\lambda)} d(\lambda)$, i.e., for all $i \equiv \deg d(\lambda) + 1 \mod 2$ one has $a_i = 0$.

Theorem 3. *For any $A \in M_k(F)$ we have the following.*
1. *The matrices A and A^t are similar.*
2. *The matrices A and $-A^t$ are similar if and only if each invariant factor of A is skew. In this case the characteristic polynomial is also skew. In particular, in the case $\operatorname{char} F \neq 2$ we have $\operatorname{tr} A = 0$ and, if k is odd, $\det A = 0$.*

Remark. Note that in the case $\operatorname{char} F = 2$ the second statement of this theorem as well as Theorem 2 are trivial.

Proof. Consider canonical decomposition (24) for A. Taking the transpose, we obtain

$$A^t - \lambda\mathbf{I} = C_2{}^t D C_1{}^t, \tag{25}$$

which is a canonical decomposition for A^t, since $C_1{}^t$, $C_2{}^t$ are clearly unimodular. Therefore, the invariant factors of A^t are the same, which, by Proposition 1, implies that A and A^t are similar.

Multiplying (25) by -1 and substituting $-\lambda$ in place of λ, we obtain

$$-A^t - \lambda\mathbf{I} = -C_2{}^t(-\lambda)D(-\lambda)C_1{}^t(-\lambda). \tag{26}$$

Denote $C_1' = -C_2{}^t(-\lambda)$, $C_2' = TC_1{}^t(-\lambda)$, and $D' = D(-\lambda)T$, where $T \in M_k(F)$ is the diagonal matrix with the entries $[T]_{ii} = (-1)^{\deg[D]_{ii}}$. From (26) we obtain the canonical decomposition $-A^t - \lambda\mathbf{I} = C_1'D'C_2'$ for $-A^t$. According to Proposition 1, A and $-A^t$ are similar if and only if $D' = D$, which says that all the invariant factors $[D]_{ii}$ of A are skew. In this case the characteristic polynomial is also skew, since from (24) it is evidently equal to the product of the invariant factors multiplied by $(-1)^k$. □

6. Necessary conditions for existence of higher conservation laws

According to Theorem 1, a necessary condition for existence of conservation laws for (9) of order greater than O is that there is a nonzero $m \times m$ matrix $\mathbf{S}_\chi$ with entries from $\mathcal{F}[q_1, \ldots, q_n]$ such that (20) holds. Let us treat A and $\mathbf{S}_\chi$ as matrices with entries from the field F of rational functions in n variables $q_1, \ldots, q_n$. Then Theorem 2 implies the following.

Theorem 4. *If $N+h$ is odd and system (9) possesses a conservation law of order greater than O, then the characteristic polynomial $d(\lambda) = \det(A - \lambda\mathbf{I})$ and the polynomial $d(-\lambda)$ have a common divisor of positive degree. Equivalently, there are eigenvalues λ_1, λ_2 of A (possibly in the algebraic closure of F) such that $\lambda_1 + \lambda_2 = 0$.*

Remark. Evidently, introducing the new dependent variables

$$u^{i,s} = \frac{\partial^s u^i}{\partial t^s}, \quad i = 1, \ldots, m, \;\; s = 0, \ldots, h-1,$$

we can rewrite (9) in the usual evolution form. But if $h > 1$ then the symbol of the right-hand side of the obtained evolution system has zero determinant and, therefore, automatically satisfies the condition in Theorem 4, even if the initial system does not meet this condition. Therefore, it is essential to consider the generalized evolution form.

Analyzing examples of known soliton equations, we can conjecture that for (9) to be integrable there must exist higher order conservation laws with nonsingular matrix $\mathbf{S}_\chi$. Therefore, it is worth formulating a necessary condition for existence of such *nonsingular* conservation laws. According to Theorem 3, we obtain the following criterion.

Theorem 5. *If* (9) *has a nonsingular conservation law of order greater than* O *then all the invariant factors of the matrix* A *are skew. In this case its characteristic polynomial is also skew. In particular,* $\operatorname{tr} A = 0$ *and, if* m *is odd,* $\det A = 0$.

If $m = 1$ then A is a nonzero 1×1 matrix, and Theorem 4 implies the following.

Theorem 6. *A scalar equation* $(m = 1)$ *of the form* (9) *with odd* $N + h$ *can not have conservation laws of order greater than* O.

Remark. To obtain stronger conditions, it is sometimes useful to write a system of PDEs in several ways in the form (9). For example, for a scalar evolution equation $u_t = u_{xxx} + u_{yy} + f(u, u_x, u_y, u_{xx})$ the condition is empty, since the sum of $h = 1$ and $N = 3$ is even. But rewriting the equation in the generalized evolution form with respect to y as follows $u_{yy} = -u_{xxx} + u_t - f(u, u_x, u_y, u_{xx})$, we see, according to Theorem 6, that there are no higher conservation laws.

7. Examples

7.1. **The viscous transonic equation.** The nonlinear viscous transonic equation

$$u_{tt} = -u_{xxx} + u_x u_{xx} - \frac{\alpha}{t} u_t \tag{27}$$

describes the asymptotic form of a gas flow in the sonic region (see [7, 9] and references therein). The following conserved currents for (27) were found in [7]

$$(u_t t^\alpha,\, u_{xx} t^\alpha - \frac{u_x^2}{2} t^\alpha), \quad (u_t t + (\alpha - 1)u,\, u_{xx} t - \frac{u_x^2}{2} t). \tag{28}$$

All other conserved currents mentioned in [7] are trivial.

Let us show that (28) span the whole space of conservation laws for (27). We have

$$L = \bar{D}_x^3 + \bar{D}_x^2 \circ u_x - \bar{D}_x \circ u_{xx} + \bar{D}_t \circ \frac{\alpha}{t} = \bar{D}_x^3 + u_x \bar{D}_x^2 + u_{xx} \bar{D}_x + \frac{\alpha}{t} \bar{D}_t - \frac{\alpha}{t^2}.$$

According to (19), one has $O = -1$. By Theorem 6, since $N + h = 5$ is odd, the characteristic χ of any conservation law is a function of x, t only. Equation (15) reads

$$\chi_{tt} = \chi_{xxx} + u_x \chi_{xx} + u_{xx} \chi_x + \frac{\alpha}{t} \chi_t - \frac{\alpha}{t^2} \chi.$$

This implies $\chi_x = 0$ and

$$\chi_{tt} = \frac{\alpha}{t} \chi_t - \frac{\alpha}{t^2} \chi. \tag{29}$$

The characteristics t^α, t of conserved currents (28) span the space of solutions to (29). Therefore, (28) span the space of conservation laws.

7.2. **Another equation for viscous transonic flows.** The equation

$$v_{yy} = 2v_{xt} + v_x v_{xx} - v_{zz} - \mu v_{xxx}, \tag{30}$$

where μ is a nonzero real constant, models nonstationary transonic flows around a thin body with effects of viscosity and heat conductivity when the velocity of the gas is close to the local speed of sound, see [5, 11] and references therein.

For this equation we obtain

$$L = \mu \bar{D}_x^3 + 2\bar{D}_x\bar{D}_t - \bar{D}_z^2 + v_x\bar{D}_x^2 + v_{xx}\bar{D}_x.$$

By definition (19), we have $O = -1$. According to Theorem 6, since $N + h = 5$ is odd, the characteristic χ of any conservation law is a function of x, y, z, t only. Equation (15) implies

$$\chi_x = 0, \;\; \chi_{yy} + \chi_{zz} = 0. \tag{31}$$

Each function χ satisfying (31) is indeed the characteristic of the conserved current

$$\bar{D}_y(\chi v_y - \chi_y v) + \bar{D}_z(\chi v_z - \chi_z v) + \bar{D}_x(\mu\chi v_{xx} - \frac{1}{2}\chi v_x^2 - 2\chi v_t) = 0. \tag{32}$$

Therefore, any conserved current for (30) is equivalent to a conserved current of the form (32).

7.3. **The Brusselator model.** The Brusselator model governing certain chemical reactions is the following multidimensional system

$$\begin{aligned} v_t &= \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} + v^2 w - (b+1)v + a, \\ w_t &= c\sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} - v^2 w + bv, \end{aligned} \tag{33}$$

where a, b, and $c \neq 0$ are real parameters [10, Section 15.4]. By definition (16) we have

$$L = \begin{pmatrix} \sum_i \bar{D}_{x_i}^2 + 2vw - (b+1) & -2vw + b \\ v^2 & c\sum_i \bar{D}_{x_i}^2 - v^2 \end{pmatrix}.$$

The symbol A is diagonal with $[A]_{11} = \sum_i q_i^2$ and $[A]_{22} = c\sum_i q_i^2$. By definition (19), we have $O = -1$. By Theorem 4, since $[A]_{11} \neq 0$, $[A]_{22} \neq 0$, and $[A]_{11} + [A]_{22} \neq 0$, the characteristic $\chi = (\chi_1, \chi_2)$ of any conservation law is a function of x_i, t only. For such χ equation (15) reads

$$\begin{aligned} -\frac{\partial \chi_1}{\partial t} &= \sum_i \frac{\partial^2 \chi_1}{\partial x_i^2} + (2vw - (b+1))\chi_1 - (2vw - b)\chi_2, \\ -\frac{\partial \chi_2}{\partial t} &= v^2\chi_1 + c\sum_i \frac{\partial^2 \chi_2}{\partial x_i^2} - v^2\chi_2. \end{aligned} \tag{34}$$

Evidently, (34) implies $\chi_1 = \chi_2$. Then (34) becomes

$$-\frac{\partial \chi_1}{\partial t} = \sum_i \frac{\partial^2 \chi_1}{\partial x_i^2} - \chi_1, \tag{35}$$
$$-\frac{\partial \chi_1}{\partial t} = c \sum_i \frac{\partial^2 \chi_1}{\partial x_i^2}.$$

Clearly, for $c = 1$ the only solution to (35) is $\chi_1 = 0$, while for $c \neq 1$ the general solution is

$$\chi_1 = \chi_2 = G(x_i) \exp(\frac{ct}{c-1}), \tag{36}$$

where $G(x_i)$ is an arbitrary solution to the equation

$$G + (c-1) \sum_i \frac{\partial^2 G}{\partial x_i^2} = 0. \tag{37}$$

Each solution (36) is indeed the characteristic of the conserved current

$$\bar{D}_t\big(G \exp(\frac{ct}{c-1})(v+w)\big) + \\ + \sum_{i=1}^{n} \bar{D}_{x_i}\Big(\exp(\frac{ct}{c-1})\big(\frac{\partial G}{\partial x_i}(v+cw) - G(v_{x_i} + cw_{x_i}) - \frac{a}{n}\int G\,\mathrm{d}x_i\big)\Big) = 0.$$

Thus if $c \neq 1$ then these conserved currents span the space of conservation laws; while if $c = 1$ then there are no nontrivial conservation laws for (33).

7.4. **The Belousov-Zhabotinskii system.** This system describes certain chemical reactions and reads [10, Section 15.4]

$$v_t = \sum_i \frac{\partial^2 v}{\partial x_i^2} + v(1 - v - rw) + Lrw,$$
$$w_t = \sum_i \frac{\partial^2 w}{\partial x_i^2} - bvw - Mw.$$

Here r, L, b, M are real constants. Evidently, Theorem 4 implies that this system does not possess conservation laws of nonnegative order. Similarly to the above examples, analysis of equation (15) for characteristics of order -1 shows that in the nonlinear case $b \neq 0$ there are no nontrivial conservation laws at all.

Acknowledgements

The author is grateful to Prof. I. S. Krasil′shchik for very stimulating attention to this work and many helpful remarks, to Prof. A. L. Onishchik for attracting author's attention to the used algebraic technique, and to Prof. R. Martini and Prof. V. V. Sokolov for useful discussions.

References

[1] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor′kova, I. S. Krasil′shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*. Edited and with a preface by Krasil′shchik and Vinogradov. Amer. Math. Soc., Providence, RI, 1999.
[2] R. L. Bryant and P. A. Griffiths. Characteristic cohomology of differential systems. I. General theory. *J. Amer. Math. Soc.* **8** (1995), 507–596.
[3] K. Foltinek. Conservation laws of evolution equations: generic non-existence. *J. Math. Anal. Appl.* **235** (1999), 356–379.
[4] A. Kurosh. *Higher algebra*. Mir Publishers, Moscow, 1972.
[5] N. Larkin. On one problem of transonic gas dynamics. *Mat. Contemp.* **15** (1998), 169–186.
[6] A. V. Mikhailov, A. B. Shabat and R. I. Yamilov. A symmetric approach to the classification of nonlinear equations. Complete lists of integrable systems. *Russian Math. Surveys* **42** (1987), 1–63.
[7] J. Mukherjee and A. Roy Choudhury. On the Lie symmetry, conservation laws and Painlevé analysis for the viscous transonic equation. *Ann. Physik (7)* **46** (1989), 6–92.
[8] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, New York, 1993.
[9] O. S. Ryzhov. Viscous transonic flows. *Annual review of fluid mechanics, Vol. 10*, 65–92. Annual Reviews, Palo Alto, Calif., 1978.
[10] M. E. Taylor. *Partial differential equations. III. Nonlinear equations.* Springer-Verlag, New York, 1996.
[11] P. A. Vel′misov, S. V. Fal′kovič. On the theory of transonic flows of a viscous gas. (Russian) *Izv. Vysš. Učebn. Zaved. Matematika* **1974**, no. 5, 52–61.
[12] A. M. Vinogradov. The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. I. The linear theory. II. The nonlinear theory. *J. Math. Anal. Appl.* **100** (1984), 1–129.

Independent University of Moscow; University of Twente, the Netherlands
E-mail address: igonin@mccme.ru