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# On one-parametric families of Bäcklund transformations

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Abstract. In the context of the cohomological deformation theory, infinitesimal description of one-parametric families of Bäcklund transformations is given. It is shown that any family of such a kind evolves in the direction of a nonlocal symmetry shadow in the sense of [10]. In addition, a formal family naturally corresponding to a shadow is described.

## Introduction

The role of Bäcklund transformations in constructing exact solutions of nonlinear partial differential equations is well known, see [1] relevant references therein, for example. A general scheme is illustrated by classical works by Bäcklund and Bianchi. Namely, for the sine-Gordon equation

$$u_{xy} = \sin u \tag{1}$$

Bäcklund constructed a system of differential relations $\mathcal{B}(u, v; \lambda) = 0$ depending on a real parameter $\lambda \in \mathbb{R}$ and satisfying the following property: if $u = u(x, y)$ is a solution of (1), then $v$ is a solution of the same equation and vice versa. Using this result, Bianchi showed that if a known solution $u_0$ is given and solutions $u_1$, $u_2$ satisfy the relations $\mathcal{B}(u_0, u_i; \lambda_i) = 0$, $i = 1, 2$, then there exists a solution $u_{12}$ which satisfies $\mathcal{B}(u_1, u_{12}; \lambda_2) = 0$, $\mathcal{B}(u_2, u_{12}; \lambda_1) = 0$ and is expressed in terms of $u_0$, $u_1$, $u_2$ in terms of relatively simple equalities. This is the so-called *Bianchi permutability theorem*, or *nonlinear superposition principle*. This scheme was successfully applied to many other "integrable" equations.

Quite naturally, a general problem arises: given an arbitrary PDE $\mathcal{E}$, when are we able to implement a similar construction? This question is closely related to another problem of a great importance in the theory of integrable systems, the problem of insertion of a nontrivial "spectral parameter" to the initial equation. In this paper, we mainly deal with the first problem referring the reader to the yet unpublished work by M. Marvan [11], where the second problem is analyzed.

Our approach to solution lies in the framework of the geometrical theory of nonlinear PDE, and the first section of the paper contains a brief introduction to this theory, including its nonlocal aspects (the theory of coverings), see [7, 9, 10]. The second section deals with cohomological invariants of nonlinear PDE naturally associated to the equation structure. Our main concern here is the relation between this cohomology theory and

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deformations of the structure [4, 5, 8]. In the third section, we give a geometrical definition of Bäcklund transformations and using cohomological techniques prove the main result of the paper describing infinitesimal part of one-parameter families of Bäcklund transformations. Finally, the last section contains discussion of the results obtained as well as some problems related to the topic of the paper.

## 1. Equations and coverings

Let us recall basic facts from the geometry of nonlinear PDE, [7, 9].

Consider a smooth manifold $M$, $\dim M = n$, and a locally trivial smooth vector bundle $\pi\colon E \to M$. Denote by $\pi_k\colon J^k(\pi) \to M$, $k = 0, 1, \dots, \infty$, the corresponding bundles of jets. A *differential equation* of order $k$, $k < \infty$, in the bundle $\pi$ is a smooth submanifold $\mathcal{E} \subset J^k(\pi)$. To any equation $\mathcal{E}$ there corresponds a series of its *prolongations* $\mathcal{E}^s \subset J^{k+s}(\pi)$ and the *infinite prolongation* $\mathcal{E}^\infty \subset J^\infty(\pi)$. We consider below *formally integrable* equations, which means that all $\mathcal{E}^s$ are smooth manifolds and the natural projection $\pi_{\mathcal{E}} = \pi_\infty|_{\mathcal{E}^\infty}\colon \mathcal{E}^\infty \to M$ is a smooth bundle. For any $s > 0$ there also exist natural bundles

$$\mathcal{E}^\infty \xrightarrow{\pi_{\mathcal{E},s}} \mathcal{E}^s \xrightarrow{\pi_{\mathcal{E},s,s-1}} \mathcal{E}^{s-1} \xrightarrow{\pi_{\mathcal{E},s-1}} M \tag{2}$$

whose composition equals $\pi_{\mathcal{E}}$.

The space $J^\infty(\pi)$ is endowed with an integrable distribution[1] denoted by $\mathcal{C}D(\pi)$. Namely, any point $\theta \in J^\infty(\pi)$ is, by definition, represented in the form $[f]_x^\infty$, $x = \pi_\infty(\theta) \in M$, where $f$ is a (local) section of $\pi$ such that the graph $M_f^\infty$ of its infinite jet passes through $\theta$ while $[f]_x^\infty$ is the class of (local) sections $f'$ satisfying the condition

$$M_{f'}^\infty \text{ is tangent to } M_f^\infty \text{ at } \theta \text{ with infinite order.}$$

Then the tangent plane $T_\theta M_f^\infty$ is independent of the choice of $f$ and we set $\mathcal{C}D(\pi)_\theta = T_\theta M_f^\infty$. The distribution $\mathcal{C}D(\pi)$ is $n$-dimensional and is called the *Cartan distribution* on $J^\infty(\pi)$. Since, by construction, all planes of the Cartan distribution are horizontal (with respect to $\pi_\infty$) and $n$-dimensional, a connection $\mathcal{C}\colon D(M) \to D(\pi)$ is determined, where $D(M)$ and $D(\pi)$ denote the modules of vector fields on $M$ and $J^\infty(\pi)$ respectively. This connection is flat and is called the *Cartan connection*.

*Remark* 1. In fact, the bundle $\pi_\infty$ possesses a stronger structure than just a flat connection. Namely, for any vector bundles $\xi$ and $\eta$ over $M$ and a linear differential operator $\Delta$ acting from $\xi$ to $\eta$, a linear differential operator $\mathcal{C}\Delta$ acting from the pullback $\pi_\infty(\xi)$ to $\pi_\infty(\eta)$ is defined in a natural way. The correspondence $\Delta \mapsto \mathcal{C}\Delta$ is linear, preserves composition, and the Cartan connection is its particular case.

Both the Cartan distribution and the Cartan connection are restricted to the spaces $\mathcal{E}^\infty$ and bundles $\pi_{\mathcal{E}}$ respectively. The corresponding objects are denoted by $\mathcal{C}D(\mathcal{E})$ and $\mathcal{C} = \mathcal{C}_{\mathcal{E}}\colon D(M) \to D(\mathcal{E})$, where $D(E)$ is the module of vector fields on $\mathcal{E}^\infty$. The characteristic property of the Cartan distribution $\mathcal{C}D(\mathcal{E})$ on $\mathcal{E}^\infty$ is that its maximal integral manifolds are solutions

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[1] Integrability in this context means that $\mathcal{C}D(\pi)$ satisfies the Frobenius condition: $[\mathcal{C}D(\pi), \mathcal{C}D(\pi)] \subset \mathcal{C}D(\pi)$.

of the equation $\mathcal{E}$ and vice versa. The connection form $U_{\mathcal{E}} \in D(\Lambda^1(\mathcal{E}))$ of the connection $\mathcal{C}_{\mathcal{E}}$ is called the *structural element* of the equation $\mathcal{E}$. Here $D(\Lambda^1(\mathcal{E}))$ denotes the module of derivations $C^\infty(\mathcal{E}^\infty) \to \Lambda^1(\mathcal{E}^\infty)$ with the values in the module of one-forms on $\mathcal{E}^\infty$.

Denote by $D_{\mathcal{C}}(\mathcal{E})$ the module

$$D_{\mathcal{C}}(\mathcal{E}) = \{\, X \in D(\mathcal{E}) \mid [X, \mathcal{C}D(\mathcal{E})] \subset \mathcal{C}D(\mathcal{E}) \,\}.$$

Then $D_{\mathcal{C}}(\mathcal{E})$ is a Lie algebra with respect to commutator of vector fields and due to integrability of the Cartan distribution $\mathcal{C}D(\mathcal{E})$ is its ideal. The quotient Lie algebra $\operatorname{sym}\mathcal{E} = D_{\mathcal{C}}(\mathcal{E})/\mathcal{C}D(\mathcal{E})$ is called the *algebra of* (*higher*) *symmetries* of the equation $\mathcal{E}$. Denote by $D^v(\mathcal{E})$ the module of $\pi_{\mathcal{E}}$-vertical vector fields on $\mathcal{E}^\infty$. Then in any coset $X \bmod \mathcal{C}D(\mathcal{E}) \in \operatorname{sym}\mathcal{E}$ there exists a unique vertical element and this element is called a (*higher*) *symmetry* of $\mathcal{E}$.

*Remark* 2. It may so happen that a coset $X \bmod \mathcal{C}D(\mathcal{E})$ contains a representative $X'$ which is projectible to a vector field $X'_s$ on $\mathcal{E}^s$ by $\pi_{\mathcal{E},s}$ for some $s < \infty$ (see (2)). Then it can be shown that $X'$ is projectible to all $\mathcal{E}^s$ and $(\pi_{\mathcal{E},s,s-1})_* X'_s = X'_{s-1}$. In this case, $X'$ is called a *classical* (*infinitesimal*) *symmetry* of $\mathcal{E}$ an possesses trajectories in $\mathcal{E}^\infty$. The corresponding diffeomorphisms preserve solutions of $\mathcal{E}$ and are called *finite symmetries*.

We now pass to a generalization of the above described geometrical theory, the theory of coverings [10]. Let $\tau\colon W \to \mathcal{E}^\infty$ be a smooth fiber bundle, the manifold $W$ being equipped with an integrable distribution $\mathcal{C}D_\tau(W) = \mathcal{C}D(W) \subset D(W)$ of dimension $n = \dim M$. Then $\tau$ is called a *covering* over $\mathcal{E}$ (or over $\mathcal{E}^\infty$), if for any point $\theta \in W$ one has $\tau_*(\mathcal{C}D(W)_\theta) = \mathcal{C}D(\mathcal{E})_{\tau(\theta)}$. Equivalently, a covering structure in the bundle $\tau$ is determined by a flat connection $\mathcal{C}_\tau\colon D(M) \to D(W)$ satisfying $\tau_* \circ \mathcal{C}_\tau = \mathcal{C}_{\mathcal{E}}$. Let $U_\tau \in D(\Lambda^1(W))$ be the corresponding connection form. We call it the *structural element* of the covering $\tau$.

**Example 1** (see [14])**.** Let $\mathcal{E} \subset J^k(\pi)$ be an equation. Consider the tangent bundle $T\mathcal{E}^\infty \to \mathcal{E}^\infty$ and the subbundle $\pi^v_{\mathcal{E}}\colon T^v\pi_{\mathcal{E}} \to \mathcal{E}^\infty$, where $T^v\pi_{\mathcal{E}}$ consists of $\pi_{\mathcal{E}}$-vertical vectors. Hence, the module of sections for $\pi^v_{\mathcal{E}}$ consists of $\pi_{\mathcal{E}}$-vertical vector fields on $\mathcal{E}^\infty$.

Then $\pi^v_{\mathcal{E}}$ carries a natural covering structure. Namely, for any vector field $X \in D(M)$ and a vertical vector field $Y$ we set $[\mathcal{C}_{\tau^v}(X), Y] = [\mathcal{C}_{\mathcal{E}}(X), Y] \lrcorner U_{\mathcal{E}}$, where $U_{\mathcal{E}}$ is the structural element of the equation $\mathcal{E}$. It is easily seen that the connection $\mathcal{C}_{\tau^v}$ is well defined in such a way and projects to the connection $\mathcal{C}_{\mathcal{E}}$.

Given two coverings $\tau_i\colon W_i \to \mathcal{E}^\infty$, $i = 1, 2$, we say that a smooth mapping $F\colon W_1 \to W_2$ is a *morphism* of $\tau_1$ to $\tau_2$, if

(i) $F$ is a morphism of fiber bundles,
(ii) $F_*$ takes the distribution $\mathcal{C}D_{\tau_1}(W_1)$ to $\mathcal{C}D_{\tau_2}(W_2)$ (equivalently, $F_* \circ \mathcal{C}_{\tau_1} = \mathcal{C}_{\tau_2}$).

A morphism $F$ is said to be an *equivalence*, if it is a diffeomorphism.

Similar to the case of infinitely prolonged equations, we can define the Lie algebra $D_{\mathcal{C}_\tau}(W)$ such that $\mathcal{C}D_\tau(W)$ is it its ideal and introduce the algebra of *nonlocal $\tau$-symmetries* as the quotient $\operatorname{sym}_\tau \mathcal{E} = D_{\mathcal{C}_\tau}(W)/\mathcal{C}D_\tau(W)$. Again, in any coset $X \bmod \mathcal{C}_\tau D(W) \in \mathrm{sym}_\tau \mathcal{E}$ there exists a unique $\pi_{\mathcal{E}} \circ \tau$-vertical representative and it is called a *nonlocal $\tau$-symmetry* of the equation $\mathcal{E}$.

Obviously, one can introduce the notion of a covering over covering, etc. In particular, the subbundle $\pi^v_{\mathcal{E}} \colon T^v\pi_{\mathcal{E}} \to \mathcal{E}^\infty$ of $\pi_{\mathcal{E}} \circ \tau$-vertical vectors (cf. Example 1) is a covering over $\mathcal{E}$ while the intermediate projection $\tau^v \colon T^v\tau \to W$ is a covering over $W$. Note also that the correspondence $\tau \Rightarrow \tau^v$ determines a covariant functor in the category of coverings.

We shall now reinterpret the concepts of a symmetry and nonlocal symmetry using the results of [14]. Namely, one has

**Proposition 1.** *Let $\mathcal{E}$ be an equation and $\tau \colon W \to \mathcal{E}^\infty$ be a covering over it. Then:*

1. *There is a one-to-one correspondence between symmetries of $\mathcal{E}$ and sections $\varphi \colon \mathcal{E}^\infty \to T^v\pi_{\mathcal{E}}$ of the bundle $\pi^v_{\mathcal{E}} \colon T^v\pi\mathcal{E} \to \mathcal{E}^\infty$ such that $\varphi_*$ takes the Cartan distribution on $\mathcal{E}^\infty$ to that on $T^v\pi_{\mathcal{E}}$.*
2. *There is a one-to-one correspondence between nonlocal $\tau$-symmetries of $\mathcal{E}$ and sections $\psi$ of the bundle $(\pi_{\mathcal{E}} \circ \tau)^v \colon T^v(\pi_{\mathcal{E}} \circ \tau) \to W$ such that $\psi_*$ takes the Cartan distribution on $W$ to that on $T^v\tau$.*

Let us say that a mapping $s \colon W \to T^v\pi_{\mathcal{E}}$ is a *$\tau$-shadow* of a nonlocal symmetry (cf. [10]), if $\pi^v_{\mathcal{E}} \circ s = \tau$ and $s_*$ preserves the Cartan distribution.

**Proposition 2** (The shadow reconstruction theorem)**.** *For any covering $\tau \colon W \to \mathcal{E}^\infty$ and a $\tau$-shadow $s \colon W \to T^v\pi_{\mathcal{E}}$ there exists a covering $\tau' \colon W' \to W$ and a nonlocal $\tau \circ \tau'$-symmetry $s' \colon W' \to T^v(\pi_{\mathcal{E}} \circ \tau \circ \tau')$ such that the diagram*

$$\begin{tikzcd}
 & T^v\pi_{\mathcal{E}} \arrow[d, "\pi^v_{\mathcal{E}}"'] & & T^v(\pi_{\mathcal{E}} \circ \tau \circ \tau') \arrow[ll, "(\tau\circ\tau')_*"'] \arrow[d, shift right, "(\pi_{\mathcal{E}}\circ\tau\circ\tau')^v"'] \\
M & \mathcal{E}^\infty \arrow[l, "\pi_{\mathcal{E}}"'] & W \arrow[l, "\tau"'] \arrow[ul, "s"'] & W' \arrow[l, "\tau'"'] \arrow[u, shift right, "s'"']
\end{tikzcd} \tag{3}$$

*is commutative. In other words, any shadow can be reconstructed up to a nonlocal symmetry in some new covering.*

*Proof.* Consider the following commutative digram:

$$\begin{tikzcd}
 & T^V\pi_{\mathcal{E}} \arrow[d, "\pi^v_{\mathcal{E}}"'] & T^v(\tau\circ\pi_{\mathcal{E}}) \arrow[l] \arrow[d, "(\tau\circ\pi_{\mathcal{E}})^v"] & T^V(\tau^v\circ\tau\circ\pi_{\mathcal{E}}) \arrow[l] \arrow[d, "(\tau^v\circ\tau\circ\pi_{\mathcal{E}})^v"] & \cdots \arrow[l] & \\
M & \mathcal{E}^\infty \arrow[l, "\pi_{\mathcal{E}}"] & W \arrow[l, "\tau"] \arrow[ul, "s"'] & T^v\tau \arrow[l, "\tau^v"] \arrow[ul, "s_*"'] & T^v\tau^v \arrow[l, "(\tau^v)^v"] \arrow[ul, "(s_*)_*"'] & \cdots \arrow[l]
\end{tikzcd}$$

and let us set $\tau_0 = \tau$, $\tau_{i+1} = \tau_i^v$, $W_0 = W$, $W_i = T^v\tau_i$, $s_0 = s$, $s_{i+1} = (s_i)_*$, where $s_* = ds$. Then the above diagram is infinitely continued to the letf, while by setting $\bar{\tau}_i = \tau_1 \circ \cdots \circ \tau_i$ passing to the inverse limit, we obtain Diagram 3 with $\tau' = \bar{\tau}_\infty$, $s' = s_\infty$, and $W' = W_\infty$. $\square$

## 2. $\mathcal{C}$-COMPLEX AND DEFORMATIONS

We now pass to describe a cohomological theory naturally related to covering structures and supplying their important invariants, cf. [5].

Let $W$ be a smooth manifold and $D(\Lambda^i(W))$ denote the $C^\infty(W)$-module of $\Lambda^i(W)$-valued derivations $C^\infty(W) \to \Lambda^i(W)$. For any element $\Omega \in \Lambda^i(W)$

one can define the *inner product* operation

$$i_\Omega\colon \Lambda^j(W) \to \Lambda^{i+j-1}(W),$$

also denoted by $\Omega \lrcorner \rho$, $\rho \in \Lambda^*(W)$, and the *Lie derivative* along $\Omega$:

$$L_\Omega = [i_\Omega, d]\colon \Lambda^j(W) \to \Lambda^{i+j}(W),$$

where $[d, i_\Omega]$ denotes the *graded* commutator.

Then for any two elements $\Omega, \Theta \in D(\Lambda^*(W))$ we can introduce their *Frölicher–Nijenhuis bracket* by setting

$$[\![\Omega, \Theta]\!](f) = L_\Omega(\Theta(f)) - (-1)^{ij} L_\Theta(\Omega(f)),$$

where $f \in C^\infty(W)$ and $i, j$ are degrees of $\Omega$ and $\Theta$ respectively[2].

*Remark* 3. In the sequel, we shall also need the following facts.

1. In the case, when $W$ is a finite-dimensional manifold, one has an isomorphism $D(\Lambda^*(W)) \simeq \Lambda^*(W) \otimes D(W)$ and thus any derivation $\Omega \in D(\Lambda^*(W))$ is representable as a finite sum of elements of the form
$$\Omega = \omega \otimes X, \tag{4}$$
where $\omega \in \Lambda^*(W)$ and $X \in D(W)$. For an arbitrary $W$, an embedding $\Lambda^*(W) \otimes D(W) \subset D(\Lambda^*(W))$ is defined by $(\omega \otimes X)f = X(f)\omega$.
2. For elements of the form (4), one has
$$(\omega \otimes X) \lrcorner \rho = \omega \wedge (X \lrcorner \rho),$$
$$L_{\omega\otimes X}\rho = \omega \wedge L_X\rho + (-1)^i d\omega \wedge (X \lrcorner \rho)$$
and
$$\begin{aligned}[\![\omega \otimes X, \theta \otimes Y]\!] &= \omega \wedge \theta \otimes [X, Y] + \omega \wedge L_x(\theta) \otimes Y + (-1)^i d\omega \wedge (X \lrcorner \theta) \otimes Y \\ &\quad - (-1)^{ij}\theta \wedge L_Y(\omega) \otimes X - (-1)^{(i+1)j} d\theta \wedge (Y \lrcorner \omega) \otimes X,\end{aligned}$$
where $X, Y \in D(W)$, $\omega \in \Lambda^i(W)$, $\theta \in \Lambda^j(W)$.
3. Note also that another two operations are defined on elements of the module $\Omega \in D(\Lambda^*(W))$: we can multiply elements of $D(\Lambda^*(W))$ by forms $\rho \in \Lambda^*(W)$ and for $\Omega = \omega \otimes X$ one has $\rho \wedge \Omega = (\rho \wedge \omega) \otimes X$. In addition, we can insert elements of $D(\Lambda^*(W))$ into each other; in representation (4) this operation is represented as
$$(\omega \otimes X) \lrcorner (\theta \otimes Y) = \omega \wedge (X \lrcorner \theta) \otimes Y.$$

The basic properties of the above introduced operations are formulated in

**Proposition 3** (see [4]). *Let $\Omega \in D(\Lambda^i(W))$, $\Theta \in D(\Lambda^j(W))$, $\rho \in \Lambda^k(W)$, and $\eta \in \Lambda^l(W)$. Then:*

(i) $i_\Omega(\rho \wedge \eta) = i_\Omega(\rho) \wedge \eta + (-1)^{(i-1)k} \rho \wedge i_\Omega(\eta)$;

(ii) $i_\Omega(\rho \wedge \Theta) = i_\Omega(\rho) \wedge \Theta + (-1)^{(i-1)k} \rho \wedge i_\Omega(\Theta)$;

(iii) $[i_\Omega, i_\Theta] = i_{[\![\Omega,\Theta]\!]^{\mathrm{rn}}}$, *where*
$$[\![\Omega, \Theta]\!]^{\mathrm{rn}} = i_\Omega\Theta - (-1)^{(i-1)(j-1)} i_\Theta\Omega$$
*is the* Richardson–Nijenhuis bracket *of $\Omega$ and $\Theta$;*

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[2] We say that $i$ is the degree of $\Omega$, if $\Omega \in D(\Lambda^i(W))$.

(iv) $L_\Omega(\rho\wedge\eta) = L_\Omega(\rho)\wedge\eta + (-1)^{ik}\omega\wedge L_\Omega(\eta)$;
(v) $L_{\rho\wedge\Omega} = \rho\wedge L_\Omega + (-1)^{i+k} d\omega\wedge i_\Omega$;
(vi) $[L_\Omega, d] = 0$;
(vii) $[L_\Omega, L_\Theta] = L_{[\![\Omega,\Theta]\!]}$;
(viii) $[\![\Omega,\Theta]\!] + (-1)^{ij}[\![\Theta,\Omega]\!] = 0$;
(ix) $[\![\Omega,[\![\Theta,\Xi]\!]]\!] = [\![[\![\Omega,\Theta]\!],\Xi]\!] + (-1)^{ij}[\![\Omega,[\![\Theta,\Xi]\!]]\!]$, *where* $\Xi\in D(\Lambda^m(W))$;
(x) $[L_\Omega, i_\Theta] = i_{[\![\Omega,\Theta]\!]} - (-1)^{i(j+1)} L_{\Theta\lrcorner\Omega}$;
(xi) $\Xi\lrcorner[\![\Omega,\Theta]\!] = [\![\Xi\lrcorner\Omega,\Theta]\!] + (-1)^{i(m+1)}[\![\Omega,\Xi\lrcorner\Theta]\!] + (-1)^i[\![\Xi,\Omega]\!]\lrcorner\Theta$
$- (-1)^{(i+1)j}[\![\Xi,\Theta]\!]\lrcorner\Omega$;
(xii) $[\![\Omega,\rho\wedge\Theta]\!] = (L_\Omega\rho)\wedge\Theta - (-1)^{(i+1)(j+k)} d\rho\wedge i_\Theta\Omega + (-1)^{ik}\rho\wedge[\![\Omega,\Theta]\!]$.

In particular, from Proposition 3 (ix) it follows that for $\Omega\in D(\Lambda^1(W))$ satisfying the *integrability property*

$$[\![\Omega,\Omega]\!] = 0 \tag{5}$$

the mapping

$$\partial_\Omega = [\![\Omega,\cdot]\!]\colon D(\Lambda^i(W)) \to D(\Lambda^{i+1}(W))$$

is a differential, i.e., $\partial_\Omega\circ\partial_\Omega = 0$, and thus we obtain the complex

$$0 \to D(W) \to \cdots \to D(\Lambda^i(W)) \xrightarrow{\partial_\Omega} D(\Lambda^{i+1}(W)) \to \cdots \tag{6}$$

Assume now that the manifold $W$ is fibered by $\xi\colon W\to M$ and a connection $\nabla$ is given in the bundle $\xi$. Then the following fact is valid:

**Proposition 4** (cf. [3])**.**

$$[\![U_\nabla, U_\nabla]\!] = 2R_\nabla,$$

*where* $U_\nabla$ *is the connection form and* $R_\nabla$ *is the curvature.*

Consequently, if $\nabla$ is a flat connection, i.e., $R_\nabla = 0$, then $\Omega = U_\nabla$ enjoys the integrability property (5) and to any flat connection a complex of the form (6) corresponds. In this case, we shall use the notation $\partial_\Omega = \partial_\nabla$.

Now, we pass to the case of our main interest: let $\xi$ be the composition $W \xrightarrow{\tau} \mathcal{E}^\infty \xrightarrow{\pi_\mathcal{E}} M$, $\tau$ being a covering over $\mathcal{E}$, and $\nabla$ be the Cartan connection $\mathcal{C}_\tau$ associated to the covering structure. We include in consideration the case $W=\mathcal{E}^\infty$, $\tau=\mathrm{id}$, and $\mathcal{C}_\tau = \mathcal{C}_\mathcal{E}$. Let us restrict complex (6) to *vertical* derivations, i.e., to derivations

$$D^v(\Lambda^i(W)) = \{\,\Omega\in D(\Lambda^i(W)) \mid \Omega(f) = 0, \forall f\in C^\infty(M)\,\}.$$

By construction, $U_\tau$ (or $U_\mathcal{E}$) lies in $D^v(\Lambda^1(W))$ (resp., in $D^v(\Lambda^1(\mathcal{E}))$), while from the definition of the Frölicher–Nijenhuis bracket it follows that the differential in (6) preserves vertical derivations. The vertical part of (6) will be denoted by

$$0 \to D^v(W) \to \cdots \to D^v(\Lambda^i(W)) \xrightarrow{\partial_\tau} D^v(\Lambda^{i+1}(W)) \to \cdots \tag{7}$$

or

$$0 \to D^v(\mathcal{E}) \to \cdots \to D^v(\Lambda^i(\mathcal{E})) \xrightarrow{\partial_\mathcal{E}} D^v(\Lambda^{i+1}(\mathcal{E})) \to \cdots, \tag{8}$$

when the equation is considered as is. The cohomology of (7) (resp., of (8)) is denoted by $H_\mathcal{C}(\mathcal{E};\tau)$ (resp., by $H_\mathcal{C}(\mathcal{E})$) and is called the $\mathcal{C}$*-cohomology* of the covering $\tau$ (resp., of the equation $\mathcal{E}$). The following fundamental result is valid:

**Theorem 1** (cf. [5]). *Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\tau\colon W \to \mathcal{E}^\infty$ be a covering over $\mathcal{E}$. Then:*

1. *The module $H^0_{\mathcal{C}}(\mathcal{E};\tau)$ is isomorphic to the Lie algebra $\operatorname{sym}_\tau \mathcal{E}$ of nonlocal $\tau$-symmetries (resp., $H^0_{\mathcal{C}}(\mathcal{E};\tau)$ is isomorphic to $\operatorname{sym}\mathcal{E}$).*
2. *The module $H^1_{\mathcal{C}}(\mathcal{E};\tau)$ is identified with equivalence classes of nontrivial infinitesimal deformations of the covering structure $U_\tau$ (resp., of the equation structure $U_{\mathcal{E}}$).*
3. *The module $H^2_{\mathcal{C}}(\mathcal{E};\tau)$ consists of obstructions to prolongation of infinitesimal deformations up to formal ones.*

Let us now consider the mapping $L_{U_\tau}\colon \Lambda^i(W) \to \Lambda^{i+1}(W)$ and denote it by $d_v$. Since the element $U_\tau$ is integrable, one has the identity $d_v \circ d_v = 0$. We call $d_v$ the *vertical*, or *Cartan differential* associated to the covering structure. Due to Proposition 3 (vi), $[d, d_v] = 0$ and consequently the mapping $d_h = d - d_v$ is also a differential and $[d_h, d_v] = 0$. The differential $d_h$ is called the *horizontal differential*, while the pair $(d_h, d_v)$ forms a bicomplex with the total differential $d$. The corresponding spectral sequence coincides with the Vinogradov $\mathcal{C}$-spectral sequence for the covering $\tau$, [15].

Denote by $\Lambda^1_h(W)$ the submodule in $\Lambda^1(W)$ spanned by $\operatorname{im} d_h$ and by $\mathcal{C}^1\Lambda(W)$ the submodule generated by $\operatorname{im} d_v$. Then the direct sum decomposition $\Lambda^1(W) = \Lambda^1_h(W) \oplus \mathcal{C}\Lambda^1(W)$ takes place and generates the decomposition

$$\Lambda^i(W) = \bigoplus_{p+q=i} \mathcal{C}^p\Lambda(W) \otimes \Lambda^q_h(W) = \bigoplus_{p+q=i} \Lambda^{p,q}(W),$$

where

$$\mathcal{C}^p\Lambda(W) = \underbrace{\mathcal{C}^1\Lambda(W) \wedge \ldots \wedge \mathcal{C}^1\Lambda(W)}_{p \text{ times}}, \qquad \Lambda^q_h(W) = \underbrace{\Lambda^1_h(W) \wedge \ldots \wedge \Lambda^1_h(W)}_{q \text{ times}}.$$

Then $d_v\colon \Lambda^{p,q}(W) \to \Lambda^{p+1,q}(W)$, $d_h\colon \Lambda^{p,q}(W) \to \Lambda^{p,q+1}(W)$ and, moreover, as it follows from Proposition 3 (xi), $\partial_\tau\colon D^v(\Lambda^{p,q}(W)) \to D^v(\Lambda^{p,q+1}(W))$.

*Remark* 4. The complex $(\Lambda^q_h(W), d_h)$ is called the *horizontal complex* of the covering $\tau$, while its cohomology is the horizontal cohomology of $\tau$. It is worth to note that $d_h$ in this case is obtained from the de Rham differential on the manifold $M$ by applying the operation $\mathcal{C} = \mathcal{C}_\tau$ (see Remark 1). From Proposition 3 (xii) it follows that the $\mathcal{C}$-cohomology of $\tau$ is a graded module over the graded algebra of horizontal cohomology.

## 3. BÄCKLUND TRANSFORMATIONS AND THE MAIN RESULT

Following [10], let us give a geometric definition of Bäcklund transformations. Let $\mathcal{E}_i \subset J^{k_i}(\pi_i)$, $i = 1, 2$, be two differential equations and

$\tau_i\colon W \to \mathcal{E}_i^\infty$ be coverings with the same total space $W$. Then the diagram

$$\begin{tikzcd} & W \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ \mathcal{E}_1^\infty & & \mathcal{E}_2^\infty \end{tikzcd}$$

is called a *Bäcklund transformation* between the equations $\mathcal{E}_1^\infty$ and $\mathcal{E}_2^\infty$. We say that it is a *Bäcklund autotransformation*, if $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty = \mathcal{E}^\infty$. Below we confine ourselves with autotransformations only.

Let $\mathcal{B} = (W, \tau_1, \tau_2, \mathcal{E})$ be a Bäcklund autotransformation. A point $w \in W$ is called $\tau_1$*-generic*, if the plane of the distribution $\mathcal{C}_{\tau_1}D(W)$ passing through $w$ has a trivial intersection with the tangent plane at $w$ to the fiber of $\tau_2$ passing through the same point. Now, if $s \subset \mathcal{E}^\infty$ is a solution of $\mathcal{E}$ and $\tau_1^{-1}(s)$ contains a $\tau_1$-generic point, then there exists a neighborhood $\mathcal{U}$ of this point such that $\tau_2(\mathcal{U} \cap \tau_1^{-1}(s))$ is fibered by solutions of $\mathcal{E}$. Thus, Bäcklund transformations really determine a correspondence between solutions.

The property of a Bäcklund transformation to be generic is naturally reformulated in global terms of structural elements. Let $U_i = U_{\tau_i}$ be the structural element of the covering $\tau_i$. Then $U_i$ may be understood as a linear mapping $U_i\colon D(W) \to D(W)$, $X \mapsto X \lrcorner\, U_i$. Moreover, $U_i$ is a projector, i.e., $U_i \circ U_i = \mathrm{id}$, and thus gives the splitting

$$D(W) = \ker U_i \oplus \operatorname{im} U_i = \mathcal{C}_{\tau_i}D(W) \oplus D^{v,i}(W),$$

where $D^{v,i}(W)$ is the module of $\tau_i$-vertical vector fields on $W$. Let us denote by

$$U_{2,1} = U_2|_{D^{v,1}(W)} : D^{v,1}(W) \to D^{v,2}(W)$$

the restriction of $U_2$ to $D^{v,1}(W)$. Then $\mathcal{B}$ is *globally* $\tau_1$*-generic*, if $U_{2,1}$ is a monomorphism. It is generic in a *strong sense*, if $U_{2,1}$ is an isomorphism.

The following construction is equivalent to the definition of Bäcklund transformations. Let $\tau_i\colon W_i \to \mathcal{E}^\infty$, $i = 1, 2$, be two coverings and $F\colon W_1 \to W_2$ be a diffeomorphism taking the distribution $\mathcal{C}_{\tau_1}D(W)$ to $\mathcal{C}_{\tau_2}D(W)$. Then $\mathcal{B} = (W, \tau_1, \tau_2 \circ F, \mathcal{E})$ is a Bäcklund transformations and any Bäcklund transformations is formally obtained in such a way.

*Remark* 5. It is important to stress here that if $F$ is an isomorphism of coverings, then the Bäcklund transformation obtained in such a way is *trivial* in the sense of its action on solutions. Thus, we are interested in mappings $F$ such that they are isomorphisms of manifolds with distributions, but not morphisms of coverings.

Assume now that a smooth family $F_\lambda\colon W_1 \to W_2$ is given, Then it generates the corresponding family $\mathcal{B}_\lambda$ of Bäcklund transformations. Our aim is to describe such families in sufficiently efficient terms. One way to construct these objects is given by the following

**Example 2** (see [10]). Consider an equation $\mathcal{E}$, a covering $\tau\colon W \to \mathcal{E}^\infty$ over it, and a finite symmetry $A\colon \mathcal{E}^\infty \to \mathcal{E}^\infty$. Let $\bar{A}\colon W \to W$ be a diffeomorphic lifting of $A$ to $W$ such that

$$\tau \circ \bar{A} = A \circ \tau. \tag{9}$$

Denote by $\bar{A}_*\mathcal{C}_\tau D(W)$ the image of the distribution $\mathcal{C}_\tau D(W)$ under $\bar{A}$. Then, by obvious reasons, $\bar{A}_*\mathcal{C}_\tau D(W)$ determines a covering structure $U_\tau^{\bar{A}}$ in $W$ and if $\tilde{A}$ is another lifting of $A$, then the structures $U_\tau^{\bar{A}}$ and $U_\tau^{\tilde{A}}$ are equivalent. Thus, $\mathcal{B}_A = (W, \tau, A \circ \tau, \mathcal{E})$ is a Bäcklund transformation for $\mathcal{E}$.

Let $X$ be a classical infinitesimal symmetry of $\mathcal{E}$ and $A_\lambda = \exp(\lambda X)\colon \mathcal{E}^\infty \to \mathcal{E}^\infty$ the corresponding one-parameter group of transformations lifted to $\mathcal{E}^\infty$. Then, using the above construction, we obtain $\lambda$-parameter family of Bäcklund transformations $\mathcal{B}_\lambda = \mathcal{B}_{A_\lambda}$.

*Remark* 6. Note that since the symmetry $X$ generating the family $\mathcal{B}_\lambda$ above cannot be lifted as a symmetry of $W$ (i.e., as a nonlocal $\tau$-symmetry), it is a shadow in the covering $\tau$, as well as in all coverings $\tau_\lambda = A_\lambda \circ \tau$.

In fact, the families of Bäcklund transformations obtained in the previous example are in a sense “counterfeit”, since, due to (9), their action on solutions reduces to the action of symmetries $A_\lambda$. To get a “real” Bäcklund transformation, one needs to add into considerations an additional mapping $F\colon W \to W$ preserving the Cartan distribution on $W$ but violating (9). Nevertheless, as it will be shown below, the construction of Example 2 is of almost general nature. Toward this end, let us do the following.

Let us denote by

$$D^g(\Lambda^i(W)) = \{\, \Omega \in D^v(\Lambda^i(W)) \mid \Omega(f) = 0 \forall f \in C^\infty(\mathcal{E}^\infty) \,\}$$

the module of $\tau$-vertical derivations.

**Lemma 1.** *The modules $D^g(\Lambda^i(W))$ are invariant with respect to the differential $\partial_\tau$:*

$$\partial_\tau\big(D^g(\Lambda^i(W))\big) \subset D^g(\Lambda^{i+1}(W)).$$

*Proof.* Let $\Omega \in D^g(\Lambda^i(W))$ and $f \in C^\infty(\mathcal{E}^\infty)$. Then due to the definition of the Frölicher–Nijenhuis bracket one has

$$(\partial_\tau(\Omega))(f) = [\![U_\tau, \Omega]\!](f) = L_{U_\tau}(\Omega(f)) - (-1)^\Omega L_\Omega(U_\tau(f)).$$

The first summand vanishes, since $\Omega \in D^g(\Lambda^i(W))$. On the other hand, $U_\tau(f) = U_\mathcal{E}(f)$ and consequently is a one-form on $\mathcal{E}^\infty$. Hence, the second summand vanishes as well. ☐

Denote by $\partial_g\colon D^g(\Lambda^i(W)) \to D^g(\Lambda^{i+1}(W))$ the restriction of $\partial_\tau$ to $D^g(\Lambda^i(W))$ and by

$$\partial_s\colon D^s(\Lambda^i(W)) \to D^s(\Lambda^{i+1}(W))$$

the corresponding quotient complex, where, by definition,

$$D^s(\Lambda^i(W)) = D^v(\Lambda^i(W))/D^g(\Lambda^i(W)).$$

Then the short exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^g(W) \arrow[r, "\partial_g"] \arrow[d] & D^g(\Lambda^1(W)) \arrow[r] \arrow[d] & \dots \arrow[r] & D^g(\Lambda^i(W)) \arrow[r, "\partial_g"] \arrow[d] & D^g(\Lambda^{i+1}(W)) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & D^v(W) \arrow[r, "\partial_\tau"] \arrow[d] & D^v(\Lambda^1(W)) \arrow[r] \arrow[d] & \dots \arrow[r] & D^v(\Lambda^i(W)) \arrow[r, "\partial_\tau"] \arrow[d] & D^v(\Lambda^{i+1}(W)) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & D^s(W) \arrow[r, "\partial_s"] \arrow[d] & D^s(\Lambda^1(W)) \arrow[r] \arrow[d] & \dots \arrow[r] & D^s(\Lambda^i(W)) \arrow[r, "\partial_s"] \arrow[d] & D^s(\Lambda^{i+1}(W)) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

is defined.

Denote by $H_g^i(\mathcal{E};\tau)$ and $H_s^i(\mathcal{E};\tau)$ the cohomology of the top and bottom lines respectively. Then one has the long exact cohomology sequence

$$0 \to H_g^0(\mathcal{E};\tau) \to H_{\mathcal{C}}^0(\mathcal{E};\tau) \to H_s^0(\mathcal{E};\tau) \xrightarrow{\phi} H_g^1(\mathcal{E};\tau) \to H_{\mathcal{C}}^1(\mathcal{E};\tau) \to H_s^1(\mathcal{E};\tau) \to \\ \cdots \to H_g^i(\mathcal{E};\tau) \to H_{\mathcal{C}}^i(\mathcal{E};\tau) \to H_s^i(\mathcal{E};\tau) \to \cdots, \qquad (10)$$

where $\phi$ is the connecting homomorphism.

Similar to Theorem 1, we have the following result:

**Proposition 5.** *In the situation above one has*:

1. *The module $H_g^0(\mathcal{E};\tau)$ consists of "gauge" symmetries in the covering $\tau$, i.e., of nonlocal $\tau$-symmetries vertical with respect to the projection $\tau$.*
2. *The module $H_s^0(\mathcal{E};\tau)$ coincides with the set of $\tau$-shadows in the covering $\tau$.*
3. *The module $H_g^1(\mathcal{E};\tau)$ consists of equivalence classes of deformations of the covering structure $U_\tau$ acting trivially on the equation structure $U_{\mathcal{E}}$.*

Now, combining the last result with exact sequence (10), we obtain the following fundamental theorem:

**Theorem 2.** *Let $\mathcal{B} = (W, \tau, \tau_\lambda, \mathcal{E})$ be a smooth family*[3] *of Bäcklund transformations such that $\tau_0 = \tau$. Then $U_{\tau_\lambda}$ is of the form*

$$U_{\tau_\lambda} = U_\tau + \lambda [\![ U_\tau, X ]\!] + O(\lambda^2), \qquad (11)$$

*where $X$ is a $\tau$-shadow, i.e., all smooth families corresponding to the covering $\tau$ are identified with* $\operatorname{im}\partial_s$.

*Proof.* In fact, let $\mathcal{B}_\lambda$ be a family under considerations. Then $\tau_\lambda$ is a deformation of $\tau$. Since we work with deformations which leave the equation structure unchanged, then, by Proposition 5, their infinitesimal parts are elements of $H_g^1(\mathcal{E};\tau)$. Let $\Omega$ be such an element.

---

[3] We say that a family is *smooth,* if the family $U_{\tau_\lambda}$ is smooth in $D^v(\Lambda^1(W))$.

Now, by Remark 5, the deformation we are dealing with is to be trivial as a deformation of $W$ endowed with the structure $U_\tau$. On the infinitesimal level, this mean that the image of $\Omega$ in $H^1(\mathcal{E};\tau)$ should vanish. But by exactness of (10) we see that $\Omega = \phi(X)$ for some $X \in H^0_s(\mathcal{E};\tau)$. It now suffices to note that by construction of the connecting homomorphism, $\phi(X) = [\![U_\tau, X]\!]$. ☐

Denote by $\mathcal{B}(\tau)$ the "manifold" of all Bäcklund transformations obtained from the covering $\tau$ by the above described way. Then from exactness of (10) it follows that the tangent plane to $\mathcal{B}(\tau)$ at $\tau$ is identified with the space $\mathrm{shad}_\tau\mathcal{E}/\overline{\mathrm{sym}}_\tau\mathcal{E}$, where $\mathrm{shad}_\tau\mathcal{E} = H^0_s(\mathcal{E};\tau)$ is the space of all $\tau$-shadows. Finally, the space $\overline{\mathrm{sym}}_\tau\mathcal{E} = \mathrm{sym}_\tau\mathcal{E}/\mathrm{sym}^g_\tau\mathcal{E}$ is the quotient of all $\tau$-symmetries over gauge ones. In particular, if a covering $\tau$ is such that any $\tau$-shadow can be reconstructed up to a nonlocal $\tau$-symmetry, then $\tau$ cannot be included in a nontrivial family of Bäcklund transformations.

**Example 3** (Universal Abelian covering)**.** An example of a covering of the above mentioned type is given by the following construction. It is known [2, 10] that to any horizontal 1-cocycle there corresponds a one-dimensional covering and cohomologous cocycles determine equivalent coverings.

Let $\mathcal{E}$ be an equation and $[\omega_1],\dots,[\omega_r],\dots$ be a basis in $H^1_h(\mathcal{E})$ consider the Whitney product of coverings corresponding to all classes $[\omega_i]$ and denote it by $\mathfrak{a}_1\colon \mathfrak{A}_1 \to \mathcal{E}^\infty$. For thus obtained object, let us repeat the construction and consider the covering $\mathfrak{a}_2\colon \mathfrak{A}_2 \to \mathfrak{A}_1$, etc. Thus we obtain the infinite sequence

$$\dots \to \mathfrak{A}_m \xrightarrow{\mathfrak{a}_m} \mathfrak{A}_{m-1} \to \dots \to \mathfrak{A}_2 \xrightarrow{\mathfrak{a}_2} \mathfrak{A}_1 \xrightarrow{\mathfrak{a}_1} \mathcal{E}^\infty,$$

and the inverse limit $\mathfrak{a}\colon \mathfrak{A} \to \mathcal{E}^\infty$ which is called the *universal Abelian covering* of $\mathcal{E}$.

**Theorem 3** ([2])**.** *Any $\mathfrak{a}$-shadow can be reconstructed up to a nonlocal $\mathfrak{a}$-symmetry.*

## 4. CONCLUDING REMARKS

We conclude the above discussion with a number of remarks of various nature.

4.1. **Bäcklund transformations and recursion operators.** As it was shown in [14], a recursion operator for symmetries of an equation $\mathcal{E}$ may be understood as a Bäcklund transformation of the form $\mathcal{R} = (W_R, \rho_1, \rho_2, T^v\pi_{\mathcal{E}})$, where $\pi^v_{\mathcal{E}}\colon T^v\pi_{\mathcal{E}} \to \mathcal{E}^\infty$ is the covering from Example 1. Let now $\mathcal{B} = (W, \tau_1, \tau_2, \mathcal{E})$ be an arbitrary Bäcklund transformation of equation $\mathcal{E}$. Then, using the fact that $T^v$ is a functor (see Section 1), we obtain the commutative diagram

$$\begin{tikzcd}
 & \tilde{W} \arrow[dl, "(\tau_1^v)^*(\tau_2^v)"'] \arrow[dd, "\tilde{\tau}"'] \arrow[dr, "(\tau_2^v)^*(\tau_1^v)"] & \\
T^v\tau_1 \arrow[d, "\tau_{1,*}"'] \arrow[dr, "\tau_1^v"] & & T^v\tau_2 \arrow[d, "\tau_{2,*}"] \arrow[dl, "\tau_2^v"'] \\
T^v\pi_{\mathcal{E}} \arrow[d, "\pi_{\mathcal{E}}^v"'] & W \arrow[dl, "\tau_1"] \arrow[dr, "\tau_2"'] & T^v\pi_{\mathcal{E}} \arrow[d, "\pi_{\mathcal{E}}^v"] \\
\mathcal{E}^\infty & & \mathcal{E}^\infty
\end{tikzcd}$$

where $\tilde{\tau} = \tau_1^v \oplus \tau_2^v$ is the Whitney product. Thus,

$$T^v\mathcal{B} = \big(\tilde{W}, \tau_{1,*} \circ (\tau_1^v)^*(\tau_2^v), \tau_{2,*} \circ (\tau_2^v)^*(\tau_1^v), T^v\pi_{\mathcal{E}}\big)$$

is a recursion operator and we obtain

**Proposition 6.** *Any Bäcklund autotransformation of an equation $\mathcal{E}$ generates a recursion operator for symmetries of this equation.*

4.2. **Bäcklund transformations and symmetries.** The following construction belongs to M. Marvan [13]. Let $\mathcal{B} = (W, \tau, \mu, \mathcal{E})$ be a Bäcklund transformation. Then one may construct the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r, "\tau_{i+3}"] & W_{i+2} \arrow[r, "\tau_{i+2}"] \arrow[d, "\mu_{i+2}"] & W_{i+1} \arrow[r, "\tau_{i+1}"] \arrow[d, "\mu_{i+1}"] & \dots \arrow[r, "\tau_2"] & W_1 \arrow[r, "\tau_1"] \arrow[d, "\mu_1"] & W_0 \arrow[d, "\mu_0"] \\
\dots \arrow[r, "\tau_{i+2}"] & W_{i+1} \arrow[r, "\tau_{i+1}"] & W_i \arrow[r, "\tau_i"] & \dots \arrow[r, "\tau_1"] & W_0 \arrow[r, "\tau_0"] & \mathcal{E}^\infty
\end{tikzcd}$$

where $\tau_0 = \tau$, $\mu_0 = \mu$, $\tau_{i+1} = \mu_i^*(\tau_i)$, $\mu_{i+1} = \tau_i^*(\mu_i)$, and $W_{i+1}$ is the total space of the Whitney product of $\tau_i$ and $\mu_i$. Passing to the inverse limit, we obtain the mapping $\mu_\infty \colon W_\infty \to W_\infty$, which is a finite symmetry of $W_\infty$.

Let now $\mathcal{B}_\lambda = (W, \tau, \tau_\lambda, \mathcal{E})$ be a smooth family of Bäcklund transformations. Then, by the above construction, we obtain a smooth family of finite symmetries $\tau_{\lambda,\infty} \colon W_\infty \to W_\infty$. Let us set $X_\infty = d\tau_{\lambda,\infty}/d\lambda|_{\lambda=0}$. Then $X_\infty \in \operatorname{sym}_{\tau_\infty} W_\infty$.

**Conjecture.** The shadow $X$ determining the infinitesimal deformation corresponding to the family $\mathcal{B}_\lambda$ is the shadow of $X_\infty$, i.e., $X_\infty|_W = X$.

4.3. **Formal deformations.** Let $X$ be a $\tau$-shadow and $U_\tau + \lambda[\![U_\tau, X]\!]$ be the corresponding infinitesimal deformation of the covering structure. In general, a possibility to continue this deformation up to a formal one is related to triviality of the group $H_g^2(\mathcal{E};\tau)$, cf. Theorem 1. Nevertheless, in the particular case under consideration there always exists such a formal continuation.

Let $U \in D(\Lambda^1(W))$ and $X \in D(W)$. Consider the formal series

$$e^{\lambda X}U = U + \lambda \operatorname{ad}_X U + \dots + \frac{\lambda^i}{i!} \operatorname{ad}_X^i U + \cdots, \tag{12}$$

where $\operatorname{ad}_X = [\![\,\cdot\,, X]\!]$.

**Lemma 2.** *One has*

$$\frac{d^i}{d\lambda^i}[\![e^{\lambda X}U, e^{\lambda X}U]\!] = \operatorname{ad}_X^i [\![e^{\lambda X}U, e^{\lambda X}U]\!]$$

*for any $U$ and $X$ of the above form.*

*Proof.* Let us first note that by definition, $d(e^{\lambda X}U)/d\lambda = \operatorname{ad}_X e^{\lambda X}U$. Then, using Proposition 3 (ix), one gets

$$\begin{aligned}\frac{d^i}{d\lambda^i}[\![e^{\lambda X}U, e^{\lambda X}U]\!] &= \sum_{j=0}^{i}\binom{i}{j}[\![\frac{d^j}{d\lambda^j}e^{\lambda X}U, \frac{d^{i-j}}{d\lambda^{i-j}}e^{\lambda X}U]\!] \\ &= \sum_{j=0}^{i}\binom{i}{j}[\![\operatorname{ad}_X^j e^{\lambda X}U, \operatorname{ad}_X^{i-j} e^{\lambda X}U]\!] = \operatorname{ad}_X^i [\![e^{\lambda X}U, e^{\lambda X}U]\!]\end{aligned}$$

□

**Corollary 1.** *If in (12) $U = U_\tau$ is the structural element of a covering $\tau$, then $[\![e^{\lambda X}U_\tau, e^{\lambda X}U_\tau]\!] = 0$, i.e., $e^{\lambda X}U_\tau$ is a formal deformation of $U_\tau$.*

*Proof.* In fact, to prove the result it suffices to show that

$$\frac{d^i}{d\lambda^i}[\![e^{\lambda X}U_\tau, e^{\lambda X}U_\tau]\!]\Big|_{\lambda=0} = 0$$

for all $i \geq 0$. But by Lemma 2 one has

$$\begin{aligned}\frac{d^i}{d\lambda^i}[\![e^{\lambda X}U_\tau, e^{\lambda X}U_\tau]\!]\Big|_{\lambda=0} &= \Big(\operatorname{ad}_X^i [\![e^{\lambda X}U_\tau, e^{\lambda X}U_\tau]\!]\Big)\Big|_{\lambda=0} \\ &= \operatorname{ad}_X^i \Big([\![e^{\lambda X}U_\tau, e^{\lambda X}U_\tau]\!]\Big|_{\lambda=0}\Big) = \operatorname{ad}_X^i [\![U_\tau, U_\tau]\!] = 0,\end{aligned}$$

since the element $U_\tau$ is integrable. □

4.4. **Some problems.** To conclude, I would like to state some problems related to the text above.

**Problem 1.** Having a rather explicit description for smooth families of Bäcklund transformations, it seems realistic now to prove the permutability theorem in general case.

**Problem 2.** Of course, formal deformations of the form (12) do not exhaust all possible deformations of the initial structure. Does there exist an efficient description for other formal deformations?

**Problem 3.** What are the conditions for convergency of (12)?

**Problem 4.** Probably, a realistic way to solve Problems 2 and 3 lies in constructing a geometrical theory of differential equations with a parameter. In fact, a formal deformation is nothing else but an infinite jet of some quantity with respect to a parameter. Thus, such a theory allows one to put the theory of deformation in a well-posed framework of geometrical and algebraic theory of PDE and apply all tools of the latter.

**Problem 5.** Finally, recall the following construction. Let $\tau_\lambda\colon W_\lambda \to \mathcal{E}^\infty$ be a smooth family of coverings with unremovable parameter $\lambda$ [11]. Then coordinates $w_r$ along the fibers of $\tau_\lambda$ (nonlocal variables), due to the covering structure, satisfy a first-order system of differential equations

$$\frac{\partial \mathbf{w}}{\partial x_i} = \mathbf{F}_i(\mathbf{x}, \mathbf{u}, \mathbf{w}; \lambda), \tag{13}$$

where $\mathbf{w} = (w^1, w^2, \dots)$, $\mathbf{x} = (x_1, \dots, x_n)$ are local coordinates in $M$ (independent variables), and $\mathbf{u}$ are internal coordinates in $\mathcal{E}^\infty$ (unknown functions and their partial derivatives). Assuming dependency of $\mathbf{w}$ on $\lambda$ and expanding both sides of (13) in formal series in $\lambda$, we obtain an infinite system of equations on coefficients of this expansion. This system determines an infinite-dimensional covering over $\mathcal{E}$. In some cases, an infinite system of elements in $H_h^1$ is associated to this covering (in the case $n = 2$ these elements are identified with — generally speaking, nonlocal — conservation laws of $\mathcal{E}$.

At least, two questions arise:

1. How to implement a purely geometric version of this construction? It is reasonable to hope that the answer lies in solving Problem 4.
2. As it follows from the above said, the construction does not need a *smooth* family of covering, but a *formal* family only. Consequently, one can apply the construction to the canonical expansion (12) associated to a shadow $X$. Thus, to any covering $\tau$ and a $\tau$-shadow $X$ we can put into correspondence, in a natural way, a new infinite-dimensional covering. What is an explicit description of the latter?

   **Conjecture.** This covering coincides with the one constructed in Proposition 2.

   Another question: if a series of conservation laws corresponds to this covering, how to describe this series in terms of the initial covering $\tau$ and the shadow $X$?

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