The Diffiety Institute Preprint Series

Preprint DIPS–8/2001 **June 7, 2001**

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by

A. M. VINOGRADOV and M. M. VINOGRADOV

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The Diffiety Institute
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

Graded multiple analogs of Lie algebras

A. M. VINOGRADOV and M. M. VINOGRADOV

Abstract. Graded analogs of (n, k, r)-Lie algebras (in particular, of Nambu–Lie algebras), introduced in [28], are defined and their general property are studied.

1. Introduction

Last years some problems in geometry and mathematical physics stimulated a growing interest to multi-analogs of Lie algebras and Poisson manifolds. The first of such analogs was introduced and studied by V.T. Filippov [7] under the name n-Lie algebra. Later and independently this class of algebras was rediscovered by physicists [24] as Nambu–Lie algebras in connection with the so-called Nambu mechanics [23]. For further developments see [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 22, 26, 27].

Around 1990 P. Michor and one of the authors [20] considered another class of multiple Lie type algebras (see also [21, 2, 4]) close to Stasheff's *sh* Lie algebras [17, 25]. These two types of multiple Lie algebras manifest quite different properties as well as the corresponding multiple Poisson structures. A natural question on what is relation between these classes of algebras was answered in [28] where for any triple nonnegative of integers n, k, r the notion of (n, k, r)-Lie algebra was introduced. (n, k, r)-Lie algebras form a "net" in a natural sense, which extremal lines, namely, $(n, n-1, 0)$ and $(n, 1, 0)$ correspond to two aforementioned types of multiple Lie algebras. In this paper we introduce graded analogs of (n, k, r)-Lie algebras and study their general properties, which are interesting by themselves. Applications are postponed to subsequent publications.

Everywhere below the proofs obtained by direct computations are omitted.

2. Graded Richardson–Nijenhuis bracket

2.1. **Preliminaries.** A *grading group* is an Abelian group G with a $\mathbb{Z}$-bilinear symmetric map $\langle\ ,\ \rangle\colon G \times G \to \mathbb{Z}_2$. The map $\langle\ ,\ \rangle_{\mathbb{Z}}\colon (p, q) \mapsto pq \bmod 2$ transforms $\mathbb{Z}$ into a grading group. If $(G_i, \langle\ ,\ \rangle_i)$, $i = 1, 2$, are grading groups, then $(G_1 \oplus G_2, \langle\ ,\ \rangle_1 \oplus \langle\ ,\ \rangle_2)$ is also a grading group. In particular, $(\mathbb{Z} \oplus G, \langle\ ,\ \rangle \oplus \langle\ ,\ \rangle_{\mathbb{Z}})$ is a grading group.

A G*-graded vector space* over a field K is the direct sum $A = \bigoplus_{g\in G} A^g$ of K-vector spaces A^g. An element $a \in A^g$ is called *homogeneous* of degree $\omega(a) = g$. A G*-graded algebra* is a graded vector space supplied with a K-algebra structure such that $A^g \cdot A^f \subset A^{f+g}$, $f, g \in G$. Such an algebra A is *commutative* with respect to $(G, \langle\ ,\ \rangle)$ if $ab = (-1)^{\langle \omega(a), \omega(b)\rangle} ba$ for any homogeneous elements $a, b \in A$.

Key words and phrases. (multi) Lie algebra, (multi) Poisson manifold, (multi) Jacobi identity, (graded) Richardson–Nijenhuis bracket, Nambu mechanics.

This work was partially supported by the Italian Ministero dell'Università e della Ricerca Scientifica and by Istituto Nazionale di Fisica Nucleare.

Denote by $\mathrm{I}^{(n)} \overset{\text{def}}{=} \{1, \dots, n\}$ the set of first n natural numbers ordered naturally. An *unshuffle* $I = (i_1, \dots, i_k)$ is a naturally ordered subset of $\mathrm{I}^{(n)}$, i.e., $1 \le i_1 < i_2 < \dots < i_k \le n$. Put also $|I| = k$. If I and J are two unshuffles such that $I \cap J = \varnothing$, $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_l\}$, then put $(I, J) \overset{\text{def}}{=} \{i_1, \dots, i_k, j_1, \dots, j_l\}$. $I + J$ stands for the unique unshuffle coinciding with (I, J) as a set and $(-1)^{(I,J)}$ denotes the sign of the permutation sending (I, J) to $I + J$. If $I = \{i\}$, then we write (i, J), $(-1)^{(i,J)}$, $i + J$ instead of (I, J), $(-1)^{(I,J)}$, $I + J$, respectively. The unshuffle $I - i$, $i \in I$, is obtained from I by cancelling i. Sometimes we write I_l instead of i_l for $I = (i_1, \dots, i_l, \dots, i_n)$.

An l-linear map $\mathcal{L}\colon A \times \dots \times A \to A$, A being a G-graded space, is called *graded* of degree $\omega(\mathcal{L}) = k \in G$, if

$$\mathcal{L}(A^{g_1} \times \dots \times A^{g_l}) \subseteq A^{g_1 + \dots + g_l + k}$$

The bidegree $\delta(\mathcal{L})$ of $\mathcal{L}$ is defined as $\delta(\mathcal{L}) \overset{\text{def}}{=} (l - 1, \omega(\mathcal{L})) \in \mathbb{Z} \oplus G$. Further on we write simply $\langle L, \mathcal{N} \rangle$ for $\langle \delta(\mathcal{L}), \delta(\mathcal{N}) \rangle$.

Let $a_1, \dots, a_n \in A$ be homogeneous and $(i_1, \dots, i_k)$ be an unshuffle. Then $a(I)$ stands for the ordered set $a_{i_1}, \dots, a_{i_n} \in A$. Put $a(\sigma) = \{a_{\sigma(1)}, \dots, a_{\sigma(n)}\}$ for a permutation σ of $\{1, \dots, n\}$ and denote by σ_i the transposition of i and $i+1$. Define the *parity* $\langle a(\sigma_i) \rangle$ of the *graded transposition* $a(\sigma_i)$ to be equal to $\langle \omega(a_i), \omega(a_{i+1}) \rangle + 1$. An n-linear graded map $\mathcal{L}$ from A to A is called *skew-symmetric* if

$$\mathcal{L}(a(\mathrm{I}^{(n)})) = (-1)^{\langle a(\sigma_i) \rangle} \mathcal{L}(a(\sigma_i)), \tag{2.1}$$

and *symmetric* if

$$\mathcal{L}(a(\mathrm{I}^{(n)})) = (-1)^{\langle \omega(a_i), \omega(a_{i+1}) \rangle} \mathcal{L}(a(\sigma_i)). \tag{2.2}$$

Denote by $\mathrm{Alt}_K^n A$ (resp., $\mathrm{Sym}_K^n A$) the K-vector space of all skew-symmetric (resp., symmetric) n-linear maps from A to A. Then

$$\mathrm{Alt}_K^* A = \bigoplus_{n=1}^{\infty} \mathrm{Alt}_K^n A, \quad \mathrm{Sym}_K^* A = \bigoplus_{n=1}^{\infty} \mathrm{Sym}_K^n A$$

become $\mathbb{Z} \times G$-graded spaces with respect to the above said bigradings.

If a permutation σ is presented as a composition of transpositions $\sigma_{k_1} \circ \dots \circ \sigma_{k_m}$ we define the parity $\langle a(\sigma) \rangle$ of the graded permutation $a(\sigma)$ as the product $\langle a(\sigma_{k_1} \rangle \circ \dots \circ \langle a(\sigma_{k_m}) \rangle$. Obviously,

$$\mathcal{L}(a(\mathrm{I}^{(n)})) = (-1)^{\langle a(\sigma) \rangle} \mathcal{L}(a(\sigma))$$

if $\mathcal{L} \in \mathrm{Alt}_K^n A$.

Notice also the following relation

$$\langle a(I, J) \rangle = \langle a(J, I) \rangle + nl + \langle \omega(a(I)), \omega(a(J)) \rangle, \tag{2.3}$$

where $\omega(a(I)) = \sum_{i \in I} \omega(a_i)$, $\omega(a(J)) = \sum_{j \in J} \omega(a_j)$.

2.2. **Graded Richardson–Nijenhuis bracket.**

Definition 2.1. Let $\mathcal{L} \in \mathrm{Alt}_K^l A$. The *action*

$$\mathcal{L}[\,\cdot\,]\colon \mathrm{Alt}_K^n A \longrightarrow \mathrm{Alt}_K^{n+l-1} A, \qquad n \ge 0,$$

is defined by

$$\mathcal{L}[\mathcal{N}](a(\mathrm{I}^{(l+n-1)})) = (-1)^{\langle \mathcal{L},\mathcal{N}\rangle} \sum_{\substack{I+J=\mathrm{I}^{(l+n-1)}\\ |I|=n,\ |J|=l-1}} (-1)^{\langle a(I,J)\rangle} \mathcal{L}(\mathcal{N}(a(I)), a(J)), \tag{2.4}$$

$a_i \in A$, $\mathcal{N} \in \mathrm{Alt}_K^n A$.

Definition 2.2. The *graded Richardson–Nijenhuis bracket* $[[\mathcal{L},\mathcal{N}]] \in \mathrm{Alt}_K^{l+n-1} A$ of $\mathcal{L} \in \mathrm{Alt}_K^l A$, $\mathcal{N} \in \mathrm{Alt}_K^n A$ is defined as,

$$[[\mathcal{L},\mathcal{N}]] \overset{\mathrm{def}}{=} \mathcal{L}[\mathcal{N}] - (-1)^{\langle \mathcal{L},\mathcal{N}\rangle}\mathcal{N}[\mathcal{L}] \tag{2.5}$$

Notice that

$$\delta([[\mathcal{L},\mathcal{N}]]) = \delta(\mathcal{L}[\mathcal{N}]) = (l+n-2, \omega(\mathcal{L})+\omega(\mathcal{N})) = \delta(\mathcal{L})+\delta(\mathcal{N}) \tag{2.6}$$

Similarly to action (2.4) associated with $\mathcal{L} \in \mathrm{Alt}_K^l A$, define a pairing (“biaction”)

$$\mathcal{L}[\cdot,\cdot]\colon\ \mathrm{Alt}_K^n A \times \mathrm{Alt}_K^p A \longrightarrow \mathrm{Alt}_K^{n+p+l-2} A$$

by putting for any $\mathcal{N} \in \mathrm{Alt}_K^n A$, $\mathcal{P} \in \mathrm{Alt}_K^p A$

$$\mathcal{L}[\mathcal{N},\mathcal{P}](a(\mathrm{I}^{(l+n+p-2)})) = (-1)^{\langle \mathcal{L},\mathcal{N}\rangle+\langle \mathcal{L},\mathcal{P}\rangle+\langle \mathcal{N},\mathcal{P}\rangle} \times \\ \times \sum_{\substack{I+J=\mathrm{I}^{(l+n+p-2)}\\ |I|=n,\ |J|=p,\ |L|=l-2}} (-1)^{\langle a(I,J,L)\rangle+\langle \omega(a(I)),\omega(\mathcal{P})\rangle} \mathcal{L}(\mathcal{N}(a(I)), \mathcal{P}(a(J)), a(L)). \tag{2.7}$$

Lemma 2.3. *Let $\mathcal{L},\mathcal{N},\mathcal{P} \in \mathrm{Alt}_K^* A$ be homogeneous. Then*

$$\mathcal{L}[\mathcal{P},\mathcal{N}] = (-1)^{\langle \mathcal{P},\mathcal{N}\rangle+p+n}\mathcal{L}[\mathcal{N},\mathcal{P}]. \tag{2.8}$$

Lemma 2.4. *Let $\mathcal{L},\mathcal{N},\mathcal{P} \in \mathrm{Alt}_K^* A$ be homogeneous. Then*

$$\big(\mathcal{P}[\mathcal{L}]\big)[\mathcal{N}] = \mathcal{P}[\mathcal{L}[\mathcal{N}]] + (-1)^{l(n-1)}\mathcal{P}[\mathcal{L},\mathcal{N}]. \tag{2.9}$$

◀ By definition we have

$$\begin{aligned} &\big(\mathcal{P}[\mathcal{L}]\big)[\mathcal{N}](a(\mathrm{I}^{(l+n+p-2)})) \\ &= (-1)^{\langle \mathcal{P}+\mathcal{L},\mathcal{N}\rangle} \sum_{\substack{J+T=\mathrm{I}^{(l+n+p-2)}\\ |J|=n,\ |T|=p+l-2}} (-1)^{\langle a(J,T)\rangle} \big(\mathcal{P}[\mathcal{L}]\big)(\mathcal{N}(a(J)), a(T)) \\ &= (-1)^{\langle \mathcal{P},\mathcal{N}\rangle+\langle \mathcal{L},\mathcal{N}\rangle+\langle \mathcal{P},\mathcal{L}\rangle} \Big(\sum_{\substack{J+I+K=\mathrm{I}^{(l+n+p-2)}\\ |J|=n,|I|=l-1,|K|=p-1}} (-1)^{\langle a(J,I+K)\rangle+\langle a(I,K)\rangle} \mathcal{P}(\mathcal{L}(\mathcal{N}(a(J)), a(I)), a(K)) \\ &\quad + \sum_{\substack{J+I+K=\mathrm{I}^{(l+n+p-2)}\\ |J|=n,|I|=l,|K|=p-2}} (-1)^{\langle a(J,I+K)\rangle+\langle a(I,K)\rangle+\langle\sigma\rangle} \mathcal{P}(\mathcal{L}(a(I)), \mathcal{N}(a(J)), a(K)) \Big), \end{aligned} \tag{2.10}$$

where σ is the permutation sending $\big(\mathcal{N}(a(J)), a(I)\big)$ to $\big(a(I), \mathcal{N}(a(J))\big)$. The first summation is identical to $\mathcal{P}[\mathcal{L}[\mathcal{N}]](a(\mathrm{I}^{(l+n+p-2)}))$, while the second one coincides up to signs with the summation defining $\mathcal{P}[\mathcal{L},\mathcal{N}](a(\mathrm{I}^{(l+n+p-2)}))$. Coincidence of the signs is checked directly by using (2.3) ▶

Theorem 2.5. *Let $\mathcal{L},\mathcal{N},\mathcal{P}\in \mathrm{Alt}^*_K A$ be homogeneous. Then*

$$[\![\mathcal{P},[\![\mathcal{L},\mathcal{N}]\!]]\!] = [\![[\![\mathcal{P},\mathcal{L}]\!],\mathcal{N}]\!] + (-1)^{\langle\mathcal{P},\mathcal{L}\rangle}[\![\mathcal{L},[\![\mathcal{P},\mathcal{N}]\!]]\!]. \tag{2.11}$$

◀ By using formulas (2.8), (2.9) we have:

$$\begin{aligned}
[\![\mathcal{P},[\![\mathcal{L},\mathcal{N}]\!]]\!] &= \mathcal{P}\big[[\![\mathcal{L},\mathcal{N}]\!]\big] - (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle}\big([\![\mathcal{L},\mathcal{N}]\!]\big)[\mathcal{P}]\\
&= \mathcal{P}\big[\mathcal{L}[\mathcal{N}] - (-1)^{\langle\mathcal{L},\mathcal{N}\rangle}\mathcal{N}[\mathcal{L}]\big]\\
&\quad - (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle}\big(\mathcal{L}[\mathcal{N}] - (-1)^{\langle\mathcal{L},\mathcal{N}\rangle}\mathcal{N}[\mathcal{L}]\big)[\mathcal{P}]\\
&= \mathcal{P}[\mathcal{L}[\mathcal{N}]] - (-1)^{\langle\mathcal{L},\mathcal{N}\rangle}\mathcal{P}[\mathcal{N}[\mathcal{L}]]\\
&\quad - (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle}\big(\mathcal{L}[\mathcal{N}]\big)[\mathcal{P}] + (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle+\langle\mathcal{L},\mathcal{N}\rangle}\big(\mathcal{N}[\mathcal{L}]\big)[\mathcal{P}]\\
&= \mathcal{P}[\mathcal{L}[\mathcal{N}]] - (-1)^{\langle\mathcal{L},\mathcal{N}\rangle}\mathcal{P}[\mathcal{N}[\mathcal{L}]]\\
&\quad - (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle}\mathcal{L}[\mathcal{N}[\mathcal{P}]] + (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle+\langle\mathcal{L},\mathcal{N}\rangle}\mathcal{N}[\mathcal{L}[\mathcal{P}]]\\
&\quad - (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle+n(p-1)}\mathcal{L}[\mathcal{N},\mathcal{P}] + (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle+\langle\mathcal{L},\mathcal{N}\rangle+l(p-1)}\mathcal{N}[\mathcal{L},\mathcal{P}]\\
&= \mathcal{P}[\mathcal{L}[\mathcal{N}]] - (-1)^{\langle\mathcal{L},\mathcal{N}\rangle}\mathcal{P}[\mathcal{N}[\mathcal{L}]]\\
&\quad - (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle}\mathcal{L}[\mathcal{N}[\mathcal{P}]] + (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+\langle\mathcal{P},\mathcal{N}\rangle+\langle\mathcal{L},\mathcal{N}\rangle}\mathcal{N}[\mathcal{L}[\mathcal{P}]]\\
&\quad - (-1)^{\langle\mathcal{P},\mathcal{L}\rangle+p(n-1)}\mathcal{L}[\mathcal{P},\mathcal{N}] + (-1)^{\langle\mathcal{P},\mathcal{N}\rangle+\langle\mathcal{L},\mathcal{N}\rangle+p(l-1)}\mathcal{N}[\mathcal{P},\mathcal{L}]
\end{aligned} \tag{2.12}$$

and similarly for $[\![[\![\mathcal{P},\mathcal{L}]\!],\mathcal{N}]\!]$ and $[\![\mathcal{L},[\![\mathcal{P},\mathcal{N}]\!]]\!]$. These three expressions put as it is required by (2.11) give the desired identity in view of formula (2.8). ▶

2.3. **Insertion operators.** Associate with a given $\mathcal{L}\in\mathrm{Alt}^l_K A$ and $a_1,\dots,a_k\in A$ a $(l-k)$-linear map $i_{a_1,\dots,a_k}(\mathcal{L})\in\mathrm{Alt}^{l-k}_K A$ defined by

$$i_{a_1,\dots,a_k}(\mathcal{L})(b_1,\dots,b_{l-k}) \stackrel{\text{def}}{=} \mathcal{L}(a_1,\dots,a_k,b_1,\dots,b_{l-k}).$$

Put also $\delta(i_a) = (-1,\omega(a))$.

Proposition 2.6. *Let $\mathcal{L}\in\mathrm{Alt}^l_K A$, $\mathcal{N}\in\mathrm{Alt}^n_K A$ and $a\in A$. Then*

$$i_a(\mathcal{L}[\mathcal{N}]) = i_a(\mathcal{L})[\mathcal{N}] + (-1)^{\langle i_a,\mathcal{L}\rangle}\mathcal{L}[i_a(\mathcal{N})]. \tag{2.13}$$

Corollary 2.7. *Under hypothesis of the proposition we have*

$$i_a([\![\mathcal{L},\mathcal{N}]\!]) = [\![i_a(\mathcal{L}),\mathcal{N}]\!] + (-1)^{\langle i_a,\mathcal{L}\rangle}[\![\mathcal{L},i_a(\mathcal{N})]\!]. \tag{2.14}$$

2.4. **Exterior multiplications.** If A is a commutative K-algebra. the space $\mathrm{Alt}^*_K A$ is supplied naturally with an *exterior product*. Namely, if $\mathcal{L}\in\mathrm{Alt}^l_K A$, $\mathcal{N}\in\mathrm{Alt}^n_K A$ we put

$$\begin{aligned}
(\mathcal{L}\wedge\mathcal{N})(a(\mathrm{I}^{(l+n)})) &\stackrel{\text{def}}{=} (-1)^{\langle\omega(\mathcal{L}),\omega(\mathcal{N})\rangle}\times\\
&\times\sum_{\substack{I+J=\mathrm{I}^{(l+n)}\\ |I|=l,\ |J|=n}}(-1)^{\langle a((I,J))\rangle+\langle\omega(\mathcal{N}),\omega(a(I))\rangle}\mathcal{L}(a(I))\mathcal{N}(a(J)).
\end{aligned} \tag{2.15}$$

Now with any $\mathcal{L}\in\mathrm{Alt}^l_K A$ the map

$$\widehat{\mathcal{L}}\colon\ \mathrm{Alt}^*_K A\to\mathrm{Alt}^*_K A,\ \mathcal{N}\mapsto\mathcal{L}\wedge\mathcal{N}.$$

is associated. Obviously, $\delta(\widehat{\mathcal{L}}) = (l, \omega(\mathcal{L}))$.

Lemma 2.8. *Let $\mathcal{L}, \mathcal{N} \in \operatorname{Alt}^*_K A$ be homogeneous and $a \in A$. Then*

$$\mathcal{L} \wedge \mathcal{N} = (-1)^{\langle \widehat{\mathcal{L}}, \widehat{\mathcal{N}} \rangle} \mathcal{N} \wedge \mathcal{L}, \tag{2.16}$$

$$i_a(\mathcal{L} \wedge \mathcal{N}) = i_a(\mathcal{L}) \wedge \mathcal{N} + (-1)^{\langle \widehat{\mathcal{L}}, i_a \rangle} \mathcal{L} \wedge i_a(\mathcal{N}). \tag{2.17}$$

Denote by $\operatorname{Md}^l_K(A) \subset \operatorname{Alt}^l_K A$ the subspace of all skew-symmetric graded multiderivations of A. Namely $\mathcal{L} \in \operatorname{Md}^l_K(A)$ iff

$$\begin{aligned} \mathcal{L}(ab, a_2, \ldots, a_l) = (-1)^{\langle \omega(a), \omega(\mathcal{L}) \rangle} a \mathcal{L}(b, a_2, \ldots, a_l) \\ + (-1)^{\langle \omega(a)+\omega(\mathcal{L}), \omega(b) \rangle} b \mathcal{L}(a, a_2, \ldots, a_l) \end{aligned} \tag{2.18}$$

(the *graded Leibniz rule*). Put also $\operatorname{Md}^*_K(A) = \oplus_l \operatorname{Md}^l_K(A)$.

Lemma 2.9. *If $\mathcal{L} \in \operatorname{Md}^l_K(A)$, $\mathcal{P} \in \operatorname{Alt}^p_K A$, $\mathcal{Q} \in \operatorname{Alt}^q_K A$, then*

$$\mathcal{L}[\mathcal{P} \wedge Q] = \mathcal{L}[\mathcal{P}] \wedge \mathcal{Q} + (-1)^{\langle \mathcal{L}, \widehat{\mathcal{P}} \rangle} \mathcal{P} \wedge \mathcal{L}[\mathcal{Q}]. \tag{2.19}$$

◀ A tedious straightforward computation by making use the graded Leibniz rule. ▶

Lemma 2.10. *Let $\mathcal{L}, \mathcal{N}, \mathcal{P} \in \operatorname{Alt}^*_K A$ be homogeneous, then*

$$(\mathcal{L} \wedge \mathcal{N})[\mathcal{P}] = \mathcal{L} \wedge \mathcal{N}[\mathcal{P}] + (-1)^{\langle \mathcal{L}, \widehat{\mathcal{P}} \rangle} \mathcal{L}[\mathcal{P}] \wedge \mathcal{N}. \tag{2.20}$$

Proposition 2.11. *If $\mathcal{L} \in \operatorname{Md}^l_K(A)$, $\mathcal{P} \in \operatorname{Alt}^p_K A$, $\mathcal{Q} \in \operatorname{Alt}^q_K A$, then*

$$[\![\mathcal{P} \wedge \mathcal{Q}, \mathcal{L}]\!] = \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L}]\!] + (-1)^{\langle \widehat{\mathcal{Q}}, \mathcal{L} \rangle} [\![\mathcal{P}, \mathcal{L}]\!] \wedge \mathcal{Q}. \tag{2.21}$$

◀ In view of the previous lemmas one has

$$\begin{aligned} [\![\mathcal{P} \wedge \mathcal{Q}, \mathcal{L}]\!] &= (\mathcal{P} \wedge \mathcal{Q})[\mathcal{L}] - (-1)^{\langle \mathcal{P} \wedge \mathcal{Q}, \mathcal{L} \rangle} \mathcal{L}[\mathcal{P} \wedge \mathcal{Q}] \\ &= \mathcal{P} \wedge \mathcal{Q}[\mathcal{L}] + (-1)^{\langle \widehat{\mathcal{Q}}, \mathcal{L} \rangle} \mathcal{P}[\mathcal{L}] \wedge \mathcal{Q} - (-1)^{\langle \mathcal{P} \wedge \mathcal{Q}, \mathcal{L} \rangle} \mathcal{L}[\mathcal{P}] \wedge \mathcal{Q} \\ &\qquad - (-1)^{\langle \mathcal{P} \wedge \mathcal{Q}, \mathcal{L} \rangle + \langle \widehat{\mathcal{P}}, \mathcal{L} \rangle} \mathcal{P} \wedge \mathcal{L}[\mathcal{Q}]. \end{aligned} \tag{2.22}$$

Now the result is obtained by observing that

$$\langle \mathcal{P} \wedge \mathcal{Q}, \mathcal{L} \rangle + \langle \widehat{\mathcal{P}}, \mathcal{L} \rangle = \langle \mathcal{Q}, \mathcal{L} \rangle, \qquad \langle \mathcal{P} \wedge \mathcal{Q}, \mathcal{L} \rangle - \langle \widehat{\mathcal{Q}}, \mathcal{L} \rangle = \langle \mathcal{P}, \mathcal{L} \rangle. \quad ▶$$

Proposition 2.12. *Under the assumptions of the previous proposition it holds*

$$[\![\mathcal{L}, \mathcal{P} \wedge \mathcal{Q}]\!] = [\![\mathcal{L}, \mathcal{P}]\!] \wedge \mathcal{Q} + (-1)^{\langle \mathcal{L}, \widehat{\mathcal{P}} \rangle} \mathcal{P} \wedge [\![\mathcal{L}, \mathcal{Q}]\!]. \tag{2.23}$$

Remark 2.1. In view of (2.16), formulas (2.21) and (2.23) can be rewritten as

$$[\![\mathcal{P} \wedge \mathcal{Q}, \mathcal{L}]\!] = \mathcal{P} \wedge [\![\mathcal{Q}, \mathcal{L}]\!] + (-1)^{\langle \mathcal{Q}, \mathcal{P} \rangle} \mathcal{Q} \wedge [\![\mathcal{P}, \mathcal{L}]\!], \tag{2.24}$$

$$[\![\mathcal{L}, \mathcal{P} \wedge \mathcal{Q}]\!] = [\![\mathcal{L}, \mathcal{P}]\!] \wedge \mathcal{Q} + (-1)^{\langle \mathcal{Q}, \mathcal{P} \rangle} [\![\mathcal{L}, \mathcal{Q}]\!] \wedge \mathcal{P}. \tag{2.25}$$

Proposition 2.13. *Let $\mathcal{L},\mathcal{N},\mathcal{P},\mathcal{Q}\in \mathrm{Md}^*_K(A)$ be homogeneous elements. Then*

$$[\![\mathcal{P}\wedge\mathcal{Q},\mathcal{L}\wedge\mathcal{N}]\!]=\mathcal{P}\wedge[\![\mathcal{Q},\mathcal{L}]\!]\wedge\mathcal{N}+(-1)^{\langle\widehat{\mathcal{Q}},\widehat{\mathcal{P}}\rangle}\mathcal{Q}\wedge[\![\mathcal{P},\mathcal{L}]\!]\wedge\mathcal{N} + (-1)^{\langle\widehat{\mathcal{L}},\widehat{\mathcal{N}}\rangle}\mathcal{P}\wedge[\![\mathcal{Q},\mathcal{N}]\!]\wedge\mathcal{L}+(-1)^{\langle\widehat{\mathcal{Q}},\widehat{\mathcal{P}}\rangle+\langle\widehat{\mathcal{L}},\widehat{\mathcal{N}}\rangle}\mathcal{Q}\wedge[\![\mathcal{P},\mathcal{N}]\!]\wedge\mathcal{L} \tag{2.26}$$

Assume now that the base algebra A is also G-graded. Denote by $\mathrm{Alt}^n_K(A,K)$ the space of all graded skew-symmetric n-covectors. In other words, these are n-linear maps from $\varphi\colon A\to K$ such that

$$\varphi(a(\mathrm{I}^{(n)}))=(-1)^{\langle a(\sigma)\rangle}\varphi(a(\sigma)),$$
$$\varphi(fa_1,\dots,a_n)=(-1)^{\langle\varphi,f\rangle}\varphi(a_1,\dots,a_n),\quad f\in K.$$

Let $\varphi\in\mathrm{Alt}^{|\varphi|}_K(A,K)$, $\psi\in\mathrm{Alt}^{|\psi|}_K(A,K)$, $\mathcal{L}\in\mathrm{Alt}^l_K A$. The exterior products $\varphi\wedge\psi\in\mathrm{Alt}^{|\varphi|+|\psi|}_K(A,K)$ and $\varphi\wedge\mathcal{L}\in\mathrm{Alt}^{|\varphi|+l}_K A$ and the action $\varphi[\mathcal{L}]\in\mathrm{Alt}^{|\varphi|+l-1}_K(A,K)$ are defined naturally:

$$(\varphi\wedge\psi)(a(\mathrm{I}^{(|\varphi|+|\psi|)}))\overset{\mathrm{def}}{=}(-1)^{\langle\omega(\varphi),\omega(\psi)\rangle}\times \times\sum_{\substack{I+J=\mathrm{I}^{(|\varphi|+|\psi|)}\\ |I|=|\varphi|,\ \ |J|=|\psi|}}(-1)^{\langle a((I,J))\rangle+\langle\omega(a(I)),\omega(\varphi)\rangle}\varphi(a(I))\psi(a(J)) \tag{2.27}$$

$$(\varphi\wedge\mathcal{L})(a(\mathrm{I}^{(|\varphi|+l)}))\overset{\mathrm{def}}{=}(-1)^{\langle\omega(\varphi),\omega(\mathcal{L})\rangle}\times \times\sum_{\substack{I+J=\mathrm{I}^{(|\varphi|+l)}\\ |I|=|\varphi|,\ \ |J|=l}}(-1)^{\langle a((I,J))\rangle+\langle\omega(a(I)),\omega(\mathcal{L})\rangle}\varphi(a(I))\mathcal{L}(a(J)) \tag{2.28}$$

$$\varphi[\mathcal{L}](a(\mathrm{I}^{(|\varphi|+l-1)}))\overset{\mathrm{def}}{=}(-1)^{\langle\varphi,\mathcal{L}\rangle}\times \times\sum_{\substack{I+J=\mathrm{I}^{(|\varphi|+l-1)}\\ |I|=|\varphi|,\|J|=l-1}}(-1)^{\langle a((I,J))\rangle}\varphi(\mathcal{L}(a(I)),a(J)),\quad a(I)\in A, \tag{2.29}$$

The following formulas are obtained in the same way as their analogs discussed above:

$$i_a(\varphi[\mathcal{L}])=i_a(\varphi)[\mathcal{L}]+(-1)^{\langle i_a,\varphi\rangle}\varphi[i_a(\mathcal{L})] \tag{2.30}$$

$$\mathcal{L}[\varphi\wedge\mathcal{N}]=(-1)^{\langle\mathcal{L},\widehat{\varphi}\rangle}\varphi\wedge\mathcal{L}[\mathcal{N}] \tag{2.31}$$

$$\psi[\varphi\wedge\mathcal{N}]=(-1)^{\langle\psi,\widehat{\varphi}\rangle}\varphi\wedge\psi[\mathcal{N}] \tag{2.32}$$

$$(\varphi\wedge\mathcal{N})[\mathcal{L}]=\varphi\wedge\mathcal{N}[\mathcal{L}]+(-1)^{\langle\widehat{\mathcal{N}},\mathcal{L})\rangle}\varphi[\mathcal{L}]\wedge\mathcal{N} \tag{2.33}$$

$$i_a(\varphi\wedge\mathcal{N})=i_a(\varphi)\wedge\mathcal{N}+(-1)^{\langle i_a,\widehat{\varphi}\rangle}\varphi\wedge i_a(\mathcal{N}) \tag{2.34}$$

In an obvious way they imply the Leibnitz type rules:

$$[\![\varphi\wedge\mathcal{L},\mathcal{N}]\!]\ =\ \varphi\wedge[\![\mathcal{L},\mathcal{N}]\!]+(-1)^{\langle\widehat{\mathcal{L}},\mathcal{N}\rangle}\varphi[\mathcal{N}]\wedge\mathcal{L} \tag{2.35}$$

$$[\![\mathcal{L},\psi\wedge\mathcal{N}]\!]\ =\ (-1)^{\langle\mathcal{L},\mathcal{N}\rangle}\psi\wedge[\![\mathcal{L},\mathcal{N}]\!]-\ (-1)^{\langle\mathcal{L},\psi\rangle}\psi[\mathcal{L}]\wedge\mathcal{N} \tag{2.36}$$

$$\begin{aligned}(2.37)\qquad [\![\varphi\wedge\mathcal{L},\psi\wedge\mathcal{N}]\!] \;&=\; (-1)^{\langle\mathcal{L},\mathcal{N}\rangle}\varphi\wedge\psi\wedge[\![\mathcal{L},\mathcal{N}]\!]\\ -(-1)^{\langle\mathcal{L},\psi\rangle}\varphi\wedge\psi[\mathcal{L}]\wedge\mathcal{N} + &\;(-1)^{\langle\widehat{\mathcal{L}},\psi\wedge\mathcal{N}\rangle+\langle\varphi,\widehat{\psi}\rangle}\psi\wedge\varphi[\mathcal{N}]\wedge\mathcal{L}\end{aligned}$$

3. Multi-Lie algebra structures

3.1. Main definition.

Definition 3.1. Let integers l,k,r be such that $0\le r\le k<l$. A multilinear map $\mathcal{L}\in \mathrm{Alt}^l_K A$ defines a *Lie algebra structure of type* (l,k,r) (or, (l,k,r)*-structure*) on A if for any $a_1,\ldots,a_k,\ b_1,\ldots b_r\in A$ the following identity holds

$$(3.1)\qquad [\![i_{b_1,\ldots,b_r}(\mathcal{L}), i_{a_1,\ldots,a_k}(\mathcal{L})]\!]=0.$$

Identity (3.1) is called (l,k,r)*-Jacobi identity.* $(l,k,0)$-structures are of a particular interest and will be called (l,k)*-structures.* Denote by $\mathrm{L}^{(n,k,r)}(A)$ (resp. $\mathrm{L}^{(n,k)}(A)$) the set of all (l,k,r)-structures (resp., (l,k)-structures) on A.

Example 3.1. In [28] it was shown that in the nongraded case $(l,l-1,0)$-structures are Filippov–Lie (=Numbu–Lie) structures. Therefore, the $(l,l-1,0)$-Jacobi identity defines graded Filippov–Lie (i.e., Nambu–Lie) structures. Similarly [28], $(2l,1,0)$-structures are graded analogs of $2l$-Lie structures in the sense of Michor–Vinogradov [21].

3.2. Hereditary structures.

It was shown in [28] that if $\mathcal{L}$ is an (l,k)-structure, then $\mathcal{L}_a \overset{\text{def}}{=} i_a(\mathcal{L})$ is an $(l-1,k-1)$-structure for any $a\in A$. The graded analog of this fact is as follow.

Proposition 3.2. *Let $\mathcal{L}$ be (l,k)-Lie algebra, $k>0$. Then for any even $a\in A$, $\mathcal{L}_a$ is an $(l-1,k-1)$-Lie algebra structure called hereditary (with respect to $\mathcal{L}$).*

◀ In fact, let $\mathcal{L}$ be an (l,k)-Lie structure. In view of (2.14) we have

$$(3.2)\qquad i_a([\![\mathcal{L}, i_{a_1,\ldots,a_k}(\mathcal{L})]\!]) = [\![i_a(\mathcal{L}), i_{a_1,\ldots,a_k}(\mathcal{L})]\!] + (-1)^{\langle i_a,\mathcal{L}\rangle}[\![\mathcal{L}, i_{a_1,\ldots,a_k,a}(\mathcal{L})]\!]=0$$

for any $a_1,\ldots,a_k$. If $\;=\;{}_1$ is even, then $[\![\mathcal{L}, i_{a_1,\ldots,a_k,a}(\mathcal{L})]\!]=0$. So,

$$[\![i_a(\mathcal{L}), i_{a,a_2\ldots,a_k}(\mathcal{L})]\!]=0.$$

In other words, $[\![i_a(\mathcal{L}), i_a(\mathcal{L})_{a_2\ldots,a_k}]\!]=0$. ▶

Diagram 1 illustrates interrelations among hereditary structures.

Similarly one can prove the following assertion.

Proposition 3.3. *Let $\mathcal{L}\in\mathrm{L}^{(l,k,r)}(A)$ with $r>0$. Then*

$$\mathcal{L}_a\in\mathrm{L}^{(l-1,k,r-1)}(A)\cap\mathrm{L}^{(l-1,k-1,r)}(A)\cap\mathrm{L}^{(l-1,k-1,r-1)}(A),\ \ \textit{if } k>r,$$
$$\mathcal{L}_a\in\mathrm{L}^{(l-1,k,k-1)}(A)\cap\mathrm{L}^{(l-1,k-1,k-1)}(A),\ \ \textit{if } k=r.$$

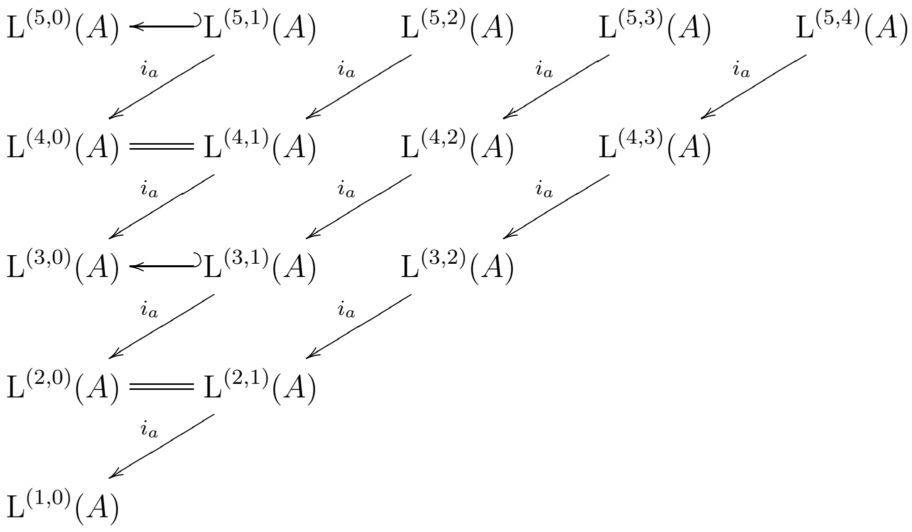

Diagram 1.

3.3. **Compatible structures.**

Definition 3.4. Let $\mathcal{L}, \mathcal{N}$ be (l,k,r)-structures. They are called *compatible* if $\alpha\mathcal{L} + \beta\mathcal{N}$ is also an (l,k,r)-structure for any $\alpha, \beta \in K$.

It results immediately from Definition 3.1 that (l,k,r)-structures $\mathcal{L}$ and $\mathcal{N}$ are compatible iff

$$[\![i_{b_1,\dots,b_r}(\mathcal{L}), i_{a_1,\dots,a_k}(\mathcal{N})]\!] + [\![i_{b_1,\dots,b_r}(\mathcal{N}),_{a_1,\dots,a_k}(\mathcal{L})]\!] = 0 \tag{3.3}$$

takes place for any $a_1,\dots,a_k,b_1,\dots,b_r \in A$. This relation generalizes the well known Magri compatibility condition [18]; (3.3) shows immediately that $\mathcal{L}$ and $\mathcal{N}$ are compatible iff $\mathcal{L}+\mathcal{N}$ is an (l,k,r)-structure.

The following simple fact is very important for various applications.

Proposition 3.5. *Let $\mathcal{L}$ be a multi-Lie algebra structure and $a, b \in A$ be even. Then $\mathcal{L}_a$ and $\mathcal{L}_b$ are compatible multi-Lie structures of the corresponding type as indicated in Propositions* 3.2 *and* 3.3.

◀ It results banally from the fact that $\mathcal{L}_a + \mathcal{L}_b = \mathcal{L}_{a+b}$. ▶

Remark 3.1. The (l,k,r)-Jacobi identity tells that an (l,k,r)-Lie algebra is compatible with itself.

3.4. **Cohereditary structures.** Put $\langle i_{a(I^{(k)})}, \varphi \rangle_j = \sum_{m=1}^{j-1} \langle i_{a_m}, \varphi \rangle$.

Lemma 3.6. *It holds*

$$\begin{aligned} i_{a(\mathrm{I}^{(k)})}(\varphi \wedge \mathcal{L}) \\ = \sum_{j=1}^{k} (-1)^{\langle a(\mathrm{I}^{(k)}), \widehat{\varphi} \rangle_j} \varphi(a(J)) \mathcal{L}_{a(\mathrm{I}^{(k)}-j)} + (-1)^{\langle a(\mathrm{I}^{(k)}), \widehat{\varphi} \rangle} \varphi \wedge \mathcal{L}_{a(\mathrm{I}^{(k)})} \end{aligned} \tag{3.4}$$

◀ The proof is by induction on k. The starting point $k = 1$ is exactly (2.34), while the general induction step is straightforward. ▶

Theorem 3.7. *If* $\varphi \in \mathrm{Alt}^1_K(A, K)$, $\mathcal{L} \in \mathrm{L}^{(n,k)}(A)$, *and* $\varphi[\mathcal{L}] = 0$, *then* $\varphi \wedge \mathcal{L} \in \mathrm{L}^{(n+1,k+1)}(A)$.

◀ It results from Lemma 3.6 that

$$[\![\varphi \wedge \mathcal{L}, i_{a(\mathrm{I}^{(k+1)})}(\varphi \wedge \mathcal{L})]\!] = \sum_{j=1}^{k} (-1)^{\langle a(\mathrm{I}^{(k+1)}, \widehat{\varphi}\rangle_j} \varphi(a(J)) [\![\varphi \wedge \mathcal{L}, \mathcal{L}_{a(\mathrm{I}^{(k+1)}\,-j)}]\!] + (-1)^{\langle a(\mathrm{I}^{(k+1)}, \widehat{\varphi}\rangle} [\![\varphi \wedge \mathcal{L}, \varphi \wedge \mathcal{L}_{a(\mathrm{I}^{(k+1)})}]\!].$$

Now, using formulas (2.35) one obtains

$$[\![\varphi \wedge \mathcal{L}, \mathcal{L}_{a(\mathrm{I}^{(k+1)}\,-j)}]\!] = \varphi \wedge [\![\mathcal{L}, \mathcal{L}_{a(\mathrm{I}^{(k+1)}\,-j)}]\!] + (-1)^{\langle \widehat{\mathcal{L}}, \mathcal{L}(a(\mathrm{I}^{(k+1)}\,-j))\rangle} \varphi[\mathcal{L}_{a(\mathrm{I}^{(k+1)}\,-j)}] \wedge \mathcal{L}.$$

The first term in the right hand side of this relation vanishes since $\mathcal{L} \in \mathrm{L}^{(n,k)}(A)$, while the second one vanishes in view of $\varphi[\mathcal{L}_{a(\mathrm{I}^{(k+1)}\,-j)}] = \varphi[\mathcal{L}]_{a(\mathrm{I}^{(k+1)}\,-j)}$. Now it remains to apply formula (2.37):

$$[\![\varphi \wedge \mathcal{L}, \varphi \wedge \mathcal{L}_{a(\mathrm{I}^{(k+1)})}]\!] = (-1)^{\langle \mathcal{L}, \mathcal{L}_{a(\mathrm{I}^{(k+1)})}\rangle} \varphi \wedge \varphi[\mathcal{L}_{a(\mathrm{I}^{(k+1)})}] \wedge \mathcal{L} - (-1)^{\langle \mathcal{L}, \varphi\rangle} \varphi \wedge \varphi[\mathcal{L}] \wedge \mathcal{L}_{a(\mathrm{I}^{(k+1)})} + (-1)^{\langle \widehat{\mathcal{L}}, \varphi \wedge \mathcal{L}_{a(\mathrm{I}^{(k+1)})}\rangle + \langle \varphi, \widehat{\varphi}\rangle} \varphi \wedge \varphi \wedge [\![\mathcal{L}, \mathcal{L}_{a(\mathrm{I}^{(k+1)})}]\!] = 0.$$

▶

This theorem allows us to obtain examples of ordinary structures.

Definition 3.8. A multi-Lie structure of the form $\varphi \wedge \mathcal{L}$ is called *cohereditary* with respect to $\mathcal{L}$.

It easy to see that any two cohereditary structures $\varphi \wedge \mathcal{L}$ and $\psi \wedge \mathcal{L}$ are compatible. Similarly, cohereditary structures $\varphi \wedge \mathcal{L}$ and $\varphi \wedge \mathcal{N}$ are compatible if multi-Lie structures $\mathcal{L}, \mathcal{N} \in \mathrm{Alt}^l_K A$ are compatible.

Interrelations among cohereditary structures are illustrated by Diagram 2.

Theorem 3.9. *Let*

1. $\mathcal{L} \in \mathrm{L}^{(l,k,r-1)}(A) \cap \mathrm{L}^{(l,k-1,r)}(A) \cap \mathrm{L}^{(l,k-1,r-1)}(A)$, $k > r$,
   or
   $\mathcal{L} \in \mathrm{L}^{(l,k,k-1)}(A) \bigcap \mathrm{L}^{(l,k-1,k-1)}(A)$,
2. $\varphi \in \mathrm{Alt}^1_K(A, K)$ $\Upsilon \varphi[\mathcal{L}] = 0$.

Then $\varphi \wedge \mathcal{L} \in \mathrm{L}^{(l+1,k,r)}(A)$.

◀ From Lemma 3.6 one finds easily that

$$[\![i_{b(\mathrm{I}^{(r)})}(\varphi \wedge \mathcal{L}), i_{a(\mathrm{I}^{(k)})}(\varphi \wedge \mathcal{L})]\!] =$$

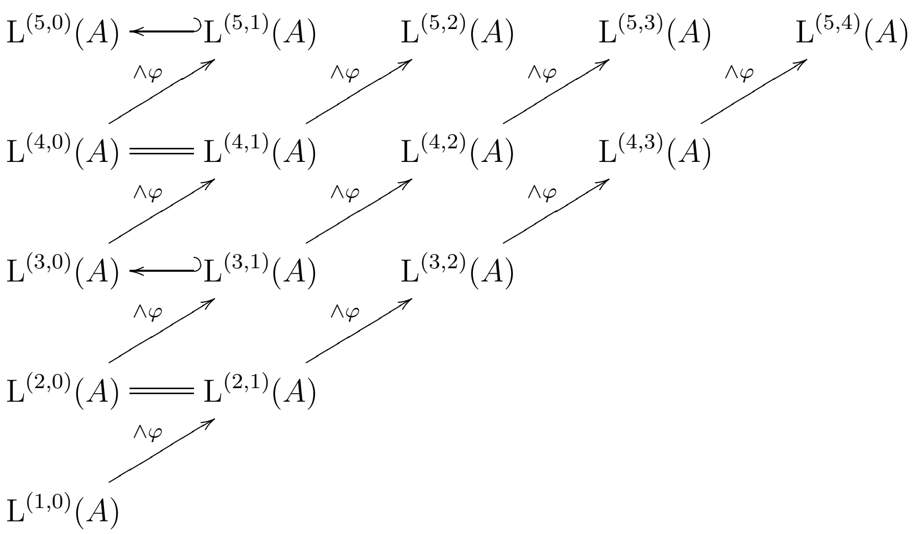

Diagram 2.

$$\begin{aligned}
&\sum_{m=1}^{r}\sum_{j=1}^{k}(-1)^{\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle_m+\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle_j}[\![\varphi(b_m)\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\varphi(a(J))\mathcal{L}_{a(\mathrm{I}^{(k)}-j)}]\!]\\
&\quad+\sum_{m=1}^{r}(-1)^{\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle_m+\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle}[\![\varphi(b_m)\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\varphi\wedge\mathcal{L}_{a(\mathrm{I}^{(k)})}]\!]\\
&\quad+\sum_{j=1}^{k}(-1)^{\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle_j+\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle}[\![\varphi\wedge\mathcal{L}_{b(\mathrm{I}^{(r)})},\varphi(a(J))\mathcal{L}_{a(\mathrm{I}^{(k)}-j)}]\!]\\
&\qquad+(-1)^{\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle+\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle}[\![\varphi\wedge\mathcal{L}_{b(\mathrm{I}^{(r)})},\varphi\wedge\mathcal{L}_{a(\mathrm{I}^{(k)})}]\!]=\\
&\sum_{m=1}^{r}\sum_{j=1}^{k}(-1)^{\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle_m+\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle_j}\varphi(b_m)\varphi(a(J))[\![\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\mathcal{L}_{a(\mathrm{I}^{(k)}-j)}]\!]\\
&\quad+\sum_{m=1}^{r}(-1)^{\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle_m+\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle}\varphi(b_m)[\![\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\varphi\wedge\mathcal{L}_{a(\mathrm{I}^{(k)})}]\!]\\
&\quad+\sum_{j=1}^{k}(-1)^{\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle_j+\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle}\varphi(a(J))[\![\varphi\wedge\mathcal{L}_{b(\mathrm{I}^{(r)})},\mathcal{L}_{a(\mathrm{I}^{(k)}-j)}]\!]\\
&\qquad\qquad+(-1)^{\langle b(\mathrm{I}^{(r)}),\widehat{\varphi}\rangle+\langle a(\mathrm{I}^{(k)}),\widehat{\varphi}\rangle}[\![\varphi\wedge\mathcal{L}_{b(\mathrm{I}^{(r)})},\varphi\wedge\mathcal{L}_{a(\mathrm{I}^{(k)})}]\!].
\end{aligned}$$

It is sufficient to show that each term in the last expression vanishes. Due to the fact that $\mathcal{L}\in\mathrm{L}^{(l,k-1,r-1)}(A)$ this is obvious for terms $[\![\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\mathcal{L}_{a(\mathrm{I}^{(k)}-j)}]\!]$. Then in view of (2.36) one has

$$\begin{aligned}
&[\![\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\varphi\wedge\mathcal{L}_{a(\mathrm{I}^{(k)})}]\!]\\
&\qquad=(-1)^{\langle\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\mathcal{L}_{a(\mathrm{I}^{(k)})}\rangle}\varphi\wedge[\![\mathcal{L}_{b(\mathrm{I}^{(r)}-m)},\mathcal{L}_{a(\mathrm{I}^{(k)})}]\!]
\end{aligned}$$

$$= (-1)^{\langle \mathcal{L}_{b(\mathrm{I}^{(r)}-m)}, \varphi\rangle} \varphi[\mathcal{L}_{b(\mathrm{I}^{(r)}-m)}] \wedge \mathcal{L}_{a(\mathrm{I}^{(k)})}.$$

The first term in this expression vanishes since $\mathcal{L} \in \mathrm{L}^{(l,k,r-1)}(A)$, while the second one vanishes in virtue of

$$\varphi[\mathcal{L}_{b(\mathrm{I}^{(r)}-m)}] = \varphi[i_{b(\mathrm{I}^{(r)}-m)}\mathcal{L}] = i_{b(\mathrm{I}^{(r)}-m)}\varphi[\mathcal{L}]. \tag{3.5}$$

Similarly, formula (2.35) shows that $[\![\varphi \wedge \mathcal{L}_{b(\mathrm{I}^{(r)})}, \mathcal{L}_{a(\mathrm{I}^{(k)}-j)}]\!] = 0$. Finally from (2.37) one sees that

$$\begin{aligned}
&[\![\varphi \wedge \mathcal{L}_{b(\mathrm{I}^{(r)})}, \varphi \wedge \mathcal{L}_{a(\mathrm{I}^{(k)})}]\!] \\
&\quad = (-1)^{\langle \mathcal{L}_{b(\mathrm{I}^{(r)})}, \mathcal{L}_{a(\mathrm{I}^{(k)})}\rangle} \varphi \wedge \varphi \wedge [\![\mathcal{L}_{b(\mathrm{I}^{(r)})}, \mathcal{L}_{a(\mathrm{I}^{(k)})}]\!] \\
&\qquad - (-1)^{\langle \mathcal{L}_{b(\mathrm{I}^{(r)})}, \varphi\rangle} \varphi \wedge \varphi[\mathcal{L}_{b(\mathrm{I}^{(r)})}] \wedge \mathcal{L}_{a(\mathrm{I}^{(k)})} \\
&\qquad\qquad + (-1)^{\langle \widehat{\mathcal{L}_{b(\mathrm{I}^{(r)})}}, \varphi \wedge \mathcal{L}_{a(\mathrm{I}^{(k)})}\rangle + \langle \varphi, \widehat{\varphi}\rangle} \varphi \wedge \varphi[\mathcal{L}_{a(\mathrm{I}^{(k)})}] \wedge \mathcal{L}_{b(\mathrm{I}^{(r)})}.
\end{aligned}$$

The first term in this expression vanishes banally. The remaining two ones are equal to zero in view of (3.5). ▶

Assume now that $\mathcal{L} \in \mathrm{L}^{(l,k,r)}(A)$, $k > 0$, $\varphi \in \mathrm{Alt}^1_K(A, K)$, $\varphi[\mathcal{L}] = 0$, and $c \in A$ is even. Then, as it results from Propositions 3.2, 3.3 and Theorems 3.7, 3.9, $\varphi \wedge \mathcal{L}_c$ is an (l, k, r)-structure.

Theorem 3.10. *Under these assumptions, (l, k, r)-structures $\mathcal{L}$ and $\varphi \wedge \mathcal{L}_c$ are compatible.*

◀ By using Lemma 3.6, the compatibility condition of the structures $\mathcal{L}$ and $\varphi \wedge \mathcal{L}_c$ can be rewritten as follows:

$$\begin{aligned}
&[\![i_{b(\mathrm{I}^{(r)})}\mathcal{L}, i_{a(\mathrm{I}^{(k)})}(\varphi \wedge \mathcal{L}_c)]\!] + [\![i_{b(\mathrm{I}^{(r)})}(\varphi \wedge \mathcal{L}_c), i_{a(\mathrm{I}^{(k)})}\mathcal{L}]\!] \\
&\quad = [\![i_{b(\mathrm{I}^{(r)})}\mathcal{L}, \sum_{j=1}^{k} (-1)^{\langle a(\mathrm{I}^{(k)}), \widehat{\varphi}\rangle_j} \varphi(a(J)) \mathcal{L}_{c, a(\mathrm{I}^{(k)}-j)} \\
&\qquad\qquad + (-1)^{\langle a(\mathrm{I}^{(k)}), \widehat{\varphi}\rangle} \varphi \wedge \mathcal{L}_{c, a(\mathrm{I}^{(k)})}]\!] \\
&\qquad + [\![\sum_{j=1}^{r} (-1)^{\langle b(\mathrm{I}^{(r)}), \widehat{\varphi}\rangle_j} \varphi(b(J)) \mathcal{L}_{c, b(\mathrm{I}^{(r)}-j)} \\
&\qquad\qquad + (-1)^{\langle b(\mathrm{I}^{(r)}), \widehat{\varphi}\rangle} \varphi \wedge \mathcal{L}_{c, b(\mathrm{I}^{(r)})}, i_{a(\mathrm{I}^{(k)})}\mathcal{L}]\!] \\
&\quad = \sum_{j=1}^{k} (-1)^{\langle a(\mathrm{I}^{(k)}), \widehat{\varphi}\rangle_j} \varphi(a(J)) [\![i_{b(\mathrm{I}^{(r)})}\mathcal{L}, \mathcal{L}_{c, a(\mathrm{I}^{(k)}-j)}]\!] \\
&\qquad + (-1)^{\langle a(\mathrm{I}^{(k)}), \widehat{\varphi}\rangle} [\![i_{b(\mathrm{I}^{(r)})}\mathcal{L}, \varphi \wedge \mathcal{L}_{c, a(\mathrm{I}^{(k)})}]\!] \\
&\quad + \sum_{j=1}^{r} (-1)^{\langle b(\mathrm{I}^{(r)}), \widehat{\varphi}\rangle_j} \varphi(b(J)) [\![\mathcal{L}_{c, b(\mathrm{I}^{(r)}-j)}, i_{a(\mathrm{I}^{(k)})}\mathcal{L}]\!] \\
&\qquad\qquad + (-1)^{\langle b(\mathrm{I}^{(r)}), \widehat{\varphi}\rangle} [\![\varphi \wedge \mathcal{L}_{c, b(\mathrm{I}^{(r)})}, i_{a(\mathrm{I}^{(k)})}\mathcal{L}]\!]
\end{aligned}$$

The summands forming both summations in the last expression are equal to zero by definition of (l,k,r)-structures. It remains to show that two single terms in this expressions vanish as well. In view of (2.36) the first of them can be represented as follows:

$$\begin{aligned}[][i_{b(\mathrm{I}^{(r)})}\mathcal{L}, \varphi \wedge \mathcal{L}_{c,a(\mathrm{I}^{(k)})}]] \\ = (-1)^{\langle i_{b(\mathrm{I}^{(r)})}\mathcal{L}, \mathcal{L}_{c,a(\mathrm{I}^{(k)})}\rangle} \varphi \wedge [[i_{b(\mathrm{I}^{(r)})}\mathcal{L}, \mathcal{L}_{c,a(\mathrm{I}^{(k)})}]] \\ - (-1)^{\langle i_{b(\mathrm{I}^{(r)})}, \varphi\rangle} \varphi[i_{b(\mathrm{I}^{(r)})}\mathcal{L}] \wedge \mathcal{L}_{c,a(\mathrm{I}^{(k)})}.\end{aligned}$$

As above, the first term in this expression vanishes since $\mathcal{L}$ is an (l,k,r)-structure, while the second one is equal to zero in virtue of

$$\varphi[i_{b(\mathrm{I}^{(r)})}\mathcal{L}] = i_{b(\mathrm{I}^{(r)})}(\varphi[\mathcal{L}]) = 0.$$

The fact that

$$[[\varphi \wedge \mathcal{L}_{c,b(\mathrm{I}^{(r)})}, i_{a(\mathrm{I}^{(k)})}\mathcal{L}]] = 0$$

is proved similarly by using (2.35) instead of (2.36). ▶

3.5. **Poisson multialgebras.** The following notion is a multiple graded analog of that of a Poisson algebra.

Definition 3.11. Let A be a graded commutative K-algebra. A map $\mathcal{L} \in \mathrm{Alt}^n_K A$ is called a *Poisson* (l,k,r)*-structure* on A if

1. $\mathcal{L}$ is an (l,k,r)-structure;
2. $\mathcal{L}$ is a multiderivation of A, i.Ξ,

$$\begin{aligned}\mathcal{L}(aa', a_2, \ldots, a_n) = (-1)^{\langle a,\mathcal{L}\rangle} a\mathcal{L}(a', a_2, \ldots, a_n) \\ + (-1)^{\langle a',\mathcal{L}\rangle + \langle \omega(a),\omega(a')\rangle} a'\mathcal{L}(a, a_2, \ldots, a_n).\end{aligned}$$

Observe that for $G = \mathbb{Z}$, $A = A^0 = C^\infty(M)$, M being a smooth manifold, a Poisson $(2,1)$-structure on A is exactly a standard Poisson structure on M. By this reason it is natural to look for a similar geometrical interpretation for general Poisson (l,k,r)-structures as Poisson (l,k,r)-manifolds. This is another task we hope to consider in a subsequent work. In the nongraded case the structure of Poisson $(n,1)$- and $(n,n-1)$-manifolds was studied in [1, 3, 4, 6, 8, 10, 13, 19, 22, 26, 27].

References

[1] D. Alexeevsky and P. Guha, *On Decomposability of Nambu–Poisson Tensor*, Acta Mathematica Universitatis Comenianae, **65** (1966), 1–9.

[2] J.A. de Azcarraga and J.C. Perez Bueno, *Higher-order simple Lie algebras*, Commun. Math. Phys. **184** (1997), 669–681.

[3] J.A. de Azcárraga, A.M. Perelomov, J.C. Pérez Bueno, *New generalized Poisson structures*, Preprint FTUV 96-1, IFIC 96–1.

[4] J.A. de Azcárraga, A.M. Perelomov, J.C. Pérez Bueno, *The Schouten-Nijenhuis bracket, cohomology and generalized Poisson structure*, J. Phys. **A29** (1996), 7993–8009.

[5] R. Chatterjee and L. Takhtajan, *Aspects of classical and quantum mechanics*, Let. Math. Phys. **37** (1996), 475–482.

[6] J.-P. Dufour and N.T. Zung, *Linearization of Nambu structures*, Composito Mathematica, **117**(1) (1999), 83–105.

[7] V.T. Filippov, *n-ary Lie algebras*, Sibirskii Math. J. **24** (1985), 126–140 (in Russian).

[8] Ph. Gautheron, *Some remarks concerning Nambu mechanics*, Lett. Math. Phys., **37** (1996), 103–116.
[9] V. Gnedbaye, *Operads of k-ary algebras*, Operads: Proceedings of Renaissance Conferences (J.-L. Loday, J. Stasheff, and A.A. Voronov, eds.), Contemporary Mathematics, vol. 202, Amer. Math. Soc., 1996, 83–114.
[10] J. Grabowski and G. Marmo, *On Filippov algebroids and multiplicative Nambu–Poisson structures*, Diff. Geom. Appl., **12**(2000), 35–50.
[11] J. Grabowski and G. Marmo, *Remarks on Nambu–Poisson and Nambu–Jacobi brackets*, Preprint math@xxx.lahl.gov DG/9902128.
[12] P. Hanlon and M.L. Wachs, *On Lie k-algebras*, Adv. in Math. **113** (1995), 206–236.
[13] R. Ibáñez, M. de León, J.C. Marrero, and D.M. de Diego, *Dynamics of generalized Poisson and Nambu–Poisson brackets*, J. Math. Phys., **38** (1997), 2332–2344.
[14] Sh.M. Kasymov, *On the n-Lie algebras theory*, Algebra and Logic, **26**, No 3 (1987), 277–297 (in Russian).
[15] Sh.M. Kasymov, *On nil-elements and nil-subsets in n-Lie algebras*, Sibirskii Math. J. **32**, No 6 (1991), 77–80 (in Russian).
[16] Sh.M. Kasymov, *Contingency of Cartan subalgebras in n-Lie algebras*, Dokl. RAN, **345**, No 1 (1996), 15–18 (in Russian).
[17] T. Lada and J.D. Stasheff, *Introduction to sh Lie algebras for physicists*, Intern'l J. Theor. Phys. **32** (1993) 1087–1103.
[18] F. Magri, J. Math.Phys. **19**, 1156 (1978).
[19] G. Marmo, G. Vilasi, A. Vinogradov, *The local structure of n-Poisson and n-Jacobi manifolds*, J.Geom.Phys. **25** (1998), 141–182.
[20] P. Michor and A.M. Vinogradov, Unpublished notes, 1990, Vienne.
[21] P. Michor and A.M. Vinogradov, *n-ary Lie and associative algebras*, Rendiconti del Seminario Matematico, Universita' e Politecnico di Torino, **53**, 4 (1996), 373–392; also ESI preprint, December 1996, E-print math.QA/9801087.
[22] N. Nakanishi, *On Nambu–Poisson manifolds*, Rev. Math. Phys., **10** (1998), 499–510.
[23] Y. Nambu, *Generalized Hamiltonian dynamics*, Phys. Rev. **D7**, (1973), 2405–2412.
[24] D. Sahoo and M.C. Valsakumar, *Nambu mechanics and its quantization*, Phys. Rev. A, **46** (1992), 4410–4412.
[25] M. Schlesinger and J.D. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl. Algebra **38** (1985), 313–322.
[26] L. Takhtajan, *On foundation of generalized Nambu mechanics*, Comm. Math. Physics, **160**, (1994), 295–315.
[27] I. Vaisman, *Nambu–Lie Groups*, Journal of Lie Theory, **10** (2000), No. 1, 181–194.
[28] A.M. Vinogradov and M.M. Vinogradov, *On multiple generalizations of Lie algebras and Poisson manifolds*, Cont. Math. **219** (1998), 273–287.

Università di Salerno, Dipartimento di Matematica ed Informatica, via S. Allende, I-84081 Baronissi (SA), Italy and Istituto Nazionale di Fisica Nucleare, sezione di Napoli-Salerno, Italy
E-mail address: vinograd@unisa.it

Diffiety Institute and Institute of Economic Forecasting, 47 Nakhimovsky prosp., Moscow, 117418, Russia
E-mail address: vin@diffiety.ac.ru