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ABSTRACT. An example of a (3+1)-dimensional integrable system is considered, infinite series of divergent forms are described. Classical symmetries for this system and self-similar exact solutions are found.

1. INTRODUCTION

A three-dimensional hydrodynamic model of the formation of sedimentary basins is considered. This model is described by quasi-steady equations of motion of incompressible fluid and by evolutionary equations of the density transfer:

$$\begin{aligned} &\nabla p = \mu \triangle \vec{v} + g\rho\delta_{i3}, \\ &\operatorname{div} \vec{v} = 0, \\ &\rho_t + \operatorname{div}(\rho\vec{v}) = 0, \end{aligned} \tag{1}$$

where $\rho_t = \partial\rho/\partial t$.

In Section 2 of the paper theoretical background is introduced. In Section 3 symmetries of system (1) are described. In Section 5 conservation laws are described and integrability of system (1) is shown. In the fifth section some exact self-similar solutions are obtained.

2. THEORETICAL BACKGROUND

Consider the space $J^k(n,m)$ with the coordinates

$$x_1, \dots, x_n, u^1, \dots, u^m, \dots, u^1_\sigma, \dots, u^m_\sigma, \dots,$$

where $\sigma = (i_1, \dots, i_n)$ and $|\sigma| \le k$. This space is called the *space of jets* of order k. Then an equation $\mathcal{E}$ can be understood as a subset in $J^k(n,m)$ defined by the relations

$$F^1(x,u,\dots,u_\sigma,\dots) = 0, \;\dots,\; F^r(x,u,\dots,u_\sigma,\dots) = 0. \tag{2}$$

If all F^i are smooth functions and the system $F^1, \dots, F^r$ is of maximal rank, then this subset is a smooth submanifold of codimension r. Thus, *differential equations are smooth submanifolds in the jet spaces.*

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Let $f = (f^1(x), \dots, f^m(x))$ be a smooth vector function. Then f determines an n-dimensional submanifold in $J^k(n,m)$ defined by

$$u^j_\sigma = \frac{\partial^{|\sigma|} f^j}{\partial x^\sigma}, \quad j = 1, \dots, m, \quad |\sigma| \le k,$$

(here and below we formally set $u_{(0,\dots,0)} = u$). This submanifold is called the *k-jet of f* and is denoted by $j_k(f)$. It easily seen that f satisfies equation (2) if and only if $j_k(f)$ lies in the corresponding submanifold in $J^k(n,m)$. In other words, solutions of $\mathcal{E}$ are identified with jets lying in the equation manifold. Hence, the following definition is justified.

Definition 1. Let $\mathcal{E} \subset J^k(n,m)$ be a differential equation. A *symmetry* of $\mathcal{E}$ is a diffeomorphism $\varphi\colon J^k(n,m) \to J^k(n,m)$ such that

(i) $\varphi(\mathcal{E}) = \mathcal{E}$ and
(ii) $\varphi(j_k(f))$ is of the form $j_k(f_\varphi)$ for every section f.

A transformation is called a *Lie transformation*, if it satisfies condition (ii).

Definition 2. A vector field X on the manifold $J^k(n,m)$ is called a *Lie field*, if the displacements along its trajectories are the Lie transformations.

Definition 3. A Lie field is called a *classical infinitesimal symmetry* of the equation $\mathcal{E} \subset J^k(n,m)$, if it is tangent to $\mathcal{E}$.

We define the operators

$$D^{[k]}_\alpha = \frac{\partial}{\partial x_\alpha} + \sum_{j=1}^{m} \sum_{|\sigma| \le k} u^j_{\sigma+1_\alpha} \frac{\partial}{\partial u^j_\sigma}. \tag{3}$$

Then one can easily see that the system

$$F = 0, \; D^{[k]}_1(F) = 0, \; \dots, \; D^{[k]}_n(F) = 0 \tag{4}$$

and initial equation (2) have the same solutions. If $\mathcal{E} \subset J^k(n,m)$ is the submanifold corresponding to (2), then the submanifold (probably, with singularities) in the jet space $J^{k+1}(n,m)$ corresponding to (4) is denoted by $\mathcal{E}^1$ and called the *first prolongation* of $\mathcal{E}$.

Now, we define by induction $\mathcal{E}^{l+1} = (\mathcal{E}^l)^1$ and see that the set $\mathcal{E}^l$ lies in $J^{k+l}(n,m)$. We call $\mathcal{E}^l$ the *l-th prolongation* of $\mathcal{E}$.

Let $J^\infty(n,m)$ be a space with coordinates

$$x_1, \dots, x_n, u^1, \dots, u^m, \dots, u^1_\sigma, \dots, u^m_\sigma, \dots,$$

where $|\sigma|$ is unlimited. This space is called the *space of infinite jets* and the whole series of prolongations of the equation $\mathcal{E}$ determines a submanifold $\mathcal{E}^\infty$ in $J^\infty(n,m)$ which is called the *infinite prolongation* of the equation $\mathcal{E}$. The infinite prolongations are expressed by relations

$$D_\sigma(F) = 0, \quad |\sigma| \ge 0,$$

where $D_\sigma = D_1^{i_1} \circ \dots \circ D_n^{i_n}$ and

$$D_\alpha = \frac{\partial}{\partial x_\alpha} + \sum_{j,\sigma} u^j_{\sigma+1_\alpha} \frac{\partial}{\partial u^j_\sigma} \tag{5}$$

are infinite counterparts of (3) and are called *total derivatives*.

Expressions of the form (5) can be considered as vector fields on $J^\infty(n,m)$. Thus, at any point $\theta \in J^\infty(n,m)$ an n-dimensional plane $\mathcal{C}_\theta$ arises spanned by vectors of these fields. It means that there exists a distribution $\mathcal{C}\colon \theta \mapsto \mathcal{C}_\theta$ on $J^\infty(n,m)$. We call it the *Cartan distribution*. Its importance to the theory of differential equations is explained by the following

Theorem 1 (see [1]). *If $\mathcal{E}^\infty \subset J^\infty(n,m)$ is the infinite prolongation, then $\mathcal{C}_\theta$ is tangent to $\mathcal{E}^\infty$ at any point $\theta \in \mathcal{E}^\infty$.*

An n-dimensional submanifold in $\mathcal{E}^\infty$ nondegenerately projecting to the space of independent variables is a solution of $\mathcal{E}$ if and only if it is a maximal integrable manifold of the Cartan distribution.

Hence, $\mathcal{E}^\infty$ carries the structure which completely determines solutions of $\mathcal{E}$ and this structure is the Cartan distribution.

Consider the sets

$$\mathcal{C}D(\mathcal{E}^\infty) = \{X \in D(\mathcal{E}^\infty) \mid X(\omega) = 0, \forall \omega \in \mathcal{C}\Lambda^1(\mathcal{E}^\infty)\},$$
$$D_\mathcal{C}(\mathcal{E}^\infty) = \{X \in D(\mathcal{E}^\infty) \mid [X, \mathcal{C}D(\mathcal{E}^\infty)] \subset \mathcal{C}D(\mathcal{E}^\infty)\}.$$

Note that the set $D_\mathcal{C}(\mathcal{E}^\infty)$ is a subalgebra in the Lie algebra of vector fields on $\mathcal{E}^\infty$ and $\mathcal{C}D(\mathcal{E}^\infty)$ is an ideal in $D_\mathcal{C}(\mathcal{E}^\infty)$.

The Lie $\mathbb{R}$-algebra

$$\operatorname{sym}\mathcal{E} = D_\mathcal{C}(\mathcal{E}^\infty)/\mathcal{C}D(\mathcal{E}^\infty)$$

is the Lie algebra of symmetries of the Cartan distribution on $\mathcal{E}^\infty$.

Definition 4. Elements of the Lie algebra $\operatorname{sym}\mathcal{E}$ are called higher (infinitesimal) symmetries of the equation $\mathcal{E}$.

Theorem 2 (see [1]). *Let $\mathcal{E} \subset J^k(n,m)$ be an equation given by the system $\{F_\alpha = 0\}$ and the functions F_α be chosen in such a way that the graph of $F = (\dots, F_\alpha, \dots)$ intersects the space of independent variables transversally, then Lie algebra $\operatorname{sym}\mathcal{E}$ is isomorphic to the Lie algebra of solutions of the equations*

$$(6) \qquad \ell^{\mathcal{E}}_{F_\alpha}(\varphi) = 0, \qquad \alpha = 1, \dots, r, \qquad \varphi = (\varphi^1, \dots, \varphi^m),$$

where φ^j are smooth functions on $\mathcal{E}^\infty$,

$$\ell_\psi = \sum_{j,\sigma} \frac{\partial \psi}{\partial u^j_\sigma} D^{(j)}_\sigma$$

and $\ell^{\mathcal{E}}_\psi$ is the restriction of ℓ_ψ to $\mathcal{E}^\infty$. This isomorphism is given by the correspondence

$$\varphi \mapsto \partial_\varphi = \sum_{\sigma,j} D_\sigma(\varphi^j)\frac{\partial}{\partial u^j_\sigma}$$

The function φ is callen the *generating function* corresponds to a symmetry of equation.

Generating functions of conservation laws correspond to solutions of the equation formally adjont to (6) (see [1]).

3. Symmetries

Consider a system of equations for the generating function of symmetries of equation (1)

$$
\begin{aligned}
D_z^2(\varphi_u) &= -\frac{1}{\mu}\big(D_x^2(\varphi_u) + D_y^2(\varphi_u)) + D_x(\varphi_p)\big),\\
D_z^2(\varphi_v) &= -\frac{1}{\mu}\big(D_x^2(\varphi_v) + D_y^2(\varphi_v)) + D_y(\varphi_p)\big),\\
D_z(\varphi_p) &= \mu(D_x^2(\varphi_w) + D_y^2(\varphi_w) - D_xD_z(\varphi_u) - D_yD_z(\varphi_v)) + g\varphi_\rho,\\
D_z(\varphi_w) &= -D_x(\varphi_x) - D_y(\varphi_y),\\
D_z(\varphi_\rho) &= -\Big(\frac{\rho_x}{\rho_z}D_x(\varphi_\rho) + \frac{\rho_y}{\rho_z}D_y(\varphi_\rho) + \frac{\rho_t}{\rho_z}D_t(\varphi_\rho) + \frac{u}{\rho_z}\varphi_u\\
&\qquad + \frac{v}{\rho_z}\varphi_v + \frac{w}{\rho_z}\varphi_w\Big).
\end{aligned}
$$

Here D_t, D_x, D_y, D_z are the total derivatives restricted to the system.

We solve this system on the space of functions in the variables t, x, y, z, ρ , p, u, v, w, u_t, u_x, u_y, u_z, v_t, v_x, v_y, v_z, w_t, w_x, w_y, p_t, p_x, p_y, ρ_t, ρ_y, u_{xz}, v_{yz}.

Theorem 3 (Classical symmetries)**.** *The system*

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}),\\
\frac{\partial p}{\partial y} &= \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}),\\
\frac{\partial p}{\partial z} &= \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) + g\rho,\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\\
&\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0
\end{aligned}
$$

has the classical symmetries

$$
\begin{aligned}
&(T) && \partial_t,\\
&(X_\alpha) && \alpha(t)\partial_x + \alpha'(t)\partial_u,\\
&(Y_\beta) && \beta(t)\partial_y + \beta'(t)\partial_v,\\
&(Z_\gamma) && \gamma(t)\partial_z + \gamma'(t)\partial_w,\\
&(PR) && \partial_\rho + gz\partial_p,\\
&(P_\varepsilon) && \varepsilon(t)\partial_p,\\
&(M_1) && t\partial_t - \rho\partial_\rho - p\partial_p - u\partial_u - v\partial_v - w\partial_w,\\
&(M_2) && x\partial_x + y\partial_y + z\partial_z - \rho\partial_\rho + u\partial_u + v\partial_v + w\partial_w,\\
&(\Theta_\delta) && \delta(t)(y\partial_x - x\partial_y + v\partial_u - u\partial_v) + \delta'(t)(y\partial_u - x\partial_v).
\end{aligned}
$$

	T	X_α	Y_β	Z_γ	P_ε	PR	M_1	M_2	Θ_δ
T	0	$X_{\alpha'}$	$Y_{\beta'}$	$Z_{\gamma'}$	$P_{\varepsilon'}$	0	T	0	$\Theta_{\delta'}$
X_α		0	0	0	0	0	$-X_{t\alpha'}$	X_α	$-Y_{\alpha\delta}$
Y_β			0	0	0	0	$-Y_{t\beta'}$	Y_β	$X_{\beta\delta}$
Z_γ				0	0	$P_{g\gamma}$	$-Z_{t\gamma'}$	Z_γ	0
P_ε					0	0	$-P_{\varepsilon+t\varepsilon'}$	0	0
PR						0	$-PR$	$-PR$	0
M_1							0	0	$\Theta_{t\delta'}$
M_2								0	0
Θ_δ									0

FIGURE 1. Commutator relations between classical symmetries

The commutators of symmetries are presented on Fig. 1. In addition, one has

$$[P_{\varepsilon_1}, P_{\varepsilon_2}] = 0, \quad [X_{\alpha_1}, X_{\alpha_2}] = 0, \quad [Y_{\beta_1}, Y_{\beta_2}] = 0,$$
$$[Z_{\gamma_1}, Z_{\gamma_2}] = 0, \quad [\Theta_{\delta_1}, \Theta_{\delta_2}] = 0.$$

The symmetry T corresponds to the translation in t. The symmetries X_α, Y_β, Z_γ correspond to generalized translations in x, y, z respectively. The symmetries P_ε, PR are gauges on trivial solution. The symmetries M_1, M_2 are scale symmetries. The symmetry Θ_δ corresponds to generalized rotation.

Let X be an infinitesimal symmetry of the equation $\mathcal{E} \subset J^k(n,m)$, f_0 be a solution of this equation, $\Gamma^k_{f_0} = j_k(f_0)(M) \subset \mathcal{E}$. Let $\{A_t\}$ be the one-parameter group of the vector field X. Then the submanifold $A_t(\Gamma^k_{f_0}) \subset J^k(n,m)$ is integrable for the Cartan distribution: if $A_t(\Gamma^k_{f_0})$ and $\Gamma^k_{f_0}$ are a horizontal with respect to the projection on base, then $A_t(\Gamma^k_{f_0}) = \Gamma^k_{f_k}$ for some section f_t and sufficiently small t. Moreover, we obtain from the definition of symmetries that $A_t(\Gamma^k_{f_0}) \subset \mathcal{E}$. The translation of an initial solution f to a family of solutions f_t is called *reproduction* of solution the f under the action of X. If φ is the generating function of symmetry $X = X_\varphi$ then f_t is a solution of the system of equations

$$\frac{\partial f^j}{\partial t} = \varphi^j|_{\Gamma^k_f}(x,t), \qquad f^j(x,0) = f^j_0(x), \qquad j = 1, \dots, m.$$

If $\varphi|_{\Gamma^k_f}(x,t) = 0$ then $f(x,t) = f(x,0)$ for all t, i.e., f_0 are stationary point for one-parameter group of transformations $\{A_t\}$ under its action on section. In other words, the manifold $\Gamma^k_{f_0}$ is invariant under the action of the vector field X_φ.

Definition 5. If $\Gamma^k_{f_0}$ is invariant under the action of the vector field X_φ, then f_0 is called a φ-*invariant* solution of $\mathcal{E}$.

Applying this technique to (1), we obtain the following

Theorem 4. *Let* $\rho = \rho(t,x,y,z)$, $p = p(t,x,y,z)$, $u = u(t,x,y,z)$, $v = v(t,x,y,z)$, $w = w(t,x,y,z)$ *be a solution of system* (1). *Then vector-functions*

$$(T) \qquad \begin{aligned} &\rho = \rho(t+\tau,x,y,z),\ p = p(t+\tau,x,y,z)\\ &u = u(t+\tau,x,y,z),\ v = v(t+\tau,x,y,z)\\ &w = w(t+\tau,x,y,z)\end{aligned}$$

$$(X_\alpha) \qquad \begin{aligned} &\rho = \rho(t,x+\tau\alpha(t),y,z),\ p = p(t,x+\tau\alpha(t),y,z)\\ &u = u(t,x+\tau\alpha(t),y,z) - \tau\alpha'(t),\ v = v(t,x+\tau\alpha(t),y,z)\\ &w = w(t,x+\tau\alpha(t),y,z)\end{aligned}$$

$$(Y_\beta) \qquad \begin{aligned} &\rho = \rho(t,x,y+\tau\beta(t),z),\ p = p(t,x,y+\tau\beta(t),z)\\ &u = u(t,x,y+\tau\beta(t),z),\ v = v(t,x,y+\tau\beta(t),z) - \tau\beta'(t)\\ &w = w(t,x,y+\tau\beta(t),z)\end{aligned}$$

$$(Z_\gamma) \qquad \begin{aligned} &\rho = \rho(t,x,y,z+\tau\gamma(t)),\ p = p(t,x,y,z+\tau\gamma(t))\\ &u = u(t,x,y,z+\tau\gamma(t)),\ v = v(t,x,y,z+\tau\gamma(t))\\ &w = w(t,x,y,z+\tau\gamma(t)) - \tau\gamma'(t)\end{aligned}$$

$$(P_\varepsilon) \qquad \begin{aligned} &\rho = \rho(t,x,y,z),\ p = p(t,x,y,z) + \tau\varepsilon(t)\\ &u = u(t,x,y,z),\ v = v(t,x,y,z)\\ &w = w(t,x,y,z)\end{aligned}$$

$$(PR) \qquad \begin{aligned} &\rho = \rho(t,x,y,z) + \tau,\ p = p(t,x,y,z) + gz\tau\\ &u = u(t,x,y,z),\ v = v(t,x,y,z)\\ &w = w(t,x,y,z)\end{aligned}$$

$$(M_1) \qquad \begin{aligned} &\rho = \tau\rho(\tau t,x,y,z),\ p = \tau p(\tau t,x,y,z)\\ &u = \tau u(\tau t,x,y,z),\ v = \tau v(\tau t,x,y,z)\\ &w = \tau w(\tau t,x,y,z)\end{aligned}$$

$$(M_2) \qquad \begin{aligned} &\rho = \tau\rho(t,\tau x,\tau y,\tau z),\ p = p(t,\tau x,\tau y,\tau z)\\ &u = u(t,\tau x,\tau y,\tau z)/\tau,\ v = v(t,\tau x,\tau y,\tau z)/\tau\\ &w = w(t,\tau x,\tau y,\tau z)/\tau\end{aligned}$$

$$(\Theta_\delta) \qquad \begin{aligned} &\rho = \rho(t,r,\theta-\tau\delta,z),\ p = p(t,r,\theta-\tau\delta,z)\\ &u = A\cos(\tau)u(t,r,\theta-\tau\delta,z) + B\sin(\tau)v(t,r,\theta-\tau\delta,z)\\ &\qquad - \frac{\delta'}{\delta} r\theta\sin\theta\\ &v = -A\sin(\tau)u(t,r,\theta-\tau\delta,z) + B\cos(\tau)v(t,r,\theta-\tau\delta,z)\\ &\qquad + \frac{\delta'}{\delta} r\theta\cos\theta\\ &w = w(t,r,\theta-\tau\delta,z)\end{aligned}$$

are solutions of the system too.

The following quotientequations correspond to the above considered symmetries:

$$(T)\qquad \frac{\partial p}{\partial x}=\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}),$$

$$\frac{\partial p}{\partial y}=\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}),$$

$$\frac{\partial p}{\partial z}=\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+g\rho,$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

$$u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}=0,$$

$$(X_\alpha)\qquad \mu(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})=0,$$

$$\frac{\partial p}{\partial y}=\mu(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}),$$

$$\frac{\partial p}{\partial z}=\mu(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+g\rho,$$

$$\frac{\alpha'(t)}{\alpha(t)}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}=0,$$

$$(Y_\beta)\qquad \frac{\partial p}{\partial x}=\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}),$$

$$\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2})=0,$$

$$\frac{\partial p}{\partial z}=\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2})+g\rho,$$

$$\frac{\partial u}{\partial x}+\frac{\beta'(t)}{\beta(t)}+\frac{\partial w}{\partial z}=0,$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+w\frac{\partial \rho}{\partial z}=0,$$

$$(Z_\gamma)\qquad \frac{\partial p}{\partial x}=\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}),$$

$$\frac{\partial p}{\partial y}=\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}),$$

$$\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+g\rho=0,$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\gamma'(t)}{\gamma(t)}=0,$$

$$(M_1)\qquad \begin{aligned}
&\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}=0,\\
&\frac{\partial p}{\partial x}=\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}),\\
&\frac{\partial p}{\partial y}=\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}),\\
&\frac{\partial p}{\partial z}=\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+g\rho,\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\\
&u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}=\rho,
\end{aligned}$$

$$(\Theta_\delta)\qquad \begin{aligned}
&v=A(t,r,z)\cos(\theta)+B(t,r,z)\sin(\theta)+\frac{\delta'}{\delta}r\theta\sin\theta,\\
&u=-A(t,r,z)\sin(\theta)+B(t,r,z)\cos(\theta)-\frac{\delta'}{\delta}r\theta\cos\theta,\\
&\frac{\partial p}{\partial r}=\mu(\frac{\partial^2 B}{\partial r^2}+\frac{1}{r}\frac{\partial B}{\partial r}-\frac{1}{r^2}B+\frac{\partial^2 B}{\partial z^2}-\frac{\delta'}{\delta}\frac{2}{r}),\\
&\frac{\partial^2 A}{\partial r^2}+\frac{1}{r}\frac{\partial A}{\partial r}-\frac{1}{r^2}\frac{\partial^2 A}{\partial z^2}=0,\\
&\frac{\partial p}{\partial z}=\mu(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2})+g\rho,\\
&\frac{\partial B}{\partial r}+\frac{1}{r}B+1+\frac{\partial w}{\partial z}=0,\\
&\frac{\partial\rho}{\partial t}+B\frac{\partial\rho}{\partial r}+w\frac{\partial\rho}{\partial z}=0.
\end{aligned}$$

4. Conservation laws

Theorem 5. *System (1) has the following conservation laws with 0-order generating functions*

$$\begin{aligned}
\psi_\rho&=(a'(\rho)+b(t))w,\\
\psi_p&=-b'(t)/g,\\
\psi_u&=c(t,x,y,z),\\
\psi_v&=d(t,x,y,z),\\
\psi_w&=a(\rho)+b(t)\rho+e(t,x,y),
\end{aligned}$$

where c, d, e are solutions of the system

$$\begin{aligned}
&\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}+\frac{\partial^2 c}{\partial z^2}=\frac{\partial e}{\partial x},\\
&\frac{\partial^2 d}{\partial x^2}+\frac{\partial^2 d}{\partial y^2}+\frac{\partial^2 d}{\partial z^2}=\frac{\partial e}{\partial y},
\end{aligned}$$

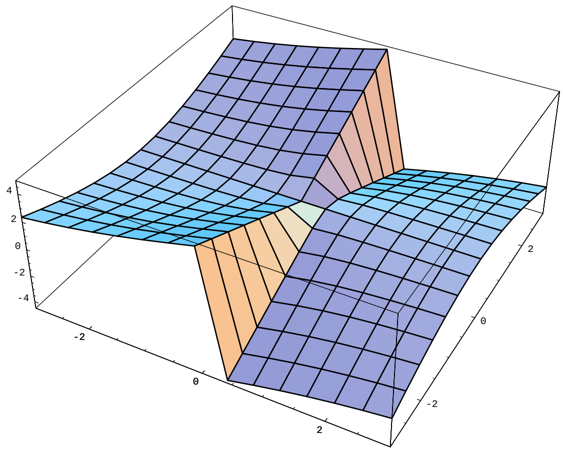


FIGURE 2. The graph of the u function

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial y} = 0.$$

Consider some divergent forms which corresponds to the generating functions in the last theorem:

$$\begin{aligned} &(a(\rho))_t + (ua(\rho))_x + (va(\rho))_y + (wa(\rho))_z = 0, \\ &(b(t)\rho)_t + (b(t)u\rho)_x + (b(t)v\rho)_y + (b(t)w\rho)_z + \frac{\mu}{g}\triangle(b'(t)w) \\ &- \frac{1}{g}(b'(t)p)_z = 0. \end{aligned}$$

The main result of the paper is formulated in the following theorem.

Theorem 6. *There exist the following higher-order generating functions of the conservation laws*

$$\psi_u = (u_y - v_x)_{\sigma y}, \qquad \psi_v = -(u_y - v_x)_{\sigma x},$$

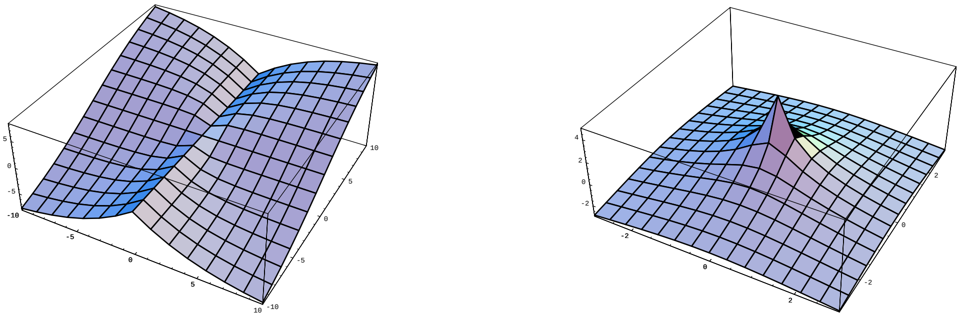
where $\sigma = (\sigma_x, \sigma_y)$, $\sigma_x + \sigma_y \equiv 1 \mod 2$.

This theorem shown that the system has divergent forms. Moreover, for every k there exists a divergent form, which contain derivatives of order higher than k.

5. EXAMPLES

Consider the invariant solution of symmetry Θ_δ and let $\delta = e^t$, $A = 0$, $B = 0$, $\rho = \rho(r)$, $w = w(t,r,z)$, $p = p(t,r)$. Then

$$\begin{aligned} u &= -r\theta \sin\theta, \\ v &= r\theta \cos\theta, \\ p &= 2\mu \ln\frac{1}{r} + p_0(t), \\ \rho &= 2\rho_0'(r^2), \end{aligned}$$



(a) (b)

FIGURE 3. The graphs of the v (a) and p (b) functions

$$w = -z - \frac{g}{\mu}\int_0^r \frac{\rho(r'^2)}{r'}dr' + w_0(t).$$

The graphs of functions u, v and p for $p_0 = 0$, $\mu = 1/2$ are at Fig. 2, and Fig. 3 a,b respectively.

This solution, being subjected to PR-deformation, transforms to:

$$\begin{aligned}
u &= -r\theta\sin\theta = -y\arctan\frac{y}{x},\\
v &= r\theta\cos\theta = x\arctan\frac{y}{x},\\
p &= 2\mu\ln\frac{1}{r} + p_0(t) + gz\tau,\\
\rho &= 2\rho_0'(r^2) + \tau,\\
w &= -z - \frac{g}{\mu}\int_0^r \frac{\rho(r'^2)}{r'}dr' + w_0(t).
\end{aligned}$$

After X_{-t}-deformation, we obtain solution with running singularity:

$$\begin{aligned}
u &= -y\arctan\frac{y}{x-t} + 1,\\
v &= (x-t)\arctan\frac{y}{x-t},\\
p &= \mu\ln\frac{1}{(x-t)^2+y^2} + p_0(t) + gz\tau,\\
\rho &= 2\rho_0'((x-t)^2+y^2) + \tau,\\
w &= -z - \frac{g}{\mu}\int_0^{(\sqrt{(x-t)^2+y^2})} \frac{\rho(r'^2)}{r'}dr' + w_0(t).
\end{aligned}$$

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