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On one-parametric family of Bäcklund autotransformations for the Liouville equation

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Abstract. Scaling symmetry of the Liouville equation is proved to provide a one-parametric family of one-dimensional non-abelian coverings determining a Bäcklund autotransformation. Deformation of the structural element is shown to be equal to the Frölicher-Nijenhuis bracket of the symmetry lifting and the structural element of the coverings. An intriguing by-formula in total derivatives is obtained.
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This paper illustrates new cohomological concepts [1, 2] in the theory of Bäcklund transformations between PDE. Notation and all definitions follow [1, 3]. First, let us introduce the notion of Bäcklund (auto)transformation in terms of coverings over PDE.

1. Let $\mathcal{E}_i \subset J^{k_i}(\pi_i)$, $i = 1, 2$, be two differential equations and $\tau_i \colon \tilde{\mathcal{E}} \to \mathcal{E}_i$ be coverings with the same total space $\tilde{\mathcal{E}}$. Then the diagram

$$\mathcal{E}_1^\infty \xleftarrow{\tau_1} \tilde{\mathcal{E}} \xrightarrow{\tau_2} \mathcal{E}_2 \tag{1}$$

is called a *Bäcklund transformation* $\mathcal{B}(\tilde{\mathcal{E}}, \tau_i, \mathcal{E}_i)$ between the equations $\mathcal{E}_i$. Diagram (1) is called a *Bäcklund autotransformation* if $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty = \mathcal{E}^\infty$.

Remark 1. Let $\tau_j \colon \tilde{\mathcal{E}}_j \to \mathcal{E}_j$, $j = 1, 2$, be two coverings and $\mu \colon \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_2$ be a diffeomorphism that maps the Cartan distribution $\mathcal{C}_{\tau_1}\mathrm{D}(\tilde{\mathcal{E}}_1)$ into $\mathcal{C}_{\tau_2}\mathrm{D}(\tilde{\mathcal{E}}_2)$. Then the diagram $\mathcal{B}(\tilde{\mathcal{E}}_j, \tau_1, \tau_2 \circ \mu, \mathcal{E}_j)$ is also a Bäcklund transformation between the equations $\mathcal{E}_j$.

Remark 2. Let $\tau \colon \tilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a covering and $\mu \colon \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ be a nontrivial diffeomorphism of manifolds preserving Cartan distribution, e.g., a discrete symmetry that cannot be restricted to $\mathcal{E}^\infty$. Then the diagram

$$\begin{array}{ccc} \tilde{\mathcal{E}} & \xrightarrow{\mu} & \tilde{\mathcal{E}}, \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & \mathcal{E}^\infty & \end{array} \tag{2}$$

is also Bäcklund autotransformation for $\mathcal{E}$. In the sequel, this construction is used for generating Bäcklund autotransformations for the Liouville equation.

2. The hyperbolic Liouville equation $\mathcal{E}$ is

$$\mathcal{E} = \{F \equiv u_{xy} - \exp(2u) = 0\}. \tag{3}$$

Consider the covering structure in $\tau_t \colon \tilde{\mathcal{E}}_t \to \mathcal{E}^\infty$ provided by the extended total derivatives

$$\tilde{D}_x = \bar{D}_x + \tilde{u}_x\,\partial/\partial\tilde{u}, \quad \tilde{D}_y = \bar{D}_y + \tilde{u}_y\,\partial/\partial\tilde{u}, \qquad [\tilde{D}_x, \tilde{D}_y] = 0, \tag{4}$$

if the partial derivatives of the nonlocal variable $\tilde{u}$ w.r.t. x and y are

$$\tilde{u}_x = u_x + \exp(-t)\cdot\exp(\tilde{u}+u), \qquad \tilde{u}_y = -u_y + 2\exp(t)\cdot\sinh(\tilde{u}-u). \tag{5}$$

Let the diffeomorphism μ be the swapping $u \leftrightarrow \tilde{u}$ of the fiber variable u and the nonlocal variable $\tilde{u}$ combined with $x \mapsto -x$ and $y \mapsto -y$. Then diagram (2) determines Bäcklund autotransformation $\mathcal{B}(\tilde{\mathcal{E}}_t, \tau_t, \tau_t \circ \mu, \mathcal{E})$ for Eq. (3), and for the Liouville equation, the equations $\tilde{\mathcal{E}}_t$ of Bäcklund autotransformation [4] are

$$(\tilde{u}-u)_x = \exp(-t)\cdot\exp(\tilde{u}+u), \tag{6}$$

$$(\tilde{u}+u)_y = 2\exp(t)\cdot\sinh(\tilde{u}-u). \tag{7}$$

Proposition 1. *Let $t_1 \neq t_2$. Then the coverings τ_{t_1} and τ_{t_2} are non-equivalent.*

Proof. Consider the Whitney sum $\tau_{t_1} \oplus \tau_{t_2}$ of the coverings τ_{t_1} and τ_{t_2} with nonlocal variables $\tilde{u}$ and $\tilde{u'}$ respectively. These coverings are equivalent if their Whitney sum is reducible. Following [1], we verify this condition in local coordinates and prove that the system

$$\left(\bar{D}_x + (u_x + \exp(\tilde{u}+u-t_1))\frac{\partial}{\partial\tilde{u}} + \left(u_x + \exp(\tilde{u'}+u-t_2)\right)\frac{\partial}{\partial\tilde{u'}}\right)\phi = 0, \tag{8}$$

$$\left(\bar{D}_y + (-u_y + 2\exp(t_1)\sinh(\tilde{u}-u))\frac{\partial}{\partial\tilde{u}} + \left(-u_y + 2\exp(t_2)\sinh(\tilde{u'}-u)\right)\frac{\partial}{\partial\tilde{u'}}\right)\phi = 0 \tag{9}$$

has only constant solution $\phi = \text{const}$ for $t_1 \neq t_2$. Put $u_k \equiv \partial^k u/\partial x^k$ and $u_{\overline{k}} \equiv \partial^k u/\partial y^k$ for any $k \in \mathbb{N}$. The characteristics of Eq. (8) are

$$\frac{d\tilde{u}}{u_x + \exp(\tilde{u}+u-t_1)} = \frac{d\tilde{u'}}{u_x + \exp(\tilde{u'}+u-t_2)} = \frac{du}{u_1} = \frac{du_k}{u_{k+1}} = dq_1,$$

$$\frac{d\tilde{u}}{-u_y + 2\exp(t_1)\sinh(\tilde{u}-u)} = \frac{d\tilde{u'}}{-u_y + 2\exp(t_2)\sinh(\tilde{u'}-u)} = \frac{du}{u_1} = \frac{du_{\overline{k}}}{u_{\overline{k+1}}} = dq_2,$$

$k \in \mathbb{N}$. We have

$$dq_1 = \frac{d(\tilde{u}-u)}{\exp(\tilde{u}+u-t_1)} = \frac{d(\tilde{u'}-u)}{\exp(\tilde{u'}+u-t_2)},$$

whence

$$\frac{d(\tilde{u}-\tilde{u'})}{\exp(\tilde{u}-t_1)-\exp(\tilde{u'}-t_2)} = 0.$$

Consequently, $\tilde{u'} - \tilde{u} = \delta = \text{const}$ and δ depends at most on t_1 and t_2. Then we have $\tilde{u}_x = \tilde{u'}_x$ and $\tilde{u}_y = \tilde{u'}_y$ as equations for δ. The first equation implies $\delta = t_1 - t_2$ and the second one gives

$$\exp(\delta)\sinh(\tilde{u'}+\delta-u) = \sinh(\tilde{u'}-u).$$

Using hyperbolic-trigonometry formulas, we obtain

$$\exp(\delta)\,\sinh(\tilde{u'}-u)\,\cosh(\delta) + \exp(\delta)\,\cosh(\tilde{u'}-u)\,\sinh(\delta) = \sinh(\tilde{u'}-u).$$

Note that $\exp(\delta)\,\cosh(\delta) - 1 = \exp(\delta)\sinh(\delta)$. Finally, we emphasize that the equation $\sinh(\tilde{u'}-u) = \cosh(\tilde{u'}-u)$ has no roots in $\mathbb{C}$. Therefore, system (8)-(9) has only constant solution $\phi = \text{const}$ for $t_1 \neq t_2$. □

Proposition 1 means that the parameter t cannot be eliminated by a coordinate transformation if the later does not depend on t itself.

3. Consider the scaling symmetry of base coordinates

$$X^0 = -x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}.$$

It corresponds to the generating function $\varphi = xu_x - yu_y$. The scaling symmetry can be extended to the entire $\mathcal{E}^\infty$:

$$\hat{X} = -x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + \sum_{k\geq 1} ku_k\,\frac{\partial}{\partial u_k} - \sum_{k\geq 1} ku_{\overline{k}}\,\frac{\partial}{\partial u_{\overline{k}}}. \tag{10}$$

Remark 3. The diffeomorphisms $A_t = \exp(t\hat{X})$ form an abelian group. Really, we see that $x(t) = \exp(-t)\,x(0)$, $y(t) = \exp(t)\,y(0)$, $u(t) = u(0)$, $u_k(t) = \exp(kt)\,u_k(0)$, $u_{\overline{k}}(t) = \exp(-kt)\,u_{\overline{k}}(0)$, $k \geq 1$. Obviously, $A_0 = \mathrm{id}$ and $A_{t_1} \circ A_{t_2} = A_{t_2} \circ A_{t_1} = A_{t_1+t_2}$.

Proposition 2. *The symmetry $\hat{X}$ cannot be extended to a symmetry of the covering equation $\tilde{\mathcal{E}}_t$.*

Proof. Assume the converse. By $\tilde{\mathfrak{z}}_\varphi$ we denote the evolutionary vector field $\sum_\sigma \tilde{D}_\sigma(\varphi) \cdot \partial/\partial u_\sigma$ on $\tilde{\mathcal{E}}_t$, $\varphi \in C^\infty(\tilde{\mathcal{E}}_t)$, and $\tilde{\ell}_F(\varphi) = \tilde{\mathfrak{z}}_\varphi(F)$. Suppose there is a smooth function $a \in C^\infty(\tilde{\mathcal{E}}_t)$ such that the system

$$\tilde{\ell}_F(\varphi) = 0, \qquad \tilde{D}_{x^i}(a) = \tilde{\mathfrak{z}}_{\varphi,a}(\tilde{u}_{x^i}) \equiv (\tilde{\mathfrak{z}}_\varphi + a\,\partial/\partial\tilde{u})\,(\tilde{u}_{x^i}), \qquad x^1 \equiv x, \quad x^2 \equiv y, \tag{11}$$

holds. This means that the field $\tilde{\mathfrak{z}}_{\varphi,a}$ is a local symmetry of the covering equation $\tilde{\mathcal{E}}_t$ and $\hat{X}$ is constructively extended onto $\tilde{\mathcal{E}}_t$. Nevertheless, system (11) is not compatible since

$$\tilde{D}_x \circ \tilde{D}_y(a) \neq \tilde{D}_y \circ \tilde{D}_x(a).$$

Really, $\tilde{D}_x \circ \tilde{D}_y(a) - \tilde{D}_y \circ \tilde{D}_x(a)$ does not depend on a at all and equals

$$\begin{gathered} xu_x^2 \exp(t + u - \tilde{u}) + u_x y u_y \exp(t + \tilde{u} - u) - xu_x \exp(2\tilde{u}) - u_x y u_y \exp(t + u - \tilde{u}) \\ -2yu_y \exp(2t + \tilde{u} + u) + 2xu_x \exp(2t + \tilde{u} + u) - xu_x^2 \exp(t + \tilde{u} - u) + x\exp(2u)u_x \\ -y\exp(2u)u_y + 2\exp(t)xu_x^2 + yu_y \exp(2\tilde{u}) - 2\exp(t)yu_yu_x \neq 0. \end{gathered}$$

This contradiction concludes the proof. □

Thus, the scaling symmetry $\hat{X}$ is a τ_t-shadow only and $\hat{X}$ provides a family of covering equations $\tilde{\mathcal{E}}_t$ over $\mathcal{E}^\infty$, $\tilde{\mathcal{E}}_t$ parametrized by $t \in \mathbb{R}$.

4. In local coordinates, the structural element U_t of the covering diffiety $\tilde{\mathcal{E}}_t$ is

$$U_t = \sum_\sigma d_{\mathcal{C}}(u_\sigma) \otimes \frac{\partial}{\partial u_\sigma} + (d\tilde{u} - (u_x + \exp(\tilde{u} + u - t))\,dx + {} \\ {} + (u_y - 2\exp(t)\sinh(\tilde{u} - u))\,dy) \otimes \frac{\partial}{\partial \tilde{u}}, \tag{12}$$

where d_C is the Cartan differential, $d_C u_\sigma = du_\sigma - \sum_i u_{\sigma+1_i}\,dx^i$.

By $[\![\cdot,\cdot]\!]^{\mathrm{FN}}$ we denote the *Frölicher-Nijenhuis bracket* [3, 2]: $[\![\Omega,\Theta]\!]^{\mathrm{FN}}(f) = \mathrm{L}_\Omega(\Theta(f))$ $-(-1)^{\mu\nu} \cdot \mathrm{L}_\Theta(\Omega(f))$, where $\Omega, \Theta \in \mathrm{D}(\Lambda^*(\mathcal{E}))$, $f \in C^\infty(\mathcal{E})$, degrees μ, ν are $\mu = \deg\Omega$,

$\nu = \deg\Theta$; $\mathrm{L}_\Omega = [i_\Omega, \mathrm{d}]\colon \Lambda^k(\mathcal{E}) \to \Lambda^{k+\deg\Omega}(\mathcal{E})$ is the Lie derivative, and $i_\Omega\colon \Lambda^k(\mathcal{E}) \to \Lambda^{k+\deg\Omega-1}(\mathcal{E})$ is the inner product.

Theorem ([2]). *Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering and $A_t\colon \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ be a smooth family of diffeomorphisms such that $A_0 = \mathrm{id}$ and $\tau_t = \tau \circ A_t\colon \tilde{\mathcal{E}} \to \mathcal{E}$ is a covering for any $t \in \mathbb{R}$. Then the structural element U_{τ_t} evolves by*

$$\frac{dU_{\tau_t}}{dt} = [\![\hat{X}_t, U_{\tau_t}]\!]^{\mathrm{FN}}, \tag{13}$$

where $\hat{X}_t$ is a τ_t-shadow for any $t \in \mathbb{R}$.

If the covering τ_t corresponds to Bäcklund autotransformation (6)-(7) for the Liouville equation (3), then we have

$$\frac{dU_t}{dt} = (\exp(\tilde{u} + u - t)\, dx - 2\exp(t)\sinh(\tilde{u} - u)\, dy) \otimes \frac{\partial}{\partial \tilde{u}}. \tag{14}$$

Now we claim that the scaling symmetry $\hat{X}$ is the τ_t-shadow such that Eq. (13) holds, where U_{τ_t} is the structural element U_t (12) of the covering τ_t defined in (4) and U_t evolves according to (14). We need Lemmas 1-4 to prove it.

We say that i is the degree of Ω, if $\Omega \in \mathrm{D}(\Lambda^i(\tilde{\mathcal{E}}))$. If $\tilde{\mathcal{E}}$ is a finite-dimensional manifold, we have an isomorphism $\mathrm{D}(\Lambda^*(\tilde{\mathcal{E}})) \simeq \Lambda^*(\tilde{\mathcal{E}}) \otimes \mathrm{D}(\tilde{\mathcal{E}})$ and thus any derivation $\Omega \in \mathrm{D}(\Lambda^*(\tilde{\mathcal{E}}))$ is representable as a finite sum, the summands are $\Omega = \omega \otimes X$, where $\omega \in \Lambda^*(\tilde{\mathcal{E}})$ and $X \in \mathrm{D}(\tilde{\mathcal{E}})$. For such elements, the Frölicher-Nijenhuis bracket is

$$\begin{split}[\![\omega \otimes X, \theta \otimes Y]\!]^{\mathrm{FN}} = \omega \wedge \theta \otimes [X, Y] + \omega \wedge \mathrm{L}_X(\theta) \otimes (Y) + (-1)^i\, d\omega \wedge (X \lrcorner\, \theta) \otimes Y -\\ - (-1)^{ij}\, \theta \wedge \mathrm{L}_Y(\omega) \otimes X - (-1)^{(i+1)j}\, d\theta \wedge (Y \lrcorner\, \omega) \otimes X,\end{split} \tag{15}$$

where X, $Y \in \mathrm{D}(\tilde{\mathcal{E}})$, $\omega \in \Lambda^i(\tilde{\mathcal{E}})$ and $\theta \in \Lambda^j(\tilde{\mathcal{E}})$.

Lemma 1. *$[\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, d\tilde{u} = (dU_t/dt) \lrcorner\, d\tilde{u}$.*

Lemma 2. *$[\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, dx = [\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, dy = [\![\hat{X}, U_t]\!]^{\mathrm{FN}} \lrcorner\, du = 0$.*

Proof. The proof of Lemmas 1 and 2 is staightforward by multiple using of (15) and thus omitted. □

However, the calculation of coefficients of $[\![\hat{X}, U_t]\!]^{\mathrm{FN}}$ at $\partial/\partial u_k$ or $\partial/\partial u_{\bar{k}}$, $k \geq 1$, is not so transparent.

Remark 4. Let $u(x)$ and $f(u)$ be smooth functions, D_x be the total derivative w.r.t. x; take an integer $n > 0$ and a positive integer $l \leq n - 1$. Then

$$D_x \left(\frac{\partial}{\partial u_l} D_x^{n-1}(f(u)) \right) = \frac{\partial}{\partial u_l} D_x^n(f(u)) - \frac{\partial}{\partial u_{l-1}} D_x^{n-1}(f(u)). \tag{16}$$

Corollary 1. Moreover,

$$(n+1)u_{n+1} \frac{\partial}{\partial u_{n+1}} D_x^{n+1}(f(u)) = (n+1)u_{n+1} \frac{\partial}{\partial u_n} D_x^n(f(u)) = (n+1)u_{n+1} \cdot f'(u). \tag{17}$$

Lemma 3. *Let $u(x)$ and $f(u)$ be smooth functions, D_x be the total derivative w.r.t. x. Denote $u_k \equiv D_x^k(u(x))$, $k \geq 0$, $u_0 \equiv u$. Then for any integer $n \geq 0$ the relation*

$$n \cdot D_x^n(f(u)) = \sum_{m=1}^{n} m\, u_m \frac{\partial}{\partial u_m} D_x^n(f(u)) \tag{18}$$

holds.

Proof. We prove (18) by induction on n. For $n = 0$ relation (18) is valid. For $n \geq 0$ one has

$$(n+1)\, D_x^{n+1}(f(u)) = D_x(n\, D_x^n(f(u)) + D_x^n(f(u))) =$$

by the inductive assumption,

$$= D_x \left(\sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_m} D_x^n(f(u)) + D_x^n(f(u)) \right) =$$

by the Leibnitz rule,

$$= \sum_{m=1}^{n} m u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) + \sum_{m=1}^{n} m u_m\, D_x \frac{\partial}{\partial u_m} D_x^n(f(u)) + D_x\, D_x^n(f(u)) =$$

by (16) applied to the second sum,

$$= \sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + \sum_{m=1}^{n} m u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) -$$
$$- \sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_{m-1}} D_x^n(f(u)) + D_x D_x^n(f(u)) =$$

by the definition of D_x and the subscript shift in the latter sum,

$$= \sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + \sum_{m=0}^{n} (m+1) u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) -$$
$$- \sum_{m=0}^{n-1} (m+1) u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) =$$

since almost all summands in the latter two sums coincide,

$$= \sum_{m=1}^{n} m u_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + (n+1) u_{n+1} \frac{\partial}{\partial u_n} D_x^n(f(u)) =$$

by (17),

$$= \sum_{m=1}^{n} mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + (n+1)u_{n+1} \frac{\partial}{\partial u_{n+1}} D_x^{n+1}(f(u)) =$$
$$= \sum_{m=1}^{n+1} mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)). \qquad \text{Q.E.D.}$$

□

Lemma 4. $[\![\hat{X}, U_t]\!]^{\text{FN}} \lrcorner\, du_k = [\![\hat{X}, U_t]\!]^{\text{FN}} \lrcorner\, du_{\overline{k}} = 0,\ k \geq 1.$

Proof. Let k be a positive integer. Taking (15) into account, consider the 1-form

$$[\![\hat{X}, U_t]\!]^{\text{FN}} \lrcorner\, du_k = \left((k-1)\bar{D}_y u_k - \sum_{l=1}^{k-1} l u_l \frac{\partial}{\partial u_l} \bar{D}_x^{k-1}(\exp(2u)) \right) \cdot dy;$$

we also see that all coefficients of dx, du, du_l, $du_{\bar{l}}$ vanish for all $l \geq 1$. Finally, note that $\bar{D}_y u_k = \bar{D}_x^{k-1}(\exp(2u))$. By Lemma 3, the coefficient of dy is trivial. Arguing as above, we see that $[\![\hat{X}, U_t]\!]^{\text{FN}} \lrcorner\, du_{\overline{k}} = 0$. This completes the proof. □

Theorem 1. *The τ_t-shadow (10) satisfies the relation*

$$[\![\hat{X}, U_t]\!]^{\text{FN}} = (\exp(\tilde{u} + u - t)\, dx - 2\exp(t)\sinh(\tilde{u} - u)\, dy) \otimes \frac{\partial}{\partial \tilde{u}},$$

i.e., the group of diffeomorfisms $A_t = \exp(t\hat{X})$ provides smooth one-parametric family (5) of non-equivalent one-dimensional non-abelian coverings over Liouville's equation (3). These coverings correspond to Bäcklund autotransformations for the Liouville equation. Bäcklund transformations are given by diagram (2). Infinitesimal part of the structural element is given by (14).

Final remark. The reader will have no difficulty in showing that quite analogous theorems are valid for Bäcklund transformations between Liouville's equation (3) and the wave equation $v_{xy} = 0$:

$$(v - u)_x = \exp(-t)\exp(u + v), \quad (v + u)_y = -\exp(t)\exp(u - v), \qquad t \in \mathbb{R},$$

and Bäcklund transformations between the Liouville equation and the scal$^+$-Liouville equation $\Upsilon_{xy} = \exp(-2\Upsilon)$:

$$(\Upsilon - u)_x = 2\exp(-t)\cosh(\Upsilon + u), \quad (\Upsilon + u)_y = -\exp(t)\exp(u - \Upsilon), \qquad t \in \mathbb{R}.$$

Scaling symmetry (10) is the required τ_t-shadow in both cases, and Lemma 3 is an efficient tool to prove these theorems.

Problems of direct integration in nonlocal variables of these Bäcklund (auto)transformations and their permutability will be the object of another paper.

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