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Full symmetry algebra for ODEs and control systems

A. Samokhin

Abstract. A description of the full symmetry algebra (i.e., including higher symmetries) for a general nonlinear system of ordinary differential equations is given in terms of its general solution and differential constants. More precisely, the full symmetry algebra of a system is a module over the ring of its differential constants; the module is generated by partial derivatives of the general solution by the independent constants. Given a general solution, this description is both effective and explicit. Special solutions, such as an envelope of a family of solutions is described naturally in this context. These results are extended to control systems; in this case the differential constants become operators on controls. Examples are provided.

1. Introduction

The study of symmetries of ordinary differential equations was initiated by Sophus Lie himself and has a long history which is described briefly in [1]. The latest results were obtained in [2] and [3].

To find symmetries for an individual equation still remains a hard task. The present paper deals, however, with another problem. We give a full description of a symmetry algebra of a system of ODE in a nondegenerate situation using the general solution whose (local) existence is guaranteed by classical theorems. For a linear system of ODEs this result was obtained in [1]. It was generalized to the normal form scalar ODEs in [3].

Given a general solution, this description is both effective and explicit. Special solutions, such as an envelope of a family of solutions is described naturally in this context.

Of course, these results are of little practical importance since there is no need in symmetries when a general solution is known. Symmetries are used to obtain new solutions, not the other way round. Yet the interconnection between differential invariants, symmetries and a general solution are quite transparent in the case of ODEs and may be used as a model aplicable in other situations.

In this paper, we give two such applications. First, we describe the symmetries of a boundary / initial value problem for a one-dimensional

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wave equation. The second application deals with symmetries of control systems. In both cases differential invariants become nonlocal ones.

The paper is organized as follows. Section 2 describes the full symmetry algebra for a general nonlinear system of ordinary differential equations. It also contains the description of special solutions as invariants of basic symmetries for a given general solution, (subsection 2.3) and examples (subsection 2.4). Section 3 is an application of this approach to control systems; examples are also provided.

2. Full symmetry algebra for a general nonlinear ordinary differential equation and a system of equations

2.1. **General solution and differential constants.** We begin with trivialities to introduce notation.

Let $\mathcal{E}$ denote a general scalar ordinary differential equation of nth order

$$y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0. \tag{1}$$

The equation's *general solution* (or a *general integral*) is of the form

$$\Phi(x, y, c_1, c_2, \dots, c_n) = 0. \tag{2}$$

When (2) is solved with respect to y, we get

$$y = f(x, c_1, \dots, c_n); \tag{3}$$

almost any solution of (1) is obtained from (3) by a proper choice of the constants c_i. (The solution that is not produced by the general one is called a *special solution*. Such solutions are discussed below.)

Differentiating (3) by x we obtain the following system of n independent equations

$$\left\{ \begin{array}{lcl} y & = & f(x, c_1, \dots, c_n) \\ y' & = & f'(x, c_1, \dots, c_n) \\ & \dots & \\ y^{(n-1)} & = & f^{(n-1)}(x, c_1, \dots, c_n) \end{array} \right. \tag{4}$$

(further differentiating produce dependent equations since $y^{(k)}, k \geq n$ are expressed in $y^{(i)}, i < n$ via (1)).

One can obtain an expression (not necessary explicit) for c_i solving (4). Thus

$$c_i = c_i(x, y, y', \dots, y^{(n-1)}), \quad i = 1, \dots, n. \tag{5}$$

In this way all c_i are differential constants of order less than n. In other words, they are differential operators of order $n-1$ or functions on the jet space $J^{n-1}(\mathbb{R})$.

In the case of a system of m differential equations,

$$\mathbf{y}^{(n)} - \mathbf{F}(x, \mathbf{y}, \mathbf{y}', \ldots, \mathbf{y}^{(n-1)}) = 0, \tag{6}$$

where $\mathbf{y} = (y_1, \ldots, y_m)$, $\mathbf{F} = (F_1, \ldots, F_m)$, the general solution is of the form

$$\Phi_k(x, \mathbf{y}, c_1, c_2, \ldots, c_{mn}) = 0, \quad k = 1, \ldots, m \tag{7}$$

or

$$\mathbf{y} = \mathbf{f}(x, c_1, \ldots, c_{mn}). \tag{8}$$

Almost any solution of (6) is obtained from (8) by a proper choice of the constants c_i.

2.2. **Full symmetry algebra.** By definition of a solution, if right-hand side of (3), $f(x, y, c_1, \ldots, c_n)$ is substituted for y in (1), we obtain an identity

$$f^{(n)} - F(x, f, f', \ldots, f^{(n-1)}) \equiv 0. \tag{9}$$

Hence

$$\forall i : \quad \frac{\partial}{\partial c_i}\left(f^{(n)} - F(x, f, f', \ldots, f^{(n-1)})\right) = 0 \tag{10}$$

or

(11)
$$\forall i : \quad \left(D_x^n - \sum_{j=1}^{n} \frac{\partial F(x, y, y', \ldots, y^{(n-1)})}{\partial y_j} D_x^j \right)\Bigg|_{y=f(x,y,c_1,\ldots,c_n)} f_{c_i} = 0,$$

where $D = \frac{d}{dx}$ is the total derivative with respect to x.

Recall that

$$\mathcal{L}_{y^{(n)}-F} \stackrel{\text{def}}{=} D_x^n - \sum_{j=1}^{n} \frac{\partial F(x, y, y', \ldots, y^{(n-1)})}{\partial y_j} D_x^j \tag{12}$$

is called an *universal linearization* of the operator $y^{(n)} - F$ and that a solution ϕ of the equation

$$\left(\mathcal{L}_{y^{(n)}-F}\right) \phi\big|_{\mathcal{E}} = 0 \tag{13}$$

is a *symmetry* of $\mathcal{E}$.

Theorem 1. *Partial derivatives $f_{c_i}, i = 1, \ldots, n$ form a full functionally independent basis of symmetries for equation (1).*

Proof. The difference between (11) and (13) is that the same operator is restricted to formally different objects. However, note that the set

$$\{y = f(x, c_1, \ldots, c_n), y' = f'(x, y, c_1, \ldots, c_n) \ldots | \forall c_i \in \mathbb{R}\} \subset J^n(\mathbb{R})$$

essentially coincides with $\mathcal{E}$. Indeed, $\dim J^n(\mathbb{R}) = n + 2$, $\operatorname{codim} \mathcal{E} \subset J^n(\mathbb{R}) = 1$, so $\dim \mathcal{E} = n + 1$. It follows from the existence theorem for an ordinary differential equation that there is a solution containing any initial value point $x_0, y_0, y'_0, \ldots, y_0^{n-1} \in \mathcal{E}$. Now, since (3) produces almost all solutions and

$$\dim\{y = f(x, y, c_1, \ldots, c_n), y' = f'(x, y, c_1, \ldots, c_n) \ldots | \forall c_i \in \mathbb{R}\} = n+1$$

we conclude that (11) coincides with the symmetry equation (13) almost everywhere on $\mathcal{E}$.

Therefore, $f_{c_i}, i = 1, \ldots, n$ are symmetries of equation (1). Moreover, they form a basis of the symmetry algebra.

Indeed, let φ be a symmetry. Then it defines a flow on a set of solutions by the formula :

$$\frac{\partial y}{\partial \tau} = \varphi|_y, \tag{14}$$

where $y = f(x, y, c_1, \ldots, c_n)$. It can be solved (see [4]) and a solution of this equation is a one-parameter family of solutions of (1). By (3), it has a form

$$y = f(x, c_1(\tau), \ldots, c_n(\tau)) \tag{15}$$

On the other hand, differentiating (15) by τ, we obtain (via (14)) that

$$\varphi|_y = \left. \left(\sum_{i=1}^{n} \frac{\partial c_i}{\partial \tau} f_{c_i} \right) \right|_y \tag{16}$$

on any solution y of equation (1). Therefore,

$$\varphi = \sum_{i=1}^{n} \frac{\partial c_i}{\partial \tau} f_{c_i} \tag{17}$$

holds everywhere on $\mathcal{E}$.

Note that the derivatives $\frac{\partial c_i}{\partial \tau}|_y$ depend on y, that is, on $c_1, \ldots, c_1$, which are functions on $J^{n-1}(\mathbb{R})$ by virtue of (5). Since any choice of arbitrary functions $c_i(\tau)$ define some symmetry by (15), the functions $\frac{\partial c_i}{\partial \tau}|_y$ are also arbitrary.

Thus, we got the general form of a symmetry for equation (1)

$$\varphi = \sum_{i=1}^{n} A_i(c_1, \ldots, c_n)\frac{\partial}{\partial c_i} f(x, y, c_1, \ldots, c_n); \tag{18}$$

here f is a general solution, A_i are arbitrary functions and c_i are functions on $J^{n-1}(\mathbb{R})$ given by system (4).

Formula (18) also completes the proof of the theorem. □

Remark 1. *A full symmetry algebra is a module over the ring of the equation's differential constants. The module is generated by partial derivatives of a general solution by the independent constants.*

Remark 2. *Formula (18) gives a representation of the algebra of vector fields on $\mathbb{R}^n$ in the full symmetry algebra of (6) by the isomorphism*

$$\sum_{i=1}^{n} A_i(c_1, \ldots, c_n)\frac{\partial}{\partial c_i} \longleftrightarrow \sum_{i=1}^{n} A_i(c_1, \ldots, c_n)\frac{\partial}{\partial c_i} f(x, c_1, \ldots, c_n)$$

(On the left-hand side, c_i are coordinates in $\mathbb{R}^n$; on the right-hand side they denote differential invariants (5) of (1) or special functions on $J^{n-1}(\mathbb{R})$).

Remark 3. *Theorem 1 generalizes easily to the case of a system of differential equations (6). Its full symmetry algebra is isomorphic to the algebra of vector fields on $\mathbb{R}^{mn}$: the representation is given by*

$$\sum_{i=1}^{mn} A_i(c_1, \ldots, c_{mn})\frac{\partial}{\partial c_i} \longleftrightarrow \partial\mathbf{f} \times \mathbf{A},$$

where $\partial\mathbf{f}, \mathbf{A}$ are respectively $m \times mn$ and $mn \times 1$ matrices with matrix elements given by the formulas

$$(\partial\mathbf{f})_{j,i} = \frac{\partial f_j}{\partial c_i}, \quad (\mathbf{A})_i = A_i$$

A variant of Theorem (1) is also valid in the case of even more general system of ordinary differential equations,

$$y_j^{(n_j)} - F_j(x, y_1, y_1', \ldots, y_1^{(n_1-1)}, \ldots, y_m, y_m', \ldots, y_1^{(n_m-1)}) = 0.$$

It is not hard to right down the correspondent isomorphism between vector fields on the solution space and symmetries in this case too. Yet the formula is awkward to read and therefore it is omitted here. See [2] *for relevant technicalities.*

Let us call $f_{c_i}, i = 1, \ldots, n$ *basic symmetries*. They correspond to the flows $y(\tau) = f(x, c_1, \ldots, c_i + \tau, \ldots, c_n)$. Thus, in the case of an explicit general solution (3) basic symmetries are $f_{c_i} = y_{c_i}$.

Remark 4. *If general solution of (1) is given in an implicit form (2) then*

$$\frac{d\Phi}{dc_i} = \frac{\partial \Phi}{\partial c_i} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial c_i} = 0.$$

It follows immediately that basic symmetries are given by

$$y_{c_i} = -\left(\frac{\partial \Phi}{\partial c_i}\right) \Big/ \left(\frac{\partial \Phi}{\partial y}\right). \tag{19}$$

This formula generalizes in a straightforward way in the case of a system of equations.

2.3. **Special and invariant solutions.** *Invariant* or *self-similar* solution y of (1) is the solution that satisfy the condition $\varphi(y) = 0$ for some symmetry φ of the form (18). Hence an invariant solution satisfy a system of equations

(20)
$$\begin{cases} \mathcal{E}(f) = y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0 \\ \phi(y) = \sum_{i=1}^{n} A_i(c_1(y), \dots, c_n(y)) \frac{\partial}{\partial c_i} f(x, y, c_1(y), \dots, c_n(y)) = 0 \end{cases}$$

Since c_i are constants on solutions of (1) so are $A_i(c_1(y), \dots, c_n(y))$. Thus (20) is simply

$$\begin{cases} \mathcal{E}(f) & = & y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0 \\ \phi(y) & = & \sum_{i=1}^{n} A_i f_{c_i}(x, y, c_1, \dots, c_n) = 0 \end{cases} \tag{21}$$

with constant A_i and c_i. The second condition in (21) means that basic symmetries are linearly dependent on an invariant solution. If $\operatorname{rank}\{f_{c_1}, \dots, f_{c_n}\}|_y = n - k$, it is natural to introduce a notion of a k-invariant solution.

Remark 5. *Recall that f_{c_i} represent independent vector fields on $\mathbb{R}^n$. In this way the structure of invariant solutions of ordinary differential equation is connected with the structure of degenerate points of a system of n independent vector fields on $\mathbb{R}^n$.*

Consider a simple case of (21),

$$\begin{cases} y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0 \\ f_{c_i} = 0 \end{cases} \tag{22}$$

Its solution is a fixed point of the flow $c_i \to c_i + \tau$. Geometrically, such a solution is an envelope for the family of solution generated by this flow, see section (2.4).

2.4. **Examples.**

Example 1.

$$y'' + \frac{9}{8}(y')^4 = 0$$

This equation is invariant with respect to the translations in both x and y, hence its symmetry algebra is obvious. Its general solution is as follows

$$\Phi(x, y, c_1, c_2) = (y + c_1)^3 - (x + c_2)^2 = 0,$$

or

$$y = f(x, c_1, c_2) = (x + c_2)^{\frac{2}{3}} - c_1$$

Therefore, its basic symmetries are $f_{c_1} = -1$, $f_{c_2} = \frac{2}{3}(x + c_2)^{-\frac{1}{3}}$. They depend on the differential constants c_1, c_2 that may be found from the system (4),

$$\begin{aligned} (y + c_1)^3 &= (x + c_2)^2, \\ 3y'(y + c_1)^2 &= 2(x + c_2). \end{aligned}$$

It follows that

$$\begin{aligned} c_1 &= (\tfrac{2}{3y'})^2 - y, \\ c_2 &= (\tfrac{2}{3y'})^3 - x. \end{aligned}$$

Now, basic symmetries come to

$$\begin{aligned} f_{c_1} &= -1, \\ f_{c_2} &= y', \end{aligned}$$

which are (not surprisingly) translation in y and x respectively.

So the general symmetry for this equation is of the form (18)

$$\varphi = A_1(c_1, c_2) f_{c_1} + A_2(c_1, c_2) f_{c_2} =$$
$$- A_1\left((\frac{2}{3y'})^2 - y, (\frac{2}{3y'})^3 - x\right) + A_2\left((\frac{2}{3y'})^2 - y, (\frac{2}{3y'})^3 - x\right) y',$$

where A_1, A_2 are arbitrary functions of two variables.

Invariant solutions have to satisfy the system (21)

$$\begin{aligned} A + y'B &= 0, \\ y'' + \tfrac{9}{8}(y')^4 &= 0, \end{aligned}$$

for some constants A, B. It follows that $y' = 0$, so $y = const$. This is a family of special solutions (in the sense they are not obtained from the general integral). Each special solution is an envelope for the family

$$(y - const)^3 - (x + c_2)^2 = 0$$

for all c_2, see figure 1.

Example 2.

$$yy'' + 2(y'^2 + 1) = 0$$

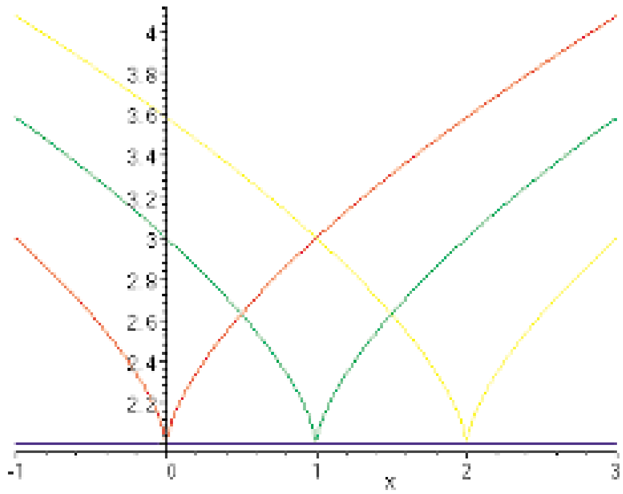

FIGURE 1. Enveloping invariant solution $y = 2$

The general integral in this case is as follows

$$\Phi = \int \frac{y^2\,dy}{\sqrt{c_1 - y^4}} \pm x + c_2.$$

Basic symmetries are obtained here by the formula (19):

$$\varphi_1 = -\frac{\Phi_{c_1}}{\Phi_y} = \frac{1}{2}\frac{\sqrt{c_1-y^4}}{y^2}\int \frac{y^2\,dy}{(\sqrt{c_1-y^4})^3},$$
$$\varphi_2 = -\frac{\Phi_{c_2}}{\Phi_y} = -\frac{\sqrt{c_1-y^4}}{y^2}.$$

To obtain a final form for these symmetries it remains to express differential constants as functions on $J^1(\mathbb{R})$ using (4):

$$\int \frac{y^a\,dy}{\sqrt{c_1-y^{2a}}} \pm x + c_2 = 0$$
$$y'\frac{\sqrt{c_1-y^4}}{y^2} \pm 1 = 0.$$

It follows immediately that

$$c_1 = y^4(y'^2 + 1),$$
$$c_2 = \pm \int dx \mp x = c_2.$$

Substituting these expressions into basic symmetries we obtain

$$\varphi_1 = \frac{y'}{2}\int \frac{dy}{y'^3 y^4},$$
$$\varphi_2 = y'.$$

Note that ϕ_1 is a nonlocal symmetry.

Example 3. *Linear equations (cf.[2])*

$$y^{(n)} + \sum_{i=0}^{n-1} a_i(x) y^{(i)} = 0$$

Here the general integral is if the form

$$y = \sum_{i=1}^{n} c_i f_i(x),$$

where $f_i(x)$ are independent solutions, i.e., their Wronskian is nonzero:

$$W = W(f_1, \dots, f_i, \dots, f_n) = \begin{vmatrix} f_1 & \dots & f_i & \dots & f_n \\ f_1' & \dots & f_i' & \dots & f_n' \\ \dots & \dots & \dots & \dots & \dots \\ f_1^{(n-1)} & \dots & f_i^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \neq 0$$

Independent solutions f_i coincide with basic symmetries in this case: $f_i = f_{c_i}$.

Differential constant c_i is given by the formula

$$c_i(y, y', \dots, y^{(n-1)}) = \frac{W_i}{W},$$

where W_i is obtained from W by changing the entries of the i-th column of W for $y, y', \dots, y^{(n-1)}$ in respective order.

The general form of the symmetry is

$$\varphi = \sum_{i=1}^{n} A_i \left(\frac{W_1}{W}, \dots, \frac{W_i}{W}, \dots, \frac{W_n}{W} \right) f_i(x).$$

Example 4. *Linear boundary problem*

$$u_{tt} - u_{xx} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0$$

This example is a rather wide generalization of the previous one. Fourier's general solution on $[0, \pi]$ for this string is

$$u = \sum_{n=0}^{\infty} \sin nx (a_n \cos nt + b_n \sin nt),$$

where a_n, b_n are constants, but neither differential nor local: the Fourier coefficient formula states that

$$a_n = \frac{2}{\pi} \int_0^{\pi} u|_{t=0} \sin nx\, dx, \quad b_n = \frac{2}{\pi n} \int_0^{\pi} u_t|_{t=0} \sin nx\, dx \tag{23}$$

A general form of the symmetry is given by

$$\varphi = \sum_{n=0}^{\infty} \sin nx \left[A_n(a_1, b_1, \dots, a_i, b_i, \dots) \cos nt + B_n(a_1, b_1, \dots, a_i, b_i, \dots) \sin nt \right].$$

Here A_n, B_n are arbitrary functions depending on any finite number of a_i, b_j which are given by (23).

3. Full symmetry algebra for a general control system

3.1. General solution and differential constants. Consider a first order control system

$$\mathbf{y}' = \mathbf{F}(x, \mathbf{y}, \mathbf{v}(x)), \tag{24}$$

where $\mathbf{y} \in \mathbb{R}^m$ is an m-vector of unknown functions and $\mathbf{v}(\mathbf{x}) \in \mathbb{R}^k$ in an k-vector of control functions.

With any fixed choice of controls, (24) comes to (6), where $n = 1$. Thus, the general solution of (24) is of the form

$$\mathbf{y} = \mathbf{f}(x, c_1, \ldots, c_m, \mathbf{v}(x)), \tag{25}$$

where c_i are constants. From (25) it follows that there exists (at least an implicit) dependence

$$c_i = c_i(x, \mathbf{y}(x), \mathbf{y}'(x), \mathbf{v}(x)), \quad i = 1, \ldots, m \tag{26}$$

of differential constants c_i on $x, \mathbf{y}(x), \mathbf{y}'(x), \mathbf{v}(x)$. Both $\mathbf{f}$ and c_i are operators on $\mathbf{v}$. Examples show that these operators may be nonlocal.

3.2. Full symmetry algebra. Technically, equation (24) is an equation with two types of dependent variables, that is, with $\mathbf{y}$ and $\mathbf{v}$. Put this equation in a form

$$\mathcal{H}(\mathbf{y}, \mathbf{v}) = \mathbf{y}' - \mathbf{F}(x, \mathbf{y}, \mathbf{v}(x)) = 0.$$

The symmetry equation in this case is as follows:

$$(D - \mathbf{F}_\mathbf{y})\mathbf{A} - \mathbf{F}_\mathbf{v}\mathbf{B}|_{\mathcal{H}=0} = 0, \tag{27}$$

where $(\mathbf{A}, \mathbf{B})$ is a symmetry (if it defines a flow, then $\mathbf{y}_\tau = \mathbf{A},\ \mathbf{v}_\tau = \mathbf{B}$). Besides, $\mathbf{F}_\mathbf{y}$ is an $m \times m$ matrix with entries $(F_i)_{y_j}$ and $\mathbf{F}_\mathbf{v}$ is an $m \times k$ matrix with entries $(F_i)_{v_j}$.

It is convenient to put (27) in a vector form,

$$(D - \mathbf{F}_\mathbf{y}, -\mathbf{F}_\mathbf{v}) \cdot \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}\Bigg|_{\mathcal{H}=0} = 0. \tag{28}$$

The left factor in this formula is the linearization of $\mathcal{H}$, denoted by $\mathcal{L}_\mathcal{H} = (D - \mathbf{F}_\mathbf{y}, -\mathbf{F}_\mathbf{v})$.

Theorem 2. *Partial derivatives vectors*

$$\begin{pmatrix} \mathbf{f_c} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{f_v} \\ \mathrm{I} \end{pmatrix} \tag{29}$$

form a full functionally independent basis of symmetries for the equation (24).

Proof. In terms of the general solution, the general form of a flow on the set of solutions of equation (24) is given by the formula

$$\mathbf{y} = \mathbf{f}(x, c_1(\tau), \ldots, c_m(\tau), \mathbf{v}(x, \tau)), \tag{30}$$

where τ is the parameter of the flow. Since (30) is a solution for any τ, we have

$$\mathbf{f}'(x, c_1(\tau), \ldots, c_m(\tau), \mathbf{v}(x, \tau)) - \mathbf{F}(x, \mathbf{f}(x, c_1(\tau), \ldots, c_m(\tau), \mathbf{v}(x, \tau)), \mathbf{v}(x, \tau)) = 0$$

Therefore,

$$\frac{d}{d\tau}[\mathbf{f}'(x, c_1(\tau), \ldots, c_m(\tau), \mathbf{v}(x, \tau)) - \mathbf{F}(x, \mathbf{f}(x, c_1(\tau), \ldots, c_m(\tau), \mathbf{v}(x, \tau)), \mathbf{v}(x, \tau))] = 0.$$

It follows that

$$\begin{aligned} &[(D - \mathbf{F_y})(\mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau) - \mathbf{F_v}\mathbf{v}_\tau]_{\mathcal{H}=0} = \\ &(D - \mathbf{F_y}, -\mathbf{F_v}) \cdot \left(\begin{array}{c} \mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau \\ \mathbf{v}_\tau \end{array} \right)\Bigg|_{\mathcal{H}=0} = \\ &\mathcal{L}_{\mathcal{H}} \left(\begin{array}{c} \mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau \\ \mathbf{v}_\tau \end{array} \right)\Bigg|_{\mathcal{H}=0} = 0. \end{aligned} \tag{31}$$

Thus, the general solution of the symmetry equation is (compare with (17))

$$\left(\begin{array}{c} \mathbf{f_c} \\ 0 \end{array} \right) \cdot \mathbf{c}_\tau + \left(\begin{array}{c} \mathbf{f_v} \\ \mathrm{I} \end{array} \right) \cdot \mathbf{v}_\tau \tag{32}$$

Here $\mathbf{f_c} = (f_i)_{c_j}$ is an $m \times m$ matrix, $\mathbf{f_v}$ is an $m \times k$ matrix and I is the $k \times k$ identity matrix.

To obtain the general form of the symmetry for equation (24) it remains to notice that

- $\mathbf{v}_\tau$ is an arbitrary vector-function;
- for any fixed $\mathbf{v}$ equation (24) coincides with (6), so $c_{i\tau}$ are the components of a vector field on the solution space for the chosen $\mathbf{v}$. Therefore, $c_{i\tau} = \mathcal{A}_i(\mathbf{c}, \mathbf{v})$ are arbitrary functions;
- c_i are differential constants on solution of (24) given by (26).

Finally, we can write down the general form of the symmetry for (24).

$$\varphi = \left(\begin{array}{c} \mathbf{f_c} \\ 0 \end{array} \right) \cdot \mathcal{A}(\mathbf{c}, \mathbf{v}(x)) + \left(\begin{array}{c} \mathbf{f_v} \\ \mathrm{I} \end{array} \right) \cdot \mathbf{u}(x) \tag{33}$$

Here $\mathcal{A}(\mathbf{c}, \mathbf{v}(x))$ and $\mathbf{u}(x)$ are arbitrary proper-sized matrices. □

Remark 6. *Generally, the solution (25) and its derivatives as well as expressions of the type $\mathcal{A}(\mathbf{c}, \mathbf{v}(x))$ or $\mathbf{u}(x)$ are operators on $\mathbf{v}(x)$. In the case these are differential operators of order l, we obtain lth order higher symmetries by the formula (33).*

3.3. Examples.

Example 5. *A linear scalar equation*

$$y' = xy + v(x). \tag{34}$$

The general solution in this case is easy to obtain:

$$y = e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} v(t)\, dt + c \cdot e^{\frac{x^2}{2}}.$$

Thus,

$$c = y \cdot e^{\frac{-x^2}{2}} - I(x), \text{ where } I(x) = \int_{x_0}^{x} e^{-\frac{t^2}{2}} v(t)\, dt,$$

is a constant on any solution of (34).

Therefore, from (33) it follows that the general form of the symmetry in this example is

(35)

$$\varphi = \begin{pmatrix} e^{\frac{x^2}{2}} \\ 0 \end{pmatrix} \cdot \mathcal{A}(y \cdot e^{\frac{-x^2}{2}} - I(x), v(x)) + \begin{pmatrix} e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} [\,\bullet\,]\, dt \\ 1 \end{pmatrix} \cdot u(x)$$

Here $\mathcal{A}(c, v(x))$ and $u(x)$ are arbitrary operator and function respectively; $f_v = e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} [\,\bullet\,]\, dt$ is an operator acting on $u(x)$ by the formula

$$\left(e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} [\,\bullet\,]\, dt \right) u(x) = e^{\frac{x^2}{2}} \int_{x_0}^{x} e^{-\frac{t^2}{2}} u(t)\, dt$$

This example shows that, since a general solution $f = f(v)$ of a control system is an operator on controls, f_v in the formula (33) is a linearization of this operator.

In Theorem 2 the flow of the control function v is arbitrary, so v is a functional parameter. Suppose it is a subject to some differential constraint $v_\tau = G(x, v, v', \ldots, v^{(r)})$. (This constitutes an alternative approach since v is then considered as an unknown on par with y, cf. [6]).

If r is the maximal order of the derivative of v entering this such a constraint, then y_τ can depend on $v^{(s)}, s \leq r-1$ only, cf. [5]. The next example is an illustration of this general statement.

Example 6. $v_\tau = v$, $c_\tau = 0$.

¿From (28) and (32) we obtain

$$(D - F_y, -F_v) \cdot \begin{pmatrix} f_v v \\ v \end{pmatrix} \Bigg|_{\mathcal{H}=0} = 0. \quad (36)$$

The highest order derivative of v entering this equation is v'. It enters linearly and its coefficient is $f_v + v f_{vv}$ so it must be zero. Solving $f_v + v f_{vv} = 0$ we obtain $f_v = 1/v$ and $y_\tau = f_v v = 1$. In particular, it does not depend on v in a perfect accordance with the result of [5].

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