CONTENTS OF THE COURSE

GEOMETRY OF DIFFERENTIAL EQUATIONS

SECOND DIFFIETY SCHOOL,
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1. Jet bundles

1.1. **Vector bundles and sections.** Smooth manifolds. Smooth locally trivial vector bundles. Sections. The $C^\infty(M)$-module structure in $\Gamma(\pi)$.

1.2. **Jets.** The jet $[f]_x^k$ of a local section f at a point $x \in M$. The space $J_x^k(\pi)$. Smooth structure in $\bigcup_{x\in M} J_x^k(\pi)$. Manifolds $J^k(\pi)$ and bundles $\pi_k\colon J^k(\pi) \to M$. The jet modules $\mathcal{J}^k(\pi) = \Gamma(\pi_k)$. Canonical coordinates u_σ^j in $J^k(\pi)$ associated to a local trivialization in π. Dimension of $J^k(\pi)$. The bundles $\pi_{k,l}\colon J^k(\pi) \to J^l(\pi)$, $k \geq l$. Graphs of jets. R-planes. Presentation of points of $J^k(\pi)$ as pairs $(\theta_{k-1}, L_{\theta_k})$, where $\theta_{k-1} \in J^{k-1}(\pi)$ and $L_{\theta_k} \subset T_{\theta_{k-1}} J^{k-1}(\pi)$ is an R-plane.

1.3. **Nonlinear differential operators.** Presentation of scalar operators as functions on $J^k(\pi)$. Pull-backs $\pi_k^*(\xi)$ and nonlinear operators $\Delta\colon \Gamma(\pi) \to \Gamma(\xi)$ as sections of the bundles $\pi_k^*(\xi)$. Presentation of operators as morphisms $J^k(\pi) \to J^0(\xi)$. The universal operator $j_k\colon \Gamma(\pi) \to \Gamma(\pi_k)$. Prolongations of nonlinear operators and their correspondence to morphisms $J^{k+l}(\pi) \to J^l(\xi)$. Composition of nonlinear operators.

1.4. **Nonlinear equations.** Differential equations as submanifolds in $\mathcal{E} \subset J^k(\pi)$. Description of equations by nonlinear operators. The first prolongation $\mathcal{E}^1 \subset J^{k+1}(\pi)$. Three definitions of the l-the prolongation, there equivalence. Solutions.

2. Geometry of the Cartan distribution in $J^k(\pi)$

2.1. **The Cartan distribution.** The Cartan plane $\mathcal{C}_\theta^k$ as the span of the set of R-planes at the point $\theta \in J^k(\pi)$. The distribution $\mathcal{C}^k\colon \theta \mapsto$

$\mathcal{C}^k_\theta$. Description of $\mathcal{C}^k_\theta$ in the form $(\pi_{k,k-1})_*^{-1}(L_{\theta_k})$. Local description of $\mathcal{C}^k$ by the Cartan forms $\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma+1_i} dx_i$. A local basis in $\mathcal{C}^k$.

2.2. **Maximal integral manifolds of the distribution $\mathcal{C}^k$.** Involutive subspaces of the Cartan distribution. The theorem on maximal integral manifolds. The type of a maximal integral manifold. Computation of dimensions for maximal integral manifolds. Integral manifolds of maximal dimension in inexceptional cases.

2.3. **The Lie–Bäcklund theorem.** Lie transformations as diffeomorphisms of $J^k(\pi)$ preserving the Cartan distribution. Lifting of Lie transformations from $J^k(\pi)$ to $J^{k+1}(\pi)$. The case $\dim \pi > 1$: correspondence between Lie transformations and diffeomorphisms of $J^0(\pi)$. The case $\dim \pi = 1$: the contact structure in $J^1(\pi)$, correspondence between Lie transformations and contact transformations of $J^1(\pi)$ (inexceptional case $\dim M \neq 1$ and exceptional case $\dim M = 1$). Local formulas for liftings of Lie transformations.

2.4. **Infinitesimal theory.** Lie fields. Local lifting formulas. Global nature of lifting for Lie fields. Infinitesimal analog for the Lie–Bäcklund theorem.

One-dimensional bundles. Generating functions of Lie fields. Correspondence between functions on $J^1(\pi)$ and Lie fields for trivial one-dimensional bundles. The jacobi bracket on $C^\infty(J^1(\pi))$. Local coordinate formulas for Lie fields and Jacobi brackets in terms of generating functions.

Bundles of higher dimensions. The element $\rho_k(\pi) \in \mathcal{J}^k(\pi_k^*(\pi))$, its definition and properties. The Spencer complexes $\cdots \to \mathcal{J}^k(\xi) \otimes \Lambda^l(N) \xrightarrow{S^l_k} \mathcal{J}^{k-1}(\xi) \otimes \Lambda^{l+1}(N) \to \cdots$ for a vector bundle $\xi \colon P \to N$, their exactness. The element $U_k(\pi) = S^0_k(\rho_k(\pi)) \in \mathcal{J}^{k-1}(\pi_k^*(\pi)) \otimes \Lambda^1(J^k(\pi))$, its properties. Generating sections $f \in \Gamma(\pi_k^*(\pi))$ as the result of construction of Lie fields with $U_1(\pi)$. Jacobi brackets for generating sections. Local coordinates.

3. Classical symmetry theory for differential equations

3.1. **Classical symmetries.** Finite and infinitesimal symmetries, definitions. “Physical meaning” of generating functions. Determining equations for coordinate computations. An example: symmetries of the Burgers equation $u_t = uu_x + u_{xx}$.

3.2. **Exterior and interior symmetries.** The restriction $\mathcal{C}(\mathcal{E})$ of the Cartan distribution to $\mathcal{E}$. Exterior $\mathrm{Lie}\,(\mathcal{E})$ and interior $\mathrm{Lie}_{\mathrm{int}}(\mathcal{E})$ symmetries of an equation $\mathcal{E}\subset J^k(\pi)$. The homomorphism $r\colon \mathrm{Lie}\,(\mathcal{E}) \to \mathrm{Lie}_{\mathrm{int}}(\mathcal{E})$. Counterexamples.

4. Perspectives

Algebraic model. The basic constructions. Cohomological invariants.