The Diffiety Institute Preprint Series

Preprint DIPS–2 March 1 — May 8, 2002

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On n-ary generalizations of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$

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Abstract. Polynomials $a_j \in \Bbbk_N[x]$, $1 \leq j \leq N$, of degree $\deg a_j = N$ are shown to be closed w.r.t. action of the N-ary bracket $[a_1, \ldots, a_N] = W(a_1, \ldots, a_N)$, where W denotes the Wronskian determinant. This bracket is proved to induce the homotopical N-Lie algebra structure on the polynomials $\Bbbk_N[x]$ of degree N, so that an extended Jacobi identity (with 2^{N-1} summands) holds; the case $N = 2$ is the Lie bracket in $\mathfrak{sl}_2(\Bbbk)$ satisfying the Jacobi identity.

The property of the Wronskian determinants to compose the homotopical N-Lie algebras is proved to hold for arbitrary analytic functions $a_j \in \Bbbk[[x]]$, thus extending the former result. The identity $[\![W^{\vec{\imath}}, W^{\vec{\jmath}}]\!]^{\text{RN}} = 0$ is discussed, where $\vec{\imath} = (0, 1, \ldots, k) \in \mathbb{Z}_+^{k+1}$, $\vec{\jmath} = (0, 1, \ldots, l) \in \mathbb{Z}_+^{l+1}$, ∂ is a derivation, $W^{\vec{\imath}} = \partial^{i_1} \wedge \ldots \wedge \partial^{i_{k+1}}$ and $W^{\vec{\jmath}} = \partial^{j_1} \wedge \ldots \wedge \partial^{j_{l+1}}$ are the Wronskians and $[\![\cdot, \cdot]\!]^{\text{RN}}$ is the Richardson–Nijenhuis bracket.

The notion of the Wronskian determinant is generalized to the case of n independent variables: $(x^1, \ldots, x^n) \in \Bbbk^n$, so that the resulting concept preserves the homotopical N-Lie Jacobi identity.

Mathematics Subject Classification (2000): 13M10, 15A15, 15A24, 26C05.

Introduction. In this paper[1], following [1, 2, 3, 4] we consider a special case of N-ary generalizations of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$, namely, the homotopical N-Lie algebra of the Nth degree polynomials $\Bbbk_N[x]$, where $2 \leq N \in \mathbb{N}$ and the field $\Bbbk$ has characteristic 0.

Remark 1. We make a conventional remark concerning the underlying field $\Bbbk$. In all the illustrative cases involving the differential calculus we assume $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$ thus investigating smooth functions $a_j \in C^\infty(\Bbbk)$. Generally, we deal with analytic functions $a_j \in \Bbbk[[x]]$ treating them as formal series over the field $\Bbbk$ of characteristic 0. In the n-dimensional case $x = (x^1, \ldots, x^n) \in \Bbbk^n$, the analytic functions are $a_j \in \Bbbk[[x^1, \ldots, x^n]]$. Still, we preserve the notation '$C^\infty(\Bbbk^n)$' from the case $\Bbbk = \mathbb{R}$ bearing in mind its applications in mathematical physics.

First let us introduce some notation. Let $\mathcal{A}$ be an associative commutative algebra over the field $\Bbbk$ such that $\operatorname{char} \Bbbk = 0$, and let ∂ be a derivation of $\mathcal{A}$. Further on, we consider the non-graded (even) case only. Let $\sigma \in S_m^k \subset S_m$ be the permutations such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(m)$ and let $[\cdot, \ldots, \cdot] \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ be an N-linear skew-symmetric bracket: $[a_{\Sigma(1)}, \ldots, a_{\Sigma(N)}] = (-1)^\Sigma [a_1, \ldots, a_N]$ for any rearrangement $\Sigma \in S_N$.

Definition 1. The algebra $\mathcal{A}$ is said to be the *homotopical N-Lie* algebra, if the Jacobi identity

$$(1) \qquad \sum_{\sigma \in S_{2N-1}^N} (-1)^\sigma [[a_{\sigma(1)}, \ldots, a_{\sigma(N)}], a_{\sigma(N+1)}, \ldots, a_{\sigma(2N-1)}] = 0$$

[1] Submitted March 1, 2002; Revised May 8, 2002.

holds for any $a_j \in \mathcal{A}$, $1 \le j \le 2N-1$.

Let $\Delta \in \operatorname{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$ and $\nabla \in \operatorname{Hom}(\bigwedge^l \mathcal{A}, \mathcal{A})$. By $\Delta[\nabla] \in \operatorname{Hom}(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A})$ we denote the *action* $\Delta[\cdot]\colon \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A}) \to \operatorname{Hom}(\bigwedge^{N+k-1} \mathcal{A}, \mathcal{A})$ of Δ on ∇:

$$\Delta[\nabla](a_1, \ldots, a_{k+l-1}) = \sum_{\sigma \in S^l_{k+l-1}} (-1)^\sigma \, \Delta(\nabla(a_{\sigma(1)}, \ldots, a_{\sigma(l)}), a_{\sigma(l+1)}, \ldots, a_{\sigma(k+l-1)}),$$

where $a_j \in \mathcal{A}$. Then, the Jacobi identity given in Eq. (1) is

$$\Delta[\Delta] = 0. \tag{2}$$

By $[\![\Delta, \nabla]\!]^{\text{RN}} \in \operatorname{Hom}(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A})$ we denote the *Richardson–Nijenhuis bracket* of Δ and ∇:

$$[\![\Delta, \nabla]\!]^{\text{RN}} \stackrel{\text{def}}{=} \Delta[\nabla] - (-1)^{(k-1)(l-1)} \nabla[\Delta].$$

Let $\Delta \in \operatorname{Hom}(\bigwedge^k \mathcal{A}, \mathcal{A})$ and $a_j \in \mathcal{A}$, $1 \le j \le k$; suppose $1 \le l \le k$. The *inner product* $\Delta_{a_1,\ldots,a_l} \in \operatorname{Hom}(\bigwedge^{k-l} \mathcal{A}, \mathcal{A})$ is defined by

$$\Delta_{a_1,\ldots,a_l}(a_{l+1}, \ldots, a_k) \stackrel{\text{def}}{=} \Delta(a_1, \ldots, a_k).$$

Now we describe the scheme that generates the (N, k, r)-Lie algebra structures [1] and treates the homotopical $2N$-Lie algebras as the $(2N, 1, 0)$-Lie algebras.

Definition 2 ([1])**.** Choose integers N, k, and r such that $0 \le r \le k < N$, and let $a_1, \ldots, a_r, b_1, \ldots, b_k \in \mathcal{A}$. The skew-symmetric map $\Delta \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ is said to determine the Lie algebra structure of the type (N, k, r) on the $\Bbbk$-vector space $\mathcal{A}$, if the (N, k, r)-Jacobi identity

$$[\![\Delta_{a_1,\ldots,a_r}, \Delta_{b_1,\ldots,b_k}]\!]^{\text{RN}} = 0 \tag{3}$$

holds for any $\vec{a}$ and $\vec{b}$. By $\text{Lie}^{(N,k,r)}(\mathcal{A})$ we denote the set of all type (N, k, r) structures $\Delta \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ on $\mathcal{A}$.

Remark 2. We claim that

$$\text{Lie}^{(N,0,0)}(\mathcal{A}) = \text{Lie}^{(N,1,0)}(\mathcal{A}) \tag{4}$$

for any even N; this is a typical instance of the heredity structures [1]. Really, the following two conditions are equivalent:

$$[\![\Delta, \Delta]\!]^{\text{RN}} = 0 \Leftrightarrow [\![\Delta, \Delta]\!]^{\text{RN}}{}_a = -2\,[\![\Delta, \Delta_a]\!]^{\text{RN}} = 0 \qquad \forall a \in \mathcal{A},$$

due to Corollary 1.1 in [1]:

$$[\![\Delta, \Delta]\!]^{\text{RN}}{}_a = (-1)^{N-1} [\![\Delta, \Delta_a]\!]^{\text{RN}} + [\![\Delta_a, \Delta]\!]^{\text{RN}}.$$

Finally, $[\![\Delta, \Delta_a]\!]^{\text{RN}} = 0$ for any $a \in \mathcal{A}$.

Thus, the Jacobi identity of the type $(N, 0, 0)$

$$[\![\Delta, \Delta]\!]^{\text{RN}} = 2\Delta[\Delta] = 0$$

implies Eq. (1) for any even N. The preceding papers [5, 6] should be regarded w.r.t. the isomorphism (4).

If N is odd, then the expression $[\![\Delta,\Delta]\!]^{\mathrm{RN}} \equiv 0$ is trivial, and we consider Eq. (2) separately from the condition (3).

In the sequel, we study the Jacobi identity (1) of the form (2), where $\Delta \in \mathrm{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$. We conclude that for even Ns the results will be equivalent to the properties of the type $(N,1,0)$ structures, while for odd Ns our results are independent from the scheme proposed in Definition 2.

1. Consider the space $\Bbbk_N[x] \ni a_j$ of polynomials a_j of degree not greater than N; on this space, there is the N-linear skew-symmetric bracket

$$[a_1,\ldots,a_N] = W(a_1,\ldots,a_N), \tag{5}$$

where W denotes the Wronskian determinant. Since N-ary bracket (5) is N-linear, we consider monomials const $\cdot x^k$ only. We choose $\{a_j^0\} = \{x^k\}$ or $\{a_j^0\} = \{x^k/k!\}$, where $0 \le k \le N$ and $1 \le j \le 2N-1$, for standard basis in $\Bbbk_N[x]$; exact choice depends on the situation: the monomials x^k are used to demonstrate the presence or absence of certain degrees in N-linear bracket (5), and the monomials $x^k/k!$ are convenient in calculations since they are closed w.r.t. the derivations. The set of monomials $a_j = x^k$ is bi-graded: $\operatorname{ord} a_j = j$ and $\deg x^k = k$.

Let $\vec{p} \in \mathbb{Z}_+^s$, $\mathbb{N} \ni s \ge 1$. By $|\vec{p}| \in \mathbb{Z}_+$ we denote the 'city-block' norm: $|\vec{p}| = \sum_{j=1}^s p_j$. By definition, put

$$\Delta_{\mathbb{Z},t}^{s-1} \equiv \{\vec{p} = (p_1,\ldots,p_s) \in \mathbb{Z}_+^s \mid |\vec{p}| = t\}. \tag{6}$$

We see that $\Delta_{\mathbb{Z},t}^{s-1}$ is the intersection $\mathbb{Z}_+^s \cap \Delta_t^{s-1}$, where Δ_t^s is the standard s-dimensional simplex: $\Delta_t^s = \{\vec{p} \in \mathbb{R}_+^{s+1} \mid \sum_{j=1}^{s+1} p_j = t\}$.

Lemma 1. *Let $0 \le k < N$; then*

1. *the inequality $0 \le \deg W\left(1,\ldots,\widehat{\frac{x^k}{k!}},\ldots,\frac{x^N}{N!}\right) \le N$ holds, i.e., the monomials are closed w.r.t. the Wronskian determinant;*
2. *([6]) the relation $W\left(1,\ldots,\widehat{\frac{x^k}{k!}},\ldots,\frac{x^N}{N!}\right) = \mathrm{const}(N,k)\cdot x^{N-k}$ holds, i.e., the Wronskian determinant is a monomial;*
3. *the coefficient $\mathrm{const}(N,k)$ is*

$$\mathrm{const}(N,k) = \frac{1}{(N-k)!}\cdot\left\{\frac{(-1)^{N-k+1}}{2} + \frac{1}{2}\sum_{l=1}^{N-k} l!\cdot\sum_{\vec{p}\in\Delta_{\mathbb{Z},N-k-l}^{l-1}}\prod_{j=1}^{l}(j)^{p_j}\right\}, \tag{7}$$

while for $k = N$ the Wronskian coefficient in (7) equals 1.

Proof. 1)[2] The Wronskian determinant is

$$\partial^{\vec{\imath}}(a_{\vec{\jmath}}) = \partial^{i_1}\wedge\ldots\wedge\partial^{i_N}(a_1,\ldots,a_N) = \det\|\partial^{i_\alpha}a_\beta\|.$$

In the case of monomials we have

$$\partial^0\wedge\ldots\wedge\partial^{N-1}(x^{j_1},\ldots,x^{j_N}) = \sum_{\sigma\in S_N}(-1)^\sigma\,\partial^0(x^{j_{\sigma(1)}})\cdot\ldots\cdot\partial^{N-1}(x^{j_{\sigma(N)}}).$$

[2]The proof of 1) was reported by A. M. Verbovetsky (private communication).

By definition, put $x^{\vec{\jmath}} = \{x^{j_1}, \ldots, x^{j_{\sharp\vec{\jmath}}}\}$ and $|x^{\vec{\jmath}}| = |\vec{\jmath}|$. Suppose $\vec{\jmath} \in [0, N]^N \cap \mathbb{Z}^N$. We claim that for every non-trivial summand $S_{\sigma,\vec{\jmath}}$:

$$S_{\sigma,\vec{\jmath}} \stackrel{\text{def}}{=} (-1)^\sigma\, \partial^0(x^{j_{\sigma(1)}}) \cdot \ldots \cdot \partial^{N-1}(x^{j_{\sigma(N)}}),$$

the degree $\deg S_{\sigma,\vec{\jmath}}$ has the upper bound N:

$$0 \le \deg S_{\sigma,\vec{\jmath}} = |x^{\vec{\jmath}}| - |W^{0,1,\ldots,N-1}| = j_1 + \ldots + j_N - \frac{N(N-1)}{2} \le N. \tag{8}$$

Really, we have

$$0 + 1 + \ldots + N - 1 = \frac{N(N-1)}{2} \le j_1 + \ldots + j_N \le 1 + 2 + \ldots + N = \frac{N(N+1)}{2}$$

and, since the derivative $\partial\colon \Bbbk_{N'}[x] \to \Bbbk_{N'-1}[x]$, $1 \le N' \le N$, $\ker \partial = \Bbbk$, decreases the degree deg by 1, we obtain (8).

This proves 1). It is not hard to prove 2) just now, but we need some additional notation and recurrence relation (11) in order to prove 3), so we apply another technique.

2) Since the degrees $\deg a_i \neq \deg a_j$, $i \neq j$, lest the Wronskian would be zero identically, rearrange the elements a_j such that $\operatorname{ord} a_i < \operatorname{ord} a_j \Leftrightarrow \deg a_i < \deg a_j$. We have

$$W\left(1, \ldots, \widehat{\frac{x^k}{k!}}, \ldots, \frac{x^N}{N!}\right) = W\left(1, \ldots, \frac{x^{k-1}}{(k-1)!}\right) \cdot W\left(x, \ldots, \frac{x^{N-k}}{(N-k)!}\right), \tag{9}$$

where the first factor in the r.h.s. of (9) equals 1 and has the degree 0. Denote the second factor, the determinant of the $(N-k) \times (N-k)$ matrix, by W_m, $m \equiv N - k$. We claim that W_m is a monomial: $\deg W_m = m$, and prove this fact by induction on $m \equiv N - k$. For $m = 1$, $\deg \det(x) = 1 = m$. Let $m > 1$; the decomposition of W_m w.r.t. the last row gives

$$W_m = W\left(x, \ldots, \frac{x^m}{m!}\right) = x \cdot W\left(x, \ldots, \frac{x^{m-1}}{(m-1)!}\right) - 1 \cdot W\left(x, \ldots, \frac{x^{m-2}}{(m-2)!}, \frac{x^m}{m!}\right), \tag{10}$$

where the degree of the first Wronskian in r.h.s. of (10) is $m - 1$ by the inductive assumption. Again, decompose the second Wronskian in r.h.s. of (10) w.r.t. the last row and proceed so iteratively using the induction hypothesis. We obtain the recurrence relation

$$W_m = \sum_{l=1}^{m-1} W_{m-l} \cdot \frac{x^l}{l!} - (-1)^m \frac{x^m}{m!}, \qquad m \ge 1, \tag{11}$$

whence $\deg W_m = m$. We see that the initial Wronskian (9) is a monomial itself of degree $m = N - k$ with yet unknown coefficient.

3) Now we calculate the coefficient $W_m(x)/x^m \in \Bbbk$ in the Wronskian determinant (9). Consider the generating function

$$f(x) \equiv \sum_{m=1}^{\infty} W_m(x) \tag{12}$$

such that

$$W_m(x) = \frac{x^m}{m!}\,\frac{d^m f}{dx^m}(0), \qquad 1 \le m \in \mathbb{N}.$$

Note that $\exp(x) \equiv \sum_{m=0}^{\infty} x^m/m!$; treating (12) as the formal sum of equations (11), we have

$$f(x) = f(x)\cdot(\exp(x)-1) - \exp(-x) + 1,$$

whence

$$f(x) = \frac{\exp(-x)-1}{\exp(x)-2} = -\frac{1}{2}\cdot\left(\exp(-x) + \frac{1}{\exp(x)-2}\right). \tag{13}$$

Note that $f(0)=0$ and $f(x)$ is smooth in the vicinity of zero. Then, differentiating (13) m times w.r.t. x, we obtain

$$\frac{d^m f}{dx^m}(x) = \frac{(-1)^{m+1}}{2}\exp(-x) - \frac{1}{2}\cdot\sum_{l=1}^{m}\underbrace{(-1)^l l!}_{\text{A}}\cdot\frac{\exp(lx)}{(\exp(x)-2)^{l+1}}\cdot\sum_{\vec{p}\in\Delta^{l-1}_{\mathbb{Z},m-l}}\overbrace{\prod_{j=1}^{l}(j)^{p_j}}^{\text{B}}, \tag{14}$$

where the underbraced factor A comes from differentiating $(\exp(x)-2)^{-1}$ l times w.r.t. x and $\binom{2(m-l)-1}{m-l-1}$ overbraced summands B of the form $1^{p_1}\cdot 2^{p_2}\cdot\ldots\cdot l^{p_l}$ come from differentiating the exponents $\exp(x)$, …, $\exp(lx)$ $m-l$ times w.r.t. x, see (6) for definition of the set $\Delta^{l-1}_{\mathbb{Z},m-l}$. Considering (14) at the point $x=0$, we finally have

$$W_m(x) = \frac{x^m}{m!}\cdot\left\{\frac{(-1)^{m+1}}{2} + \frac{1}{2}\sum_{l=1}^{m} l!\cdot\sum_{\vec{p}\in\Delta^{l-1}_{\mathbb{Z},m-l}}\prod_{j=1}^{l}(j)^{p_j}\right\}$$

for any integer $m \ge 1$, whence follows (7). Also, we see that $W_m(x) \rightsquigarrow (\ln 2)^{-m}\cdot x^m$ as $m\to\infty$, since series (12) converges within $|x| < \ln 2$.

The proof is complete. □

From Lemma 1 follows

Theorem 1 ([6])**.** *Polynomials $a_1, \ldots, a_N \in \Bbbk_N[x]$ of degree $\deg a_j \le N$, $1 \le j \le N$, are closed with respect to action of N-ary bracket (5): $[a_1, \ldots, a_N] = W(a_1, \ldots, a_N) \in \Bbbk_N[x]$, while the structural constants are (7).*

2. Now we start to construct the homotopical N-Lie generalizations of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$. In this section, we introduce some notation and describe the number $\sharp$ of summands in the relations we deal with. Our starting point is the following

Example 1 ([5]). The polynomials $\Bbbk_2[x] = \{\alpha x^2 + \beta x + \gamma \mid \alpha,\beta,\gamma\in\Bbbk\}$ of degree 2 with bracket (5) form a Lie algebra isomorphic to $\mathfrak{sl}_2(\Bbbk)$. The commutation relations can be easily checked in the basis $\langle 1, -2x, -x^2\rangle$:

$$[-2x, 1] = 2, \qquad [-2x, -x^2] = 2x^2, \qquad [1, -x^2] = -2x,$$
$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$

whence the representation $\varrho\colon \mathfrak{sl}_2(\Bbbk) \to \Bbbk_2[x]$ is $\varrho(e) = 1$, $\varrho(h) = -2x$, and $\varrho(f) = -x^2$.

Definition 3. The system $(a_1, \ldots, a_{2N-1})$ of monomials is called *the standard system of monomials* $\vec{a}(k_1, k_2)$ if $0 \le \deg a_j \le N$ for all $1 \le j \le 2N-1$ and the set $\{a_j\}$ of elements a_j, ungraded by ord, is

$$\{a_j\} = \{1, \ldots, x^N\} \cup \{1, \ldots, x^N\} \setminus \{x^{k_1}, x^{k_2}, x^{N-k_1}\} \quad \text{where} \quad 0 \le k_1, k_2 \le N. \tag{15}$$

By $J^{N,\infty}_{\vec{a}}(x)$ we denote the l.h.s. in (1) with N-ary bracket (5) acting on analytic functions a_j of $x \in \Bbbk$, and by $J^N_{\vec{a}(k_1,k_2)}(x)$ we denote the same l.h.s. in (1) with N-ary bracket (5) acting on the standard system of monomials $\vec{a}(k_1, k_2)$.

Lemma 2 ([6])**.** *The number of summands* $\sharp J^N_{\vec{a}(k_1,k_2)}(x) = 2^{N-1}$*. Also, the number of summands* $\sharp J^{N,\infty}_{\vec{a}}(x) = \binom{2N-1}{N-1} = \binom{2N-1}{N}$*, see* [5].

Proof. Consider three cases: $k_1 \ne k_2 \ne N - k_1$, $k_2 = k_1$, and $k_2 = N - k_1$.

Case 1. Let $k_1 \ne k_2 \ne N - k_1$; then there are two types of the internal Wronskians in (1). We say that the summands with the internal Wronskians $W\left(\ldots, \widehat{x^{k_1}}, x^{k_2}, x^{N-k_1}, \ldots\right)$ belong to type 1, and the summands containing $W\left(\ldots, x^{k_1}, x^{k_2}, \widehat{x^{N-k_1}}, \ldots\right)$ belong to type 2, where $k_2 \ne N - k_1$ and $k_2 \ne k_1$ by assumption. Other elements x^m are met twice in the set $\{a_j\}$ and their choice for position into the internal (resp., external) Wronskian is arbitrary. So, we have 2 types $\times$ 2^{N-2} arrangements $= 2^{N-1}$ summands.

Case 2. Let $k_1 = k_2 \ne N - k_1$; then there are 2^{N-1} type 1 internal Wronskians $W\Big(\ldots, \widehat{x^{k_1}}, x^{N-k_1}, \ldots\Big)$, necessarily containing the element x^{N-k_1}; the type 2 summands are not realized for this $\vec{a}(k_1, k_1)$ since these Wronskians are zero identically.

Case 3. This case $k_2 = N - k_1$ repeats literally Case 2 after the transposition type 1 $\leftrightarrow$ type 2.

Thus, in all cases the number of summands $\sharp J^N_{\vec{a}(k_1,k_2)}(x)$ equals 2^{N-1}.

Obviously, the number of summands $\sharp J^{N,\infty}(x) = \sharp S^N_{2N-1} = \binom{2N-1}{N}$, if no three elements a_{m_1}, a_{m_2}, and $a_{m_3} \in \vec{a}$ have equal degrees. □

Corollary 1. By the Stirling formula,

$$\frac{\sharp J^{N,\infty}_{\vec{a}}(x)}{\sharp J^N_{\vec{a}(k_1,k_2)}(x)} = \frac{\binom{2N-1}{N-1}}{2^{N-1}} \rightsquigarrow \frac{2^N}{\sqrt{\pi N}} \qquad \text{as } N \to \infty.$$

We see that the numbers of summands grow exponentially but the case $0 \le \deg a_j \le N$ provides exponentially less summands than the general case of analytic functions.

3. Our goal is to prove that $J^N_{\vec{a}(k_1,k_2)}(x) = J^N(x; k_1, k_2) \equiv 0$, not depending on rearrangements of $\vec{a}$, if the elements x^{k_1}, x^{k_2}, and $x^{N-k_1} \notin \vec{a}$.

Lemma 3 ([6])**.** *Let the orders* ord *and* deg *coincide:* $\operatorname{ord} a_i \le \operatorname{ord} a_j \Leftrightarrow \deg a_i \le \deg a_j$*, i.e.,*

$$(a_1, \ldots, a_{2N-1}) = (1, 1, x, x, \ldots, \widehat{x^{k_1}}, \widehat{x^{k_2}}, \widehat{x^{N-k_1}}, \ldots, x^N, x^N) \quad \leftarrow 2N-1 \textit{ elements.}$$

Then $J^N_{\vec{a}(k_1,k_2)}(x) \equiv 0$.

Proof. Consider a pair of neighbouring elements a_m, a_{m+1} such that $\deg a_m = \deg a_{m+1} = k$; for $N \geq 2$, such a pair exists. Now consider two kinds of summands in (1): the ones with direct order, i.e., $(m = \sigma(i)$ and $m+1 = \sigma(j)) \Rightarrow i < j$, and the ones with the reverse order, $i > j$. Obviously, the algebraic expressions $W\left(W\left(\ldots, x^k, \ldots\right), \ldots, x^k, \ldots\right)$ are identical to each other, but we claim that these two kinds of summands in $J^N_{\vec{a}(k_1,k_2)}(x)$ differ by a sign -1 that comes from $(-1)^\sigma$. Really, all elements a_j with $j < m$ of $j > m+1$ do not acquire additional inversions spoiling the sign $(-1)^\sigma$ after the transposition $a_m \leftrightarrow a_{m+1}$, yet the unique additional inversion between a_{m+1} and a_m appeares in all arrangements $\sigma \in S^N_{2N-1}$, giving the sign -1. So, to every summand in (1) there corresponds the unique opposite one, and the pairs of opposite summands do not intersect. Thus, $J^N_{\vec{a}(k_1,k_2)}(x) = 0$. □

Now we maximize the number of summands in $J^N_{\vec{a}(k_1,k_2)}(x)$ in order to note its skew-symmetry w.r.t. the transpositions $a_j \mapsto a_{\Sigma(j)}$, $\Sigma \in S_{2N-1}$.

Remark 3. The l.h.s. of Jacobi identity (1) equals

$$J^N_{\vec{a}} = \frac{1}{N!(N-1)!} \sum_{\sigma \in S_{2N-1}} (-1)^\sigma [[a_{\sigma(1)}, \ldots, a_{\sigma(N)}], a_{\sigma(N+1)}, \ldots, a_{\sigma(2N-1)}], \tag{16}$$

where *all* elements $\sigma \in S_{2N-1}$ are taken into consideration; see [2, 4].

Remark 4. The r.h.s. in (16) is skew-symmetric w.r.t. any rearrangement Σ of the elements $a_j \in \vec{a}$:

$$\frac{1}{N!(N-1)!} \sum_{\sigma \in S_{2N-1}} (-1)^\sigma [[a_{(\sigma\circ\Sigma)(1)}, \ldots, a_{(\sigma\circ\Sigma)(N)}], a_{(\sigma\circ\Sigma)(N+1)}, \ldots, a_{(\sigma\circ\Sigma)(2N-1)}] =$$
$$= \frac{(-1)^\Sigma}{N!(N-1)!} \sum_{\sigma \in S_{2N-1}} (-1)^\sigma [[a_{\sigma(1)}, \ldots, a_{\sigma(N)}], a_{\sigma(N+1)}, \ldots, a_{\sigma(2N-1)}].$$

Consequently, the l.h.s in (16) is skew-symmetric also with no restrictions on degrees $\deg a_j \in \mathbb{Z}$.

From Remarks 3 and 4 follows

Lemma 4 ([6]). *Let $0 \leq k_1, k_2 \leq N$. Suppose $\vec{a}(k_1,k_2)$ is a standard system of monomials (15) and $\Sigma\vec{a}(k_1,k_2) = \{a_{\Sigma(1)}, \ldots, a_{\Sigma(2N-1)}\}$ is its arbitrary rearrangement, $\Sigma \in S_{2N-1}$. Suppose $J^N_{\vec{a}(k_1,k_2)}(x) \equiv 0$; then $J^N_{\Sigma\vec{a}(k_1,k_2)}(x) \equiv 0$.*

From Lemmas 3 and 4 we obtain

Theorem 2 ([6]). *Let an integer $N \geq 2$, $1 \leq j \leq 2N-1$, and the elements a_j be such that $0 \leq \deg a_j \leq N$; then $J^N_{\vec{a}}(x) \equiv 0$.*

From Theorems 1 and 2 we have the main theorem:

Theorem 3. *Polynomials $\Bbbk_N[x]$ of degree not greater than N form the homotopical N-Lie algebra with N-linear skew-symmetric bracket (5) for any integer $N \geq 2$.*

Thus, we have generalized the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ of quadratic polynomials $\Bbbk_2[x]$ to the homotopical N-Lie algebra of the Nth degree polynomials $\Bbbk_N[x]$ for arbitrary integer $N \geq 2$. Still, the dimension n of the base $\Bbbk \equiv \Bbbk^1 \ni x$ equals 1. In Section 6, we generalize the concept of the homotopical N-Lie algebra of polynomials $\Bbbk_N[x]$ to the case $x \in \Bbbk^n$, where integer $n \geq 1$ is arbitrary.

4. The case when the monomials x^k have an arbitrary degree $k \in \mathbb{N}$ (the Taylor series) or $k \in \mathbb{Z}$ (the Laurent series) is a natural succession of the preceding research.

Remark 5 ([5]). Let $\vec{a}$ be a system of arbitrary analytic functions; then $J^{3,\infty}_{\vec{a}}(x) \equiv 0$. The proof is by direct calculation of 20 Wronskians.

Conjecture 1. Choose an integer $N \geq 2$ and let $\vec{a}$ be a system of $2N-1$ analytic functions; then the relation $J^{N,\infty}_{\vec{a}}(x) \equiv 0$ is expected to hold.

Remark 6. In fact, the homotopical N-Lie algebra of the polynomials of degree N must not be the unique subalgebra in the homotopical N-Lie algebra of analytic functions. E.g., let $k \in \mathbb{Z}$ and choose an integer $p \geq 0$; then the monomials x^{pk+1} are closed w.r.t. the $(2,1,0)$-Lie bracket, the Wronskian (5) with $N = 2$; see also [8] for mod p-Lie algebras.

We also note that the monomials x^k, $k \geq k_0 > 0$, form the homotopical N-Lie subalgebra $\mathcal{L}_{k_0} = \{\text{span}\langle x^k\rangle, [\cdot,\ldots,\cdot] \mid k \geq k_0 > 0\}$ in the homotopical N-Lie algebra $\mathcal{L}_{\mathbb{N}}$ of the Taylor polynomials. The homotopical N-Lie factoralgebra $\mathcal{L}_{\mathbb{N}}/\mathcal{L}_{k_0}$ is k_0-dimensional for any $k_0 \in \mathbb{N}$.

Conjecture 2. The homotopical N-Lie algebra of monomials x^k, $0 \leq k \leq N$, is expected to be the unique finite-dimensional subalgebra in the homotopical N-Lie algebra of analytic functions.

The preceding results were submitted before January 25, 2002, for publication [6] in the Proceedings of the Young Scientists Conference at the Faculty of Mechanics and Mathematics, Lomonosov MSU, April 8–13, 2002. Recently, Conjecture 1 was proved in [7]. In this section, we sketch the proof for consistency.

Consider the generalized Wronskians $W^{\vec{\imath}} = \partial^{i_1} \wedge \ldots \wedge \partial^{i_N} \in \text{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$. Let a multiindex $\vec{\imath} \in \mathbb{Z}^N_+$ be such that $0 \leq i_1 < \ldots < i_N$. By $\text{Hom}_t(\bigwedge^N \mathcal{A}, \mathcal{A})$ we denote the linear span of the generalized Wronskians $W^{\vec{\imath}}$ such that $|\vec{\imath}| = t$. By definition, put $|W^{\vec{\imath}}| = |\vec{\imath}|$.

Lemma 5 ([7]). *Let $t < N(N-1)/2$, then* $\text{Hom}_t(\bigwedge^N \mathcal{A}, \mathcal{A}) = 0$.

Proof [7]. If $0 \neq \partial^{i_1} \wedge \ldots \wedge \partial^{i_N} \in \text{Hom}_t(\bigwedge^N \mathcal{A}, \mathcal{A})$, then $t = \sum_{j=1}^N i_N \geq 0+1+2+\ldots+(N-1) = N(N-1)/2$. □

Lemma 6 ([8]). *Let $\vec{\imath}$ and $\vec{\jmath}$ be multiindexes: $\vec{\imath} \in \mathbb{Z}^k_+$ and $\vec{\jmath} \in \mathbb{Z}^l_+$; then the relation*

$$W^{\vec{\imath}}[W^{\vec{\jmath}}] \in \text{Hom}_{|\vec{\imath}|+|\vec{\jmath}|}(\bigwedge^{k+l-1} \mathcal{A}, \mathcal{A})$$

holds.

Corollary 2 ([7]). Let k and l be positive integers, then the identity

$$W^{0,1,\ldots,k}[W^{0,1,\ldots,l}] = 0$$

holds.

Proof [7]. First note that $|W^{0,1,\ldots,k}| = k(k+1)/2$ for any k. Thence,

$$W^{0,1,\ldots,k}[W^{0,1,\ldots,l}] \in \mathrm{Hom}_{\frac{k(k+1)+l(l+1)}{2}}(\bigwedge^{k+l+1}\mathcal{A},\mathcal{A}).$$

Nevertheless, $k^2+k+l^2+l < (k+l+1)(k+l)$ for any k and l. Consequently, by Lemma 5,

$$\mathrm{Hom}_{\frac{k^2+k+l^2+l}{2}}(\bigwedge^{k+l+1}\mathcal{A},\mathcal{A}) = 0 \quad \text{and} \quad W^{0,1,\ldots,k}[W^{0,1,\ldots,l}] = 0.$$

This completes the proof. □

Taking into account the definition of $J^{N,\infty}_{\vec{a}}$, we observe that the identity $J^{N,\infty}_{\vec{a}}(x) \equiv 0$ in Conjecture 1 follows from Corollary 2.

Remark 7. We also see that Theorem 3 follows from Theorem 1 in virtue of Corollary 2.

From Corollary 2 we obtain

Theorem 4. *Let k and l be positive integers, then the relation*

$$[\![W^{0,1,\ldots,k}, W^{0,1,\ldots,l}]\!]^{\mathrm{RN}} = 0$$

holds.

5. We see that the proof of Theorem 3 is non-inductive, i.e., a property of the polynomials of degree not greater than $N' < N$ to form the homotopical N'-Lie algebra is not used if $\mathbb{N} \ni N \geq 3$. In this section, we study the scheme that generates the Wronskian $W^{0,1,\ldots,N+1}$ starting from the Wronskians $W^{0,1,\ldots,N-1}$ and $W^{N,N+1}$.

In [1], the notion of the exterior multiplication $\wedge$ in $\mathrm{Hom}(\bigwedge^*\mathcal{A},\mathcal{A})$ is introduced.

Definition 4. Let $\Delta \in \mathrm{Hom}(\bigwedge^k\mathcal{A},\mathcal{A})$ and $\nabla \in \mathrm{Hom}(\bigwedge^l\mathcal{A},\mathcal{A})$ be two operators; by definition, put

$$(\Delta\wedge\nabla)(a_1,\ldots,a_{k+l}) = \sum_{\sigma\in S^k_{k+l}} (-1)^\sigma\,\Delta(a_{\sigma(1)},\ldots,a_{\sigma(k)})\cdot\nabla(a_{\sigma(k+1)},\ldots,a_{\sigma(k+l)})$$

for any $a_1,\ldots,a_{k+l}\in\mathcal{A}$.

We see that the Wronskians $W^{0,1,\ldots,N+1} = \partial^0\wedge\ldots\wedge\partial^{N+1}$ are obtained in the fasion of Definition 4, e.g., $W^{0,1,\ldots,N+1} = W^{0,1,\ldots,N-1}\wedge W^{N,N+1}$.

Definition 5. The structure $\Delta\in\mathrm{Hom}(\bigwedge^N\mathcal{A},\mathcal{A})$ is called a *multi-derivation*, if the Leibnitz rule

$$\Delta(ab,a_2,\ldots,a_N) = a\,\Delta(b,a_2,\ldots,a_N) + \Delta(a,a_2,\ldots,a_N)\,b$$

is valid for any a, b, $a_j\in\mathcal{A}$.

Taking into account Remark 2, we restrict ourselves to the case of even $N = 2N'$.

Proposition 1 ([1])**.** *Let* $\Delta \in \mathrm{Lie}^{(2k,0,0)}(\mathcal{A})$ *and* $\nabla \in \mathrm{Lie}^{(2l,0,0)}(\mathcal{A})$ *be multi-derivations. If* $[\![\Delta, \nabla]\!]^{\mathrm{RN}} = 0$*, then* $\Delta \wedge \nabla \in \mathrm{Lie}^{(2k+2l,0,0)}(\mathcal{A})$*.*

From Corollary 2 we know that the Wronskians $W^{0,1,\ldots,N+1}$ do satisfy the identity (2) for any even N, in particular. Thus, the conclusion of Proposition 1 is valid; still, we stress that the Wronskian determinants are *not* multi-derivations; besides, by a simple argument we observe the discouraging fact that, generally, $[\![W^{0,1,\ldots,N-1}, W^{N,N+1}]\!]^{\mathrm{RN}} \neq 0$. So, the example of the Wronskians demonstrates that there is another mechnism that generated higher order structures by using the exteriour multiplication rather than the reasoning in Proposition 1.

The task to describe all pairs of the generalized Wronskians $W^{\vec{\imath}}, W^{\vec{\jmath}} \in \mathrm{Hom}(\bigwedge^* \mathcal{A}, \mathcal{A})$, such that the identity $[\![W^{\vec{\imath}}, W^{\vec{\jmath}}]\!]^{\mathrm{RN}} = 0$ holds, remains an independent problem.

6. In this section, we generalize the concept of the Wronskian determinants to the multidimensional case of the base $\Bbbk^n$: further on, we consider the kth order jets $J^k(n,1)$ over the bundle $\pi\colon \Bbbk^n \oplus \Bbbk \to \Bbbk^n$, where the base dimension is $n \geq 1$ and the algebra $\mathcal{A}$ is the associative commutative algebra $C^\infty(\Bbbk^n)$ of smooth functions.

In order to construct a natural n-dimensional base generalization of the Wronskians, we pass to the geometrical standpoint and make an experimental observation first.

Remark 8 ([5]). Consider the infinite jets $J^\infty(\pi)$ over the bundle $\pi\colon \Bbbk \oplus \Bbbk \to \Bbbk$. Let $x \in \Bbbk$ be the independent base variable, u be the dependent fiber variable, D_x be the total derivative w.r.t. x, and $u^{(k)} \equiv D_x^k u$ be the coordinates in $J^\infty(\pi)$ for any $k \geq 0$. By $d_{\mathcal{C}}$ we denote the Cartan differential, $d_{\mathcal{C}}\colon C^\infty(J^\infty(\pi)) \to \mathcal{C}\Lambda(J^\infty(\pi))$, that maps $u^{(k)} \mapsto du^{(k)} - D_x u^{(k)}\, dx$. The Wronskian determinants (5) can be interpreted as action of the N-forms $d_{\mathcal{C}} u \wedge \ldots d_{\mathcal{C}} u^{(N-1)} \in \mathcal{C}\Lambda^*(J^{N-1}(\pi)) \subset \mathcal{C}\Lambda^*(J^\infty(\pi))$ upon the evolutionary vector fields $\partial_{a_j} \equiv \sum_{k=0}^{\infty} D_x^k(a_j)\, \partial/\partial u^{(k)}$:

$$[a_1, a_2] = du \wedge d(u')\,(\partial_{a_1}, \partial_{a_2}),$$
$$[a_1, a_2, a_3] = du \wedge d(u') \wedge d(u'')\,(\partial_{a_1}, \partial_{a_2}, \partial_{a_3}), \qquad \text{etc.}$$

for any $a_j \in C^\infty(\Bbbk)$.

Remark 9. Consider the ternary bracket $\square_3 \in \mathrm{Hom}(\bigwedge^3 C^\infty(\mathbb{R}^2), C^\infty(\mathbb{R}^2))$:

$$\square_3(a_1, a_2, a_3) = d_{\mathcal{C}} u \wedge d_{\mathcal{C}} u_x \wedge d_{\mathcal{C}} u_y(\partial_{a_1}, \partial_{a_2}, \partial_{a_3}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ D_x(a_1) & D_x(a_2) & D_x(a_3) \\ D_y(a_1) & D_y(a_2) & D_y(a_3) \end{vmatrix}.$$

For the bracket $\square_3$, the homotopical ternary Jacobi identity $\square_3[\square_3] = 0$ of the form (2) holds. We prove this fact by direct calculations using the Jet software [9].

Proposition 2 ([10])**.** *The dimension of the jets space* $J^k(n,1)$ *vertical part* $J^k(n,1)/\Bbbk^n$ *is*

$$\dim \frac{J^k(n,1)}{\Bbbk^n} = \dim J^k(n,1) - n = \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n+k}{n}.$$

We also note that the dimension $N \equiv \binom{n+k}{n}$ is such that the inequality

$$\dim J^{2k}(n,1) - n - 1 > 2(\dim J^k(n,1) - n - 1)$$

is valid; in what follows, we need to substract the dimension $\dim J^0(n,1) = n+1$ in order to deal with non-trivial multiindexes $\sigma \neq \varnothing$ such that $|\sigma| > 0$ and $u_\sigma \notin T^*J^0(n,1)$.

Choose arbitrary positive integers n and k; then $N \equiv \binom{n+k}{n}$ is the dimension $\dim(J^k(n,1)/\Bbbk^n)$. Let $\mathcal{A} = C^\infty(\Bbbk^n)$ be the algebra of smooth functions $a_j \in \mathcal{A}$, $1 \le j \le N$. Now we define the N-linear skew-symmetric bracket $\square \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$: by definition, put

$$\square(a_1, \ldots, a_N) = \bigwedge_{l=0}^{k} \Big(\bigwedge_{|\sigma|=l} d_C \cdot D_\sigma u \Big) (\ni_{a_1}, \ldots, \ni_{a_N}). \tag{17}$$

Theorem 5. *The N-linear skew-symmetric bracket $\square \in \operatorname{Hom}(\bigwedge^N \mathcal{A}, \mathcal{A})$ defined in (17) satisfies the homotopical N-Lie Jacobi identity*

$$\square[\square] = 0. \tag{2}$$

Proof. In contrast with the reasoning in Section 4, we deal with $D_{\vec{\sigma}} = D_{\sigma_1} \wedge \ldots \wedge D_{\sigma_N}$, where σ_j is a multiindex $(\sharp x^1, \ldots, \sharp x^n) \in \mathbb{Z}^n_+$ for any j, $1 \le j \le N = \binom{n+k}{n}$. By definition, put $|D_{\vec{\sigma}}| = |\vec{\sigma}| = \sum_{j-1}^{N} |\sigma_j|$. We see that $|\square[\square]| = 2|\square|$, c.f. Lemma 6.

Now we note that the non-trivial skew-symmetric $(2N-1)$-linear bracket $\square_{\min} \in \operatorname{Hom}(\bigwedge^{2N-1} \mathcal{A}, \mathcal{A})$ with the minimal norm is

$$\square_{\min} = \square \wedge \Big(\sum_{\bar{\jmath} \in \Lambda^{N-1}\left(J^{2k}(n,1)/J^k(n,1)\right)} \operatorname{const}(\bar{\jmath}) \cdot D_{\sigma_{\bar{\jmath}}} \Big), \tag{18}$$

where $\operatorname{const}(\bar{\jmath}) \in \Bbbk$ are some constant coefficients.

We claim that $|\square_{\min}| > 2|\square|$, and thence $\square[\square] = 0$. Really, consider the r.h.s. in (18) and note that $|\Delta \wedge \nabla| = |\Delta| + |\nabla|$. The set of N different derivatives in the first wedge factor $\square$ admits the canonical splitting:

$$\square = D_{\vec{\tau}} = \mathbf{1} \wedge \underbrace{D_{\tau_2} \wedge \ldots \wedge D_{\tau_N}}_{N-1 \text{ factors}},$$

where $\vec{\tau}$ contains all multiindexes in $J^k(n,1)$, and those underbraced derivatives are in bijective correspondence with $N-1$ different derivatives in any summand in the second wedge factor (there is the correspondence due to the equal numbers of elements). Still,

$$1 \le |D_{\tau_i}| = |\tau_i| \le k < k+1 \le |\sigma_j| = |D_{\sigma_j}| \le 2k \qquad \forall i \neq 1, \quad \forall j.$$

Indeed, if a multiindex σ_j is such that $u_{\sigma_j} \in T^*(J^{2k}(n,1)/J^k(n,1))$, then σ_j is *longer* than any multiindex τ_i such that $u_{\tau_i} \in T^*J^k(n,1)$. Consequently, the norm $|\cdot|$ of the second wedge factor in the r.h.s. of (18) is strictly greater than $|\square|$, and thus $\square[\square]$ is trivial. This completes the proof. $\square$

In addition, we give an example of the binary homotopical 3-Lie algebra of polynomials:

Example 2. The polynomials $\mathrm{span}_{\Bbbk}\langle 1, x, y, xy\rangle \subset \Bbbk_2[x, y]$ endowed with the ternary bracket $\mathbf{1} \wedge D_x \wedge D_y$ form a homotopical 3-Lie algebra.

The commutation relations of this algebra are

$$[1, x, y] = 1, \qquad [1, x, xy] = x, \qquad [1, y, xy] = -y, \quad \text{and} \quad [x, y, xy] = -xy,$$

and we see that the case under consideration is somehow different from the scheme of Example 1.

7. The nontrivial one-dimensional central extension of the holomorphic vector fields $a(x)\,d/dx$ Lie algebra Vect^1 on the circumpherence $x^2 = 1$ of unit raduis is well known [11] to be the Virasoro algebra. In the basis $\langle d_n = -x^{n+1}\,d/dx,\ c\rangle$, the generating relations are

$$\begin{aligned} [d_m, d_n] &= (m-n)d_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12}\cdot c, \\ [d_m, c] &= 0, \qquad \forall m \in \mathbb{Z}. \end{aligned}$$

Now we discuss a possibility to construct a nontrivial one-dimensional central extension of the homotopical 3-Lie algebra. Choose $d_n \equiv x^{n+3/2}\,d/dx$ for the basis and write down the decomposition of the ternary bracket $[\,\cdot\,,\,\cdot\,,\,\cdot\,]$ w.r.t. the extended basis:

$$[d_k, d_l, d_m] = \pi(k, l, m)d_{k+l+m} + f(k, l, m)\cdot c,$$

where $\pi(k, l, m) \equiv (k-l)(l-m)(m-k)$ and

$$f(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = (-1)^\sigma f(a_1, a_2, a_3),$$

due to the skew-symmetry of the bracket for any permutation $\sigma \in S_3$ and any set of arguments $\vec{a}$, $\sharp a = 5$. The condition $c \in \mathcal{Z}$ for c to belong to the center of the extended algebra is naturally understood as

$$[d_i, d_j, c] \equiv 0, \qquad \forall i, j \in \mathbb{Z}.$$

For any $\vec{a}$, the Jacobi identity defined in (1) implies the functional equation [5]

$$\sum_{\sigma\in S_5^2} (-1)^\sigma \pi(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})\cdot f(a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)}, a_{\sigma(4)}, a_{\sigma(5)}) = 0 \tag{19}$$

for the function f. Nontrivial solutions to Eq. (19) are unknown.

Final remarks.

1. To each Bäcklund transformation between equations of the f-Gordon type $u_{xy} = f(u)$, where $d^2f/du^2 = \mathrm{const}\cdot f$, Shadwick assigned [12] the one-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ in smooth vector fields $a(x)\,d/dx$. Given a representation of the homotopical N-Lie generalization of $\mathfrak{sl}_2(\mathbb{R})$ in monomials x^k, $0 \le k \le N$, can we generalize Bäcklund transformations as relations between N PDE? We assume that these relations are non-trivial, i.e., they do not split into pairs of initial Bäcklund transformations (in general, those binary initial Bäcklund transformations may even not exist).

If such N-ary Bäcklund transformations existed, the question of their deformations by using the Frölicher-Nijenhuis bracket [13, 14] would naturally arise.

2. Suppose $\vec{a}$ is a system of $\sharp\vec{a} = 2N - 1$ analytic functions a_j and $\vec{a}$ contains a standard subsystem of monomials (15). Consider the N-ary bracket (5) satisfying the homotopical N-Lie Jacobi identity (1), as $N \to \infty$. In this case, it is interesting to study whether bracket (5) has an integral limit or any relation to super-power series. Quantization of bracket (5) is a matter of further study, also.

3. In fact, generalizations of the PDE conformal symmetry algebras are provided by Theorems 3, 4, and 5. So, one can try to describe standard objects in CFT in the general homotopical N-Lie case now; e.g., non-trivial one-dimensional central extensions of the homotopical 3-Lie algebra of smooth functions depend on solutions of Eq. (19).

4. Treating $a_j \in \vec{a}$ as state functions of $N+1$ (or $2N-1$) particles, we see that bracket (5) (resp., the Jacobi identity (1)) describes a collective effect, their interaction. Intuitively, in descriptions of the condensed matter models in statistical physics, the role of large $N \gg 2$ is supposed to grow as the temperature $\theta \to 0$. Also, the homotopical N-Lie dynamics with different Ns may describe different phase states and phase state transformations, e.g., a process starting from binary interactions and passing through near ordered matter to far ordered states. Study of the $\mathfrak{sl}_2(\Bbbk)$-based models (or $a(x)\,d/dx$-based models, $a \in C^\infty(\Bbbk)$) with respect to their homotopical N-Lie generalizations would be of great interest.

Acknowledgement. The author thanks A. V. Ovchinnikov, A. M. Verbovetsky, and Prof. E. B. Vinberg for discussion and remarks, and also Prof. I. S. Krasil'shchik for pointing out the paper [7]. The work was partially supported by the INTAS Grant YS 2001/2-33.

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