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Abstract. A general geometric method for studying mechanical systems whose phase space may undergo an abrupt reduction is proposed. This method is based on a variant of the Transition Principle, proposed for the first time in [3] for elastic collisions.

1. Introduction

Mechanical phenomena involving discontinuities of motion, such as impacts or refractions, are traditionally considered to fall outside the realm of analytical mechanics, or at least to be a very marginal part of it. In fact, usually, most of the modern books on classical mechanics do not contain even a minimal reference to this subject. The usual approach to such problems consists in making some reasonable assumptions on the physical nature of impulsive forces and deducing from them the "conservation laws" necessary to calculate the velocity jumps after the impact. In this kind of approach the symplectic structure of the phase space is not taken into consideration, and consequently the powerful techniques of Hamiltonian formalism cannot be applied. Furthermore, such an approach works only when a sufficient number of conservation laws for the considered system is known, which is not always the case.

A completely different method not presenting such drawbacks, based on the so-called *Transition Principle,* was proposed for the first time in [3]. This is a prescription on the behaviour of dynamical systems when a drastic change in one or more of the basic ingredients composing the Hamiltonian formalism occurs. For instance, it could be a discontinuity of the Hamiltonian, a reduction of the phase or configuration space, "switching on (off)" of some kind of constraints, etc. Things of this kind occur in the mathematical description of such phenomena as:

1. Reflection and refraction processes for light rays and electromagnetic waves at the separation surface between two different media
2. Collisions between rigid bodies
3. Explosions or disintegration processes

   etc.

It is remarkable that the transition principle can be applied to such a wide range of different physical situations. For instance, reflection and refraction phenomena are described by means of Hamiltonian functions which are discontinuous along a hypersurface Γ of the phase space. In

this case (see [3], [6]) the transition principle prescribes how a moving phase point reaching Γ can leave it. Thus, the degrees of freedom of the system do not change after an impact with Γ.
A different situation occurs if one considers non elastic collisions when the dimension of the phase space changes abruptly after the impact. This happens, for instance, in totally inelastic collisions between rigid bodies, or in the sudden connection between two vehicles (think of the coupling of two spacecrafts). In this case the system is forced to move along a submanifold of the original phase space.
The version of the transition principle, allowing one to describe the latter situation is given below. It is worth stressing that, though the principle needs slightly different formulations depending on the physical context to which it is applied, its philosophy is quite simple and natural. Namely, it states that whenever a Hamiltonian system reaches a critical situation, *it escapes from it by jumping along a suitable characteristic leaf.*

2. The Transition Principle for systems with reducing phase space.

In order to make the geometrical ideas more transparent, we will start by describing the transition principle at a general symplectic level. Then, we will pass to the configuration level, i.e. to the case when the considered symplectic manifold is the cotangent bundle of the configuration space of a mechanical system. This is the most important case for applications, and we will show how inelastic collisions can be included in it.

2.1 Symplectic scheme. First, we need the following basic definition. Let (Φ^{2n}, Ω) be a symplectic manifold, and let $P^m \subset \Phi$ be an m-dimensional submanifold in it, $m \leq 2n$. Assume the restriction $\Omega|_P$ of the canonical 2-form on P to be of a constant rank $2k \leq m$. Then the map

$$x \longmapsto Ker\ (\Omega|_P)_x\,, \tag{1}$$

with $Ker\ (\Omega|_P)_x = \{\xi \in T_x(P) |\ \Omega_x(\xi, \eta) = 0, \forall \eta \in T_x(P)\}$, is an $(m-2k)$-dimensional integrable distribution on P. Hence, a foliation $\mathcal{F}$ of P corresponds to (1). The leaves of $\mathcal{F}$ are called *characteristics* of P.

Definition. A *symplectic reduction* of P is any submanifold $R \subset P$ transversal to the characteristic foliation and intersecting all its leaves.

Note that $\Omega|_R$ is non-degenerate, so that $\widetilde{\Phi} = (R, \Omega|_R)$ is a symplectic manifold.

Remark. Note that the term "symplectic reduction" in the sense of the above definition has a slightly different meaning with respect to the

commonly used one, which refers to the canonical symplectic structure on the quotient manifold $P/\mathcal{F}$ (if the latter exists). Obviously, the two meanings are locally equivalent. But a reduction in our sense may exist even if $P/\mathcal{F}$ does not globally exist.

The general symplectic scheme is now described as follows. Let $\Gamma^{2n-1} \subset \Phi$ be a hypersurface, and suppose that on it is given an m-dimensional foliation $\mathcal{F} = \{P^m_\alpha\}_{\alpha \in A}$, with A being a $(2n-1-m)$-dimensional parameter. We assume that the restriction Ω_α of Ω to any leaf P_α has a constant rank $2k$ and that k does not depend on α. Finally, suppose that for any $\alpha \in A$ a symplectic reduction $R_\alpha \subset P_\alpha$ is chosen in such a way that R_α depends (locally) smoothly on α.

Remark. In the above scheme one can substitute Φ with a closed regular domain $W \subset \Phi$ and Γ with ∂W.

Now, consider a Hamiltonian system on Φ, with the Hamiltonian $H \in C^\infty(\Phi)$. Let $X_H \in D(\Phi)$ be the corresponding Hamiltonian vector field, i.e.

$$X_H \lrcorner \Omega = -dH$$

Suppose that a trajectory of X_H reaches Γ at a point $\overline{x} \in P_{\overline{\alpha}}$ and that after this "impact" the system is constrained to move along the corresponding symplectic reduction $R_{\overline{\alpha}}$ under the control of the Hamiltonian $\overline{H} = H_{\overline{\alpha}} = H|_{R_{\overline{\alpha}}}$. The problem to be solved is: from which point $x^* \in R_{\overline{\alpha}}$ does the reduced system begin its motion along $R_{\overline{\alpha}}$? In other words, how can one determine the jump in the passage from the original Hamiltonian system to the reduced one? The answer is given by the transition principle.

Transition Principle. *Suppose that a trajectory γ of X_H, $H \in C^\infty(\Phi)$, reaches the critical hypersurface Γ at a point $\overline{x} \in P_{\overline{\alpha}}$, and let $\beta_{\overline{x}}$ be the characteristic leaf of $P_{\overline{\alpha}}$ passing through $\overline{x}$. Then γ can be prolonged starting from any point $x^* \in R_{\overline{\alpha}} \cap \beta_{\overline{x}}$ as a trajectory of the reduced Hamiltonian system $X_{\overline{H}}$, with $\overline{H} = H|_{R_{\overline{\alpha}}} \in C^\infty(R_{\overline{\alpha}})$.*

In the scheme above there are three basic "ingredients", namely: the critical hypersurface Γ, the foliation $\mathcal{F}$ and the corresponding family $\{P_\alpha\}$ of symplectic reductions. Such data must be extracted from the physical context to which the transition principle is applied. As we now shall see, in the case of inelastic collisions the first two data emerge naturally from the geometry of the configuration space.

2.2 Configuration scheme. Let $\Phi = T^*(M)$ with M being the configuration space of a mechanical system, $\dim M = n$. Then the symplectic structure on Φ is given by $\Omega = d\rho$ with $\rho \in \Lambda^1(\Phi)$ being the Liouville form.

Denote by $\pi : T^*(M) \to M$ the canonical projection. Given a submanifold $K \subset M$, $dim\, K = k < n$, let

$$T^*_K(M) \stackrel{def}{=} \pi^{-1}(K) = \underset{q\in K}{\cup} T^*_q(M)$$

Denote by Ω_K the restriction of Ω to $T^*_K(M)$. Then Ω_K has constant rank $2k$ and $Ker_x\,\Omega_K \subset T_x\left(T^*_{\pi(x)}(M)\right)$ for any $x \in T^*_K(M)$. Namely, it is easy to see that

$$Ker_x\,\Omega_K = \alpha_x\left(Ann\, T_{\pi(x)}(K)\right),$$

with $Ann\, T_{\pi(x)}(K) = \left\{\phi \in T^*_{\pi(x)}(M) \,\middle|\, \phi\left(T_{\pi(x)}(K)\right) = 0\right\}$ being the annihilator of $T_{\pi(x)}(K)$ and

$$\alpha_x : T^*_{\pi(x)}(M) \to T_x\left(T^*_{\pi(x)}(M)\right) \subset T_x(\Phi)$$

being the natural identification of the vector space $T^*_{\pi(x)}(M)$ with its tangent space at point x. Therefore, the characteristic leaves of $T^*_K(M)$ are the affine subspaces of fibres $T^*_q(M)$ which are parallel to $Ann\, T_q(K)$, $q \in K$.
In terms of local coordinates the above situation is described as follows. Let K be locally described by equations

$$f_i(q) = 0, \qquad i = 1, ..., n-k\ , \tag{2}$$

where $q = (q_1, ..., q_n)$ is a local chart on M and the functions $f_i \in C^\infty(M)$ are supposed to be such that the differentials $d_q f_1, ..., d_q f_{n-k}$ are linearly independent at any point $q \in K$. These differentials span $Ann\, T_q(K)$. So, denoting with (q, p) the natural coordinates on $\Phi = T^*(M)$, the characteristic leaf passing through a point $\overline{x} \equiv (\overline{q}, \overline{p}) \in T^*_K(M)$ is described by parametric equations

$$\begin{aligned} q &= \overline{q} \\ p &= \overline{p} + J^T\ \lambda, \end{aligned} \tag{3}$$

with $J = (\partial f/\partial q)_{q=\overline{q}} = \|\partial f_i/\partial q_j\|_{q=\overline{q}}$ being the $(n-k)\times n$ Jacobian matrix associated with equations (2) and $\lambda = (\lambda_1, ..., \lambda_{n-k})^T \in \mathbf{R}^{n-k}$ being the column vector of affine parameters.

Now, let $N \subset M$ be a hypersurface (also, one can assume, as in the general symplectic case, that M has a boundary and take $N = \partial M$). Then $\Gamma = T^*_N(M)$ is a hypersurface of Φ. Suppose that on N is given a regular k-dimensional foliation $\widetilde{\mathcal{F}} = \{S_\alpha\}_{\alpha\in A}$. Then the family $\mathcal{F} = \{P_\alpha\}_{\alpha\in A}$, with $P_\alpha = T^*_{S_\alpha}(M)$, is a regular foliation of Γ of dimension $m = n + k$.
Suppose that at an instant $\overline{t}$ the dynamical system reaches N at a point $\overline{q} \in S_{\overline{\alpha}}$ and that afterwards the configuration space abruptly reduces to $S_{\overline{\alpha}}$. Then, at the same instant the phase space reduces to $T^*(S_{\overline{\alpha}})$

and, as before, the problem is to find the phase p^* of the system "immediately after" the impact with N. The transition principle answers this question, provided a family $\{R_\alpha\}$ of symplectic reductions for the leaves of $\mathcal{F}$ is specified such that any R_α is naturally identified with the reduced phase space $T^*(S_\alpha)$. Below it will be shown how such a natural reduction is associated with Hamiltonian systems of mechanical type. But, before, we explain in what sense inelastic collisions, as well as many other impulsive phenomena involving an abrupt reduction of the degrees of freedom, are included in our geometrical scheme.

Main example (totally inelastic collisions). To fix the ideas, let us consider the case of two rigid bodies C_1, C_2. Suppose that at a certain instant they collide in a totally inelastic way, i.e. that after the collision they bind together so as to form a unique rigid body $C = C_1 \cup C_2$. Denote by M the n-dimensional, $n = 12$, configuration space of the system before the collision: it is an open domain of $(R^3 \times SO(3)) \times (R^3 \times SO(3))$. Let $N = \partial M$ be the boundary of M. Its points correspond to the states of the system in which the two bodies touch each other. Obvioususly, N is invariant under the natural effective action $\mathcal{R}$ of the group of proper isometries of the Euclidean space on M. Thus, the orbits of $\mathcal{R}$ in N form a regular foliation $\widetilde{\mathcal{F}} = \{S_\alpha\}$ of N, whose leaves are diffeomorphic to $R^3 \times SO(3)$. So, in this case $k = 6$.
Now, suppose $\widetilde{\mathcal{F}}$ to be reducible. Then, the quotient manifold $N/\widetilde{\mathcal{F}}$ may be described in the following way. Consider the manifold $\widetilde{A}$ of glueings of ∂C_1 and ∂C_2. By a *glueing* we mean a triple (P_1, P_2, ϕ) with $P_i \in \partial C_i$ and $\phi : T_{P_1}(\partial C_1) \to T_{P_2}(\partial C_2)$ being an orientation-preserving isometry. There is a natural fibration $\mu : \widetilde{A} \to \partial C_1 \times \partial C_2$, $(P_1, P_2, \phi) \mapsto (P_1, P_2)$, whose standard fibre is S^1. It is easy to see that the manifold of all rigid bodies that may be formed by glueing together C_1 and C_2 is a closed domain A in $\widetilde{A}$ whose boundary generally has some singularities. Obviously, A coincides with $N/\widetilde{\mathcal{F}}$ and may be taken as the parameter space for the foliation:

$$\widetilde{\mathcal{F}} = \{S_\alpha\}_{\alpha \in A}$$

In terms of local coordinates, A is described as follows. Let (x_1,x_2) and (y_1, y_2) be local coordinates on ∂C_1 and ∂C_2, respectively. Denote by g_i be the natural Riemannian metric on ∂C_i. Then a local chart on A is given by (x, y, B) with B being the 2×2 matrix which describes ϕ with respect to the bases $(\partial/\partial x_1, \partial/\partial x_2)$, $(\partial/\partial y_1, \partial/\partial y_2)$. So, B must satisfy the condition

$$B^t G_2(y) B = G_1(x),$$

with B^t being the transposed matrix of B and $G_1(x), G_2(y)$ being the matrices of g_1, g_2, respectively.

Clearly, the above discussion can be repeated word by word for other, more general kinds of "collisions". Let us consider the following two examples

Example 1. Suppose that C_1 and C_2 collide at an instant $\overline{t}$, and let $\alpha \equiv (P_1, P_2, \phi)$ be the corresponding glueing. Suppose that "immediately after" $\overline{t}$ the system is subjected to the following holonomic constraints: i) $P_1(t) \equiv P_2(t)$, ii) $T_{P_1(t)}(\partial C_1) \equiv T_{P_2(t)}(\partial C_2)$ $\forall t > \overline{t}$. In other words, after the impact the two bodies are compelled to remain in contact and rotate independently from each other around the axis perpendicular to the common tangent plane. In this case the foliation on N is $\widehat{\mathcal{F}} = \left\{\Sigma_{(P_1,P_2)} \times (R^3 \times SO(3))\right\}_{(P_1,P_2)\in\partial C_1\times\partial C_2}$ with $\Sigma_{(P_1,P_2)} = \{(P_1, P_2, \phi) \in A\}$. In this case the foliation is of dimension $k = 7$.

Example 2. Suppose that after the impact C_1 and C_2 are constrained to slide over each other without friction. This situation corresponds to the "trivial foliation" $\{N\}$ of N.

In both these cases the only difference with the main example is the choice of the foliation on N. So, our general symplectic scheme includes practically all the cases when some holonomic constraints cause an abrupt reduction of the configuration space of the system.

2.3 Choice of the reduction. It remains to fix in any leaf P_α of $\mathcal{F}$ a symplectic reduction R_α. As we observed above, since such a reduction must be the phase space of the system after the impact, the only reasonable choice is $R_\alpha = T^*(S_\alpha)$. First of all, note that for a given submanifold $W \subset M$ there exists no natural immersion $T^*(W) \subset T^*(M)$. On the other hand, such an embedding is associated naturally with a (pseudo-)Riemannian metrics G on M. Namely, we define

$$i_G : T^*(W) \to T^*(M)$$

by putting

$$i_G|_{T^*_q(W)} = \sigma_q^* : T^*_q(W) \to T^*_q(M) \subset T^*(M),$$

with σ_q^* being the dual to the orthogonal (with respect to G_q) projection $\sigma_q : T_q(M) \to T_q(W)$. Now, a Hamiltonian function H of a mechanical type is of the form

$$H(q,p) = \widehat{T}(q,p) + V(q), \tag{4}$$

where $\widehat{T} \in C^\infty(T^*(M))$ is the kinetic energy of the system expressed in terms of momenta, while $V \in C^\infty(M)$ is the potential energy. It is

easy to see that, if

$$T(q,v) = \sum_{i,j=1}^{n} g_{ij}(q)\, v_i v_j = v^t G v, \tag{5}$$

with $G(q) = G_q = \|g_{ij}(q)\|$ Riemannian metric on M, is the usual expression of the kinetic energy in terms of velocities, then

$$\widehat{T}(q,p) = \frac{1}{4} p^t G_q^{-1} p$$

So, in the case of a Hamiltonian of mechanical type, it is natural to use the corresponding "kinetic" tensor T to embed $T^*(W)$ in $T^*(M)$. It is worth noticing that the image of $T^*(W)$ in such an embedding coincides with $\mathcal{L}(T(W))$, where

$$\mathcal{L} : T(M) \to T^*(M)$$

is the Legendre map associated with the Lagrangian function

$$L(q,v) = T(q,v) - V(q)$$

describing the system in the tangent bundle. In terms of local coordinates $\mathcal{L}$ is expressed by formulae

$$\begin{aligned} q &= q \\ p &= 2\, G v \end{aligned} \tag{6}$$

For any submanifold $W \subset M$, put $\widetilde{T}^*(W) = i_G(T^*(W))$. Denote by $\overline{q} \in S_{\overline{\alpha}} \subset N$ an impact configuration of the system. Let $\overline{p}$ be the impulse "immediately before" the collision. In general, the point $\overline{x} \equiv (\overline{q}, \overline{p}) \in P_{\overline{\alpha}} = T^*_{S_{\overline{\alpha}}}(M)$ does not belong to $R_{\overline{\alpha}} = \widetilde{T}^*(S_{\overline{\alpha}})$, whereas the impulse $x^* \equiv (\overline{q}, p^*)$ "immediately after" the impact must belong to it. So, the problem is to calculate p^* as a function of $\overline{q}$ and $\overline{p}$.
The solution of this problem is given by the transition principle, which is specialized in the case of totally inelastic collisions as follows.

The transition principle for totally inelastic collisions. *Let the system of the two bodies C_1, C_2 be described by the Hamiltonian function (4). Suppose C_1 and C_2 collide. Let $\overline{q} \in S_{\overline{\alpha}} \subset N$ be the corresponding impact configuration, and let $\overline{p}$ be the impulse immediately before the impact. Then, after the impact the configuration space reduces to $S_{\overline{\alpha}}$ and the system is described by the reduced Hamiltonian $\overline{H} = H|_{\widetilde{T}^*(S_{\overline{\alpha}})} = \widehat{T}\Big|_{\widetilde{T}^*(S_{\overline{\alpha}})} + \pi^*\left(V|_{S_{\overline{\alpha}}}\right)$. Further, the initial impulse for the reduced system is $p^* = \widetilde{T}^*_{\overline{q}}(S_{\overline{\alpha}}) \cap \overline{\beta}$, with $\overline{\beta} \subset T^*_{\overline{q}}(M)$ being the affine subspace passing through $\overline{x} = (\overline{q}, \overline{p})$ and parallel to $Ker\ (\Omega_K)_{\overline{x}}$.*

In terms of local coordinates the expression of p^* is the following. Let $S_{\overline{\alpha}}$ be described by equations (2). Denote, as before, with $G = \|g_{ij}(q)\|$

the Riemannian metric on M associated with the kinetic energy via formula (5). Then, $R_{\overline{\alpha}} = \widetilde{T}^* (S_{\overline{\alpha}})$ is described by the linear equations

$$J_{\overline{q}}\, G_{\overline{q}}^{-1}\; p = 0, \tag{7}$$

with $J = \partial f / \partial q$ being the matrix $\|\partial f_i / \partial q_j\|$. Therefore, from (7), (3) it follows that

$$p^* = (\mathbf{1} - D)\; \overline{p},$$

with $\mathbf{1}$ being the identity matrix, $D = J^t B^{-1} J G^{-1}$, $B = J G^{-1} J^t$, each matrix being evaluated at $q = \overline{q}$. In terms of velocities, keeping in mind (6), the above formula becomes

$$v^* = G^{-1}\; (\mathbf{1} - D)\; G\, \overline{v}$$

3. A comparison with the classical approach

In this final section we will show that the results obtained by applying the transition principle agree with those obtained by traditional methods (see, for example, [2], [7] for elementary and classical results, and [1] for more recent and sophisticated models). However, we would like to srtress that our scheme, which relies only on naturality of the geometrical background of the problem, has, in comparison with the traditional approach, the following two advantages:

1. It does not require any additional, more or less ad hoc, assumptions on the nature of impulsive forces;
2. No symmetry arguments are needed (see below)

It is clear from the above discussion that the phase jump described by way of the transition principle does not depend on the potential energy but only on the kinetic energy of the system. This means that, during the collision process, internal and external conservative forces acting on the system do not play any role. What really matters is the *geometry and the distribution of mass* in the colliding bodies.

This perfectly agrees with the classical theory of impact, according to which the collision is schematized as an instantaneous process. More exactly, denote by t_0 the instant at which the two bodies C_1, C_2 touch each other, and let τ be the time interval during which the impact takes place. Denote by $P_i \in \partial C_i$, $i = 1, 2$, the contact points. Let $\mathbf{F}_{ij}(t)$, $t \in [t_0, t_0 + \tau]$ be the impulsive force exerted by C_i on C_j at the point P_j during the impact. Obviously, $\mathbf{F}_{12}(t) = -\mathbf{F}_{21}(t)$, for any t. Apart from this, the law of force $t \mapsto \mathbf{F}_{ij}(t)$ is uknown, and one only assumes that

$$\mathbf{I}_{ij} = \lim_{\tau \to 0} \int_{t_0}^{t_0 + \tau} \mathbf{F}_{ij}(t)\; dt \neq 0$$

This practically means that the $\mathbf{F}_{ij}$'s magnitudes diverge as τ tends to zero (formally, they could be modelled as two delta functions centered

at t_0). On the other hand, forces deriving from a potential function, having finite magnitudes, generate infinitesimal impulses during the impact, and hence produce irrelevant effects.
Thus, summing up:

Let a mechanical system be described by the Hamiltonian (4), and suppose that at an instant t_0 it is subjected to an impulsive process. Then the jump of velocity due to the impact is the same as if the motion was free, i.e. if the system was subjected only to constraint reactions.

But, in the latter case, the total linear and angular momenta, $\mathbf{Q}$ and $\mathbf{K}$, of the system would be constant vectors, and their projections along coordinate axes would be 6 independent first integrals of the system. Therefore, from the above reasoning it follows that such scalar quantities are always preserved after the impact. Thus, one has 6 independent equations, allowing to determine the phase of the system immediatly after the impact.
Let us note that, in the case of free motion, the above first integrals are associated via Noether's theorem to infinitesimal symmetries of the system, which are 1-parameter subgroups of the group of rigid motions of the ambient Euclidean space. The corresponding vector fields on M are tangent to N. Therefore, in order to show that our results agree with the classical ones, it is sufficient to prove the theorem below.
Given a vector field X on M, denote by $\widehat{X} \in D\left(T^*\left(M\right)\right)$ its lifting to the phase space. Recall (see [4]) that $\widehat{X}$ is a Hamiltonian vector field, and that the corresponding Hamiltonian function is $\rho\left(\widehat{X}\right)$, with $\rho = \sum_{i=1}^{n} p_i dq_i$ being the Liouville 1-form on $T^*\left(M\right)$. In other words

$$\widehat{X} \lrcorner\, \Omega = -d\left(\rho\left(\widehat{X}\right)\right)$$

Theorem. Let $X \in D\left(M\right)$ be an infinitesimal non-hidden symmetry of the Hamiltonian system (4), i.e. $\widehat{X}\left(H\right) = 0$. Suppose that X is tangent to N. Then the corresponding first integral $h = \rho\left(\widehat{X}\right)$ is constant along the leaves of the distribution $\mathcal{D}$.

Proof. Let $Z \in D\left(T_N^*\left(M\right)\right)$ be tangent to the characteristic distribution $\mathcal{D}$. This is equivalent to

$$Z \lrcorner\, \Omega_N = 0$$

Denote by $\sigma : T_N^*\left(M\right) \to T^*\left(M\right)$ the inclusion map. Let $\widetilde{h} = \sigma^*\left(h\right)$ be the restriction of h to $T_N^*\left(M\right)$. We have to prove that

$$Z\left(\widetilde{h}\right) = 0\ ,$$

First, note that $\widehat{X}$ is tangent to $T^*_N(M)$. This immediately follows from the relation

$$\widehat{X} \circ \pi^* = \pi^* \circ X$$

and the fact that X is tangent to N. Denote by $\widetilde{X}$ the restriction of X to $T^*_N(M)$, then

$$\widetilde{X} \circ \sigma^* = \sigma^* \circ \widehat{X}$$

Hence

$$\begin{aligned} Z\left(\widetilde{h}\right) &= Z \lrcorner \sigma^*(dh) = -Z \lrcorner \sigma^*\left(\widehat{X} \lrcorner \Omega\right) \\ &= -Z \lrcorner \left(\widetilde{X} \lrcorner \Omega_N\right) = \widetilde{X} \lrcorner (Z \lrcorner \Omega_N) = 0 \end{aligned}$$

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