Lectures on Linear Differential Operators over Commutative Algebras
The 1st Diffiety School

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Foreword

These Lecture Notes contain introductory material to algebraic theory of linear differential operators over commutative algebras. The basics of this theory were exposed in a short paper by A.M. Vinogradov [**10**]. Later, a much more extensive exposition was published in the first chapter of [**3**].

The initial course, held in Forino (Summer 1998) at the *First Italian Diffiety School*, consisted of ten lectures and a number of practical lessons, but in preparation of the printed version it became clear that a more logical structure should contain five parts (which are also called *lectures* below):

- general introduction to *commutative algebra* (rings, algebras, and modules) together with main concepts of the *category theory* — first two lectures,
- main functors of *differential calculus* over commutative algebras — lecture 3,
- their *representative objects* (jets and differential forms) — lecture 4,
- relations to geometry serving as a bridge to the next course in *geometry of differential equations* — the last lecture.

We conclude these notes with a series of exercises which are an essential part of the main text. A lot of them were analyzed and solved during practical lessons held by M.M. Vinogradov and V.A. and V.N. Yumaguzhins.

As additional reading, we strongly recommend the books by I. MacDonald and M. Atiyah [**6**] and S. Lang [**5**] (commutative algebra), S. MacLane [**7, 8**] (category theory and homology), M. Atiyah [**1**] and D. Husemoller [**2**] (vector bundles), and Jet Nestruev [**9**] (relations between commutative algebra and geometry of smooth manifolds). An extended discussion of algebraic theory of differential calculus together with its relation to differential equations can be found in [**4**].

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This text is based on the notes made by Barbara Prinari during the lectures. She not only fixed the material with great accuracy, but also typeset the initial TEXfile. Without her help these lectures would hardly be published.

Lecture 1

We start with the maim concept lying in the base of all our future constructions. This is a notion of a *ring*.

Let A be an Abelian group. The operation in the group will be denoted by the symbol "+":

$$+ \colon A \times A \to A, \qquad (a, b) \mapsto a + b.$$

Suppose there is another operation in A denoted by "dot":

$$\cdot \colon A \times A \to A, \qquad (a, b \mapsto a \cdot b \equiv ab,$$

such that one has the distributivity law

$$a(b + c) = ab + ac \tag{1}$$

$$(b + c)a = ba + ca. \tag{2}$$

for all $a, b, c \in A$. Then one says that A is a *ring*. If in addition we have associativity

$$a(bc) = (ab)c,$$

then A is called an *associative* ring. If there is commutativity

$$ab = ba,$$

it is called *commutative*. If there exists an element $1 \in A$ such that

$$1 \cdot a = a, \qquad a \cdot 1 = a$$

for all $a \in A$, the ring is called *unitary* and the element 1 is called the *unity* of the ring. It is easy to check that if it exists it is unique.

Example 1. The groups $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ are rings with respect to "ordinary" operations of summation and multiplication of numbers. They are all associative, unitary and commutative rings.

Remark 1. The rings $\mathbb{R}$ and $\mathbb{C}$ possess an additional property: any element $a \neq 0$ is invertible in them, i.e., there exists an element $b \equiv a^{-1}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Such rings are called *fields*. Obviously, $\mathbb{Z}$ is not a field.

Example 2. Let V be a vector space over $\mathbb{R}$. The set of all linear operators acting on V,

$$L \colon V \to V,$$

forms a ring with composition playing the role of the product and it is denoted by $\mathrm{Mat}(V)$ (since, in a chosen basis, linear operators correspond to matrices). Then we know that

$$L_1 \circ (L_2 \circ L_3) = (L_1 \circ L_2) \circ L_3,$$

i.e., $\mathrm{Mat}(V)$ is an associative ring, but since in general,

$$L_1 \circ L_2 \neq L_2 \circ L_1$$

it is not commutative.

Example 3. Let E be a set and consider the set of all subsets of E denoted by 2^E. It can be shown that 2^E forms a commutative ring with respect to the operations

$$a + b \stackrel{\text{def}}{=} a \cap b, \quad a + b \stackrel{\text{def}}{=} \bar{a} \cap b) \cup (a \cap \bar{b}), \qquad a, b \in 2^E,$$

where $\bar{a} \stackrel{\text{def}}{=} E \setminus a$ (see Exercise 2).

Example 4. Let $C(\mathbb{R})$ be the set of all continuous functions on the real line. With respect to the operations

$$(f+g)(x) \stackrel{\text{def}}{=} f(x) + g(x)$$
$$(f \cdot g)(x) \stackrel{\text{def}}{=} f(x)g(x)$$

$C(\mathbb{R})$ is a commutative and unitary ring (the same is valid for the set of smooth functions $C^\infty(\mathbb{R})$).

Example 5. Let G be a finite group and A a commutative ring with unit. Consider the set $A[G]$ consisting of all formal sums

$$A[G] = \Big\{ \sum_{g_i \in G} a_i g_i \mid a_i \in A \Big\}.$$

Then, having two such expressions, we can define their sum and product

$$\sum_{g_i \in G} a_i g_i + \sum_{g_i \in G} b_i g_i \stackrel{\text{def}}{=} \sum_{g_i \in G} (a_i + b_i) g_i$$
$$\sum_{g_i \in G} a_i g_i \cdot \sum_{g_j \in G} b_i g_j \stackrel{\text{def}}{=} \sum (a_i b_j) g_i g_j.$$

Then this object becomes a ring which is associative but in general not commutative (if G is not commutative). $A[G]$ is called the *group ring* of G (with coefficients in A).

Example 6. Let V be a vector space and assume that there exists a multiplication in V denoted by $[\ ,\ ]$ such that

1. $[a, b] = -[b, a]$,
2. $[a, b + c] = [a, b] + [a, c]$,
3. $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$.

Then V is called a *Lie algebra* (it is neither commutative, nor associative).

If A is a ring, a Lie algebra structure can naturally be induced by defining

$$[a,b] \stackrel{\text{def}}{=} ab - ba.$$

for all $a, b \in A$. It is easy to check that it satisfies properties 1-3. If A is a commutative ring, then $[\ ,\ ]$ will be trivial, i.e., $[a,b] = 0$ for any $a, b \in A$.

EXAMPLE 7. Let V be a vector space over $\mathbb{R}$ (or any other field). Then we can define the *tensor product* $V \otimes V$. This is a vector space. In a similar way we can consider the spaces $V \otimes V \otimes V = (V \otimes V) \otimes V$ and so on:

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} = V^{\otimes n}$$

which is called the n-th *tensor power* of V and, if we put $V^{\otimes 0} \stackrel{\text{def}}{=} \mathbb{R}$, we can define

$$T(V) = \sum_{n \geq 0} V^{\otimes n}.$$

Elements of $V^{\otimes n}$ are linear combinations of *homogeneous* elements $v = v_1 \otimes \cdots \otimes v_n$, $v_i \in V$. If $w = w_1 \otimes \cdots \otimes w_m \in V^{\otimes m}$ is another homogeneous element, we define their product $v \otimes w$ as a simple concatenation:

$$v \otimes w \stackrel{\text{def}}{=} v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes v_m \in V^{\otimes (n+m)}.$$

This operation is extended to arbitrary elements of $T(V)$ in an obvious way. Thus $T(V)$ becomes an associative algebra with respect to the tensor product. It is called the *tensor algebra* of the space V. In a sense, all rings can be obtained from tensor algebras by a regular procedure.

Let now A and B be two rings and consider a map

$$f\colon A \to B \tag{3}$$

such that for all $a, a' \in A$ one has

$$f(a + a') = f(a) + f(a') \tag{4}$$

$$f(aa') = f(a)f(a') \tag{5}$$

(if A and B are unitary we should add also

$$f(1_A) = 1_B, \tag{6}$$

where 1_A are 1_B units in A and B respectively.). Then f is called a *ring homomorphism.*

The subset of B

$$\operatorname{im} f \stackrel{\text{def}}{=} \{b \in B \mid b = f(a), \quad a \in A\}$$

is a ring and is called the *image* of f and the subset of A

$$\ker f \stackrel{\text{def}}{=} \{a \in A \mid \ f(a) = 0\}$$

is a ring too, and is called the *kernel* of f. If $\ker f = 0$, the homomorphism f is called a *monomorphism*. In the case $\operatorname{im} f = B$, it is called an *epimorphism*. If a homomorphism is both epi- and monomorphism, it is called an *isomorphism*.

Elements of $\ker f$ satisfy the following property: for all $a \in A$ and $a' \in \ker f$ one has $aa' \in \ker f$ and $a'a \in \ker f$.

Now let A be a ring and $P \subset A$ be a subgroup of A with respect to addition. If

$$A \cdot P \subset P \tag{7}$$

(i.e., $ap \in P$ for all $a \in A$ and $p \in P$), P is called a *left ideal* of A. If

$$P \cdot A \subset P, \tag{8}$$

then P is called a *right ideal* of A. If both (7) and (8) are valid, then P is called a *two-sided ideal*. If A is a commutative ring, all ideals are two-sided ones.

Consider a pair, a ring A and a two-sided ideal P. Then the quotient group A/P is defined

$$A/P = \{[a] = a \bmod P \mid a \in A\}.$$

Two elements a, a' are in the same coset in A/P if and only if $a - a' \in P$. Then we can define sum and product of cosets in the natural way

$$[a] + [b] \overset{\text{def}}{=} [a + b], \qquad [a] \cdot [b] \overset{\text{def}}{=} [a \cdot b].$$

If P is a two-sided ideal, these operation are well defined. With respect to these operations the quotient set is a ring. It is the *quotient ring* of A with respect to the two-sided ideal P.

There is a natural epimorphism $\pi \colon A \to A/P$ taking any element $a \in A$ to the corresponding coset $[a] \in A/P$. Thus we have the sequence of homomorphisms

$$0 \to P \xrightarrow{i} A \xrightarrow{\pi} A/P \to 0,$$

where i is the natural embedding and the kernel of π coincides with the image of i. This is the so-called *short exact sequence* of rings.

EXAMPLE 8. Consider the real line $\mathbb{R}$ and the ring of smooth functions $C^\infty(\mathbb{R})$. Fix a point $x \in \mathbb{R}$ and consider the set

$$\mu_x = \{f \in C^\infty(\mathbb{R}) \mid f(x) = 0\} \tag{9}$$

It is easy to check that μ_x is an ideal. The quotient ring $C^\infty(\mathbb{R})/\mu_x$ is isomorphic to $\mathbb{R}$ (see Exercise 7). If we choose two points $x, y \in \mathbb{R}$ the set $\mu_{x,y}$ of functions vanishing at both points is an ideal too, and so on.

EXAMPLE 9. Consider an integer $n \in \mathbb{Z}$ and the set

$$n\mathbb{Z} = \{na \mid a \in \mathbb{Z}\} \tag{10}$$

It is an ideal in the ring of integer numbers so we can consider the quotient ring $\mathbb{Z}_n \overset{\text{def}}{=} \mathbb{Z}/n\mathbb{Z}$. This ring consists of n elements

$$\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$$

which are residues modulo n. In fact, we can prove (see Exercise 9) that any ideal of $\mathbb{Z}$ is of the form (10), that is the following fact is valid:

PROPOSITION 1. *A subset $P \subset \mathbb{Z}$ is an ideal of the ring $\mathbb{Z}$ if and only if there exists $n \in \mathbb{Z}$ such that $P = n\mathbb{Z}$.*

Note that $\mathbb{Z}_n$ is a field if and only if n is a prime number.

EXAMPLE 10. Consider the tensor algebra $T(V)$ (see Example 7) and the two-sided ideal P_- of this algebra generated by the elements of the form $a \otimes b - b \otimes a$, $a, b \in V = T^1(V)$. This means that P_- consists of the elements of the form

$$\sum_i v_i \otimes (a_i \otimes b_i - b_i \otimes a_i) \otimes w_i, \quad v_i, w_i \in T(V).$$

Then the quotient ring is commutative and is called the *symmetric algebra* of V and is denoted by $S(V)$. If we chose a basis $x_1, \dots, x_n$ in V, then any element of $S(V)$ can be uniquely represented in the form of finite sum

$$\sum_{0 \le i_1, \dots, i_n} a_{i_1,\dots,i_n} x_1^{i_1} \dots x_n^{i_n},$$

where $a_{i_1,\dots,i_n}$ are elements of the basic field, while $S(V)$ is identified with the *ring of polynomials* in the variables $x_1, \dots, x_n$.

EXAMPLE 11. Consider another ideal P_+ in $T(V)$ generated by the elements $a \otimes b + b \otimes a$. The corresponding quotient algebra is denoted by $\bigwedge(V)$ and is called the *external algebra* of V. The multiplication in this algebra is denoted by $\wedge\colon \bigwedge(V) \otimes \bigwedge(V) \to \bigwedge(V)$. Under the natural projection $\pi\colon T(V) \to \bigwedge(V)$, the images of elements from $V^{\otimes n}$ constitute elements of *degree n*. If $v, w \in \bigwedge(V)$ are elements of degrees n and m respectively, the one can show that $v \wedge w = (-1)^{nm} wv$.

If $x_1, \dots, x_n$ is a basis in V, then any element of $\bigwedge(V)$ is uniquely represented in the form

$$\sum_{1 \le i_1 < \dots i_k \le n} a_{i_1,\dots,i_n} x_{i_1} \wedge \dots \wedge x_{i_k}.$$

EXAMPLE 12. Now let A be the ring of all real polynomials in one variable $A = \mathbb{R}[x]$. Any ideal P in A is of the form $A \cdot f$, i.e., is generated by some polynomial. The quotient ring is a field in this case if and only if the polynomial f is indecomposable. This field is $\mathbb{R}$, if $f = x - \lambda$, $\lambda \in \mathbb{R}$, and $\mathbb{C}$, if f has complex roots.

The last example, together with Example 9, leads to the following

DEFINITION 1. Let A be a ring and $P \subset A$ be its ideal. P is called *maximal*, if for any other ideal P' the embedding $P' \supset P$ implies $P' = P$.

THEOREM 2. *Let A be a commutative and unitary ring. Let $P \subset A$ be an ideal. Then the quotient ring A/P is a field if and only if P is a maximal ideal.* (*For the proof see Exercise* 15).

Let A be a commutative ring with unity (from now on we shall always consider such rings only) and let $a, b \in A$. We say that $a \neq 0$ is a *zero divisor*, if

$$ab = 0 \tag{11}$$

for some $b \neq 0$.

DEFINITION 2. An ideal $P \subset A$ is called a *prime ideal*, if $ab \in P$ implies that either $a \in P$ or $b \in P$.

PROPOSITION 3. *An ideal $P \subset A$ is prime if and only if the quotient ring A/P possesses no zero divisors* (*see Exercise* 16).

PROPOSITION 4. *Any maximal ideal $P \subset A$ is a prime ideal* (*see Exercise* 17).

PROPOSITION 5. *Any ideal of A is contained in a maximal ideal* (*for the proof Exercise* 18).

EXAMPLE 13. Let A be the ring of polynomials over $\mathbb{R}$ in three variables, $A = \mathbb{R}[x, y, z]$. Consider the ideals

$$P_1 = \{x - \lambda\}, \quad P_2 = \{x - \lambda, y - \mu\}, \quad P_3 = \{x - \lambda, y - \mu, z - \nu\}.$$

Then both P_1 and P_2 are prime ideals, P_3 is a maximal and consequently a prime ideal too.

Let us now consider two rings A and B and a ring homomorphism $\varphi\colon A \to B$. Then we can define an action of A over B in this way: for any $a \in A$ and $b \in B$ we set

$$a \circ b \stackrel{\text{def}}{=} \varphi(a)b. \tag{12}$$

Usually we shall omit the "$\circ$" symbol and denote the action of a over b simply as ab.

It is easy to check, using (1), (2) and (4), (5), that this action possesses the following properties

$$a \circ (b + b') = a \circ b + a \circ b' \tag{13}$$

$$(a \circ b)b' = a \circ (bb') \tag{14}$$

$$(aa') \circ b = a \circ b + a\prime \circ b \tag{15}$$

$$(a + a') \circ b = a \circ b + a' \circ b \tag{16}$$

and, if A is a unitary ring, one has also

$$1_A \circ b = b. \tag{17}$$

In this case, B is called an *algebra* over A or an A-algebra.

EXAMPLE 14. The ring of smooth functions $C^\infty(\mathbb{R}^n)$ is an algebra over the ring $\mathbb{R}$.

EXAMPLE 15. The polynomial ring $\Bbbk[x_1,\dots,x_n]$ is an algebra over the field $\Bbbk$.

Take now instead of B just an Abelian group and let us denoted by P. Let us define an action of A over P in a reasonable way. "Reasonable" in this context means that for all $a \in A$ and $p \in P$ one has a correspondence

$$(a,p) \mapsto ap \in P$$

and this correspondence possesses the following properties: for all $a, a' \in A$ and $p, q \in P$ one has

$$a(p+q) = ap + aq \tag{18}$$

$$(a+a')p = ap + a'p \tag{19}$$

$$(a+a')p = ap + a'p. \tag{20}$$

Then we say that P is an A-*module*. We shall always assume the modules under consideration to be *unitary*, which means that

$$1_A p = p. \tag{21}$$

EXAMPLE 16. Any vector space over a field $\Bbbk$ is a module over this field.

EXAMPLE 17. Consider the algebra $C^\infty(\mathbb{R})$ and the set of vector valued functions $A = \{(f_1,\dots,f_s) \mid f_i \in C^\infty(\mathbb{R})\}$. We can multiply such functions by any function $g \in C^\infty(\mathbb{R})$ in a natural way

$$g(f_1,\dots,f_s) \stackrel{\text{def}}{=} (gf_1,\dots,gf_s)$$

and add them to each other as vectors. So A is a $C^\infty(\mathbb{R})$-module, that is a module over the ring of smooth functions.

EXAMPLE 18. Let V be a real vector space (i.e., a space over $\mathbb{R}$) and $\varphi\colon V \to V$ be a linear operator. Consider the ring of all real polynomials in one variable, $\mathbb{R}[x]$. If $p \in \mathbb{R}[x]$,

$$p = a_0 + a_1 x + \cdots + a_n x^n, \quad a_0,\dots,a_n \in \mathbb{R},$$

then we can define the action of this polynomial over V by

$$p(v) \stackrel{\text{def}}{=} (a_0 + a_1\varphi + \cdots + a_n\varphi^n)(v)$$

that is

$$p(v) = a_0 v + a_1\varphi(v) + \cdots + a_n\varphi^n(v).$$

It is easy to check that V is a module over the ring $\mathbb{R}[x]$ with respect to this action.

Example 19. Let now G be a finite group and V be a vector space over a field $\Bbbk$. We say that G is *represented* in V, if a group homomorphism of G to the group of endomorphisms of V is given, that is for any $g \in G$ one has $\varphi(g): V \to V$ satisfying

$$\varphi(gg') = \varphi(g) \circ \varphi(g').$$

To say that G is represented in V is the same as to say that we have a $\Bbbk[G]$-module structure on V (see Example 5).

Let A be a ring and P and Q be two A-modules. A mapping $f: P \to Q$ is called an A-*homomorphism* if for any elements $p, p' \in P$ and $a \in A$ one has

$$f(p + p') = f(p) + f(p') \tag{22}$$

$$f(ap) = af(p). \tag{23}$$

If A is a field and P, Q are vector spaces over A, an A-homomorphism is just a linear operator. The set of all A-homomorphisms is denoted by $\mathrm{Hom}_A(P, Q)$. If $f \in \mathrm{Hom}_A(P, Q)$ and $a \in A$, we can define an action of a over f as

$$(af)(p) \stackrel{\text{def}}{=} af(p) \tag{24}$$

and the sum of two homomorphisms by

$$(f + f'(p) \stackrel{\text{def}}{=} f(p) + f'(p).$$

This definition introduces an A-module structure in $\mathrm{Hom}_A(P, Q)$. The ring A itself is an A-module, so we can consider the module $\mathrm{Hom}_A(A, P)$ It is easy to check that

$$\mathrm{Hom}_A(A, P) \simeq P. \tag{25}$$

Let us also introduce the module $P^* \stackrel{\text{def}}{=} \mathrm{Hom}_A(P, A)$ which is called the *dual* or *adjoint* module of P.

Lecture 2

At the end of the previous lecture we introduced the notion of a homomorphism between A-modules and showed that the set $\mathrm{Hom}_A(P, Q)$ is an A-module as well.

Let us now consider three A-modules P, Q, and R and two A-module homomorphisms f, g

$$P \xrightarrow{f} Q \xrightarrow{g} R.$$

Then we can consider the composition $h = g \circ f \colon P \to R$. It is easy to check that h is an A-homomorphism from P to R. A way to express that h is the composition of f and g is to say that the diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "h"'] & Q \arrow[dl, "g"] \\ R & \end{tikzcd}$$

is *commutative*. We shall often use the diagram language to make exposition more clear and intuitive.

Let us consider a more complicated situation: namely let A be a ring and B be an A-algebra. Let $\varphi \colon A \to B$ be the corresponding homomorphism and P, Q be B-modules. Then they are A-modules as well: the A-module structure is defined by

$$ap \stackrel{\text{def}}{=} \varphi(a)p, \qquad a \in A,\ p \in P.$$

EXAMPLE 20. Obviously, the field of complex numbers $\mathbb{C}$ is an $\mathbb{R}$-algebra. Therefore any complex vector space can be considered as a real vector space using the previous definition with $\varphi \colon \mathbb{R} \hookrightarrow \mathbb{C}$ being the standard embedding.

So, P and Q are both A- and B-modules and, consequently, $\mathrm{Hom}_A(P, Q)$ is an A-module while $\mathrm{Hom}_B(P, Q)$ is both a B-module and an A-module as well according to the previous definition. But we can define also a B-module structure in $\mathrm{Hom}_A(P, Q)$: given $f \in \mathrm{Hom}_A(P, Q)$ and $b \in B$, we can set

$$(bf)(p) \stackrel{\text{def}}{=} bf(p), \qquad p \in P \tag{26}$$

and we can also introduce a different action, which we shall denote by $^+$:

$$(b^+f)(p) \stackrel{\text{def}}{=} f(bp). \tag{27}$$

It is easy to check that both bf and b^+f are A-homomorphism from P to Q and they are different since f is an A-homomorphism and not a B-homomorphism.

EXAMPLE 21. Let $B = \mathbb{R}[x]$. Then B an $\mathbb{R}$-algebra and is a module over itself. As for any ring, $\mathrm{Hom}_B(B, B) = B$. On the other hand, elements of $\mathrm{Hom}_{\mathbb{R}}(B, B)$ are uniquely determined by images of $1, x, x^2, \dots$ and are arbitrary infinite-dimensional $\mathbb{R}$-matrices.

Thus, two module structures exist in $\mathrm{Hom}_A(P, Q)$ defined by the actions (26) and (27), and these structures commute in the following sense:

$$b_1(b_2^+f) = b_2^+(b_1f).$$

for any $b_1, b_2 \in B$ and $f \in \mathrm{Hom}_A(P, Q)$. In this case we say that a *bimodule structure* is given.

Now let A be a ring and P, Q two A-modules. We say that a mapping $f \in \mathrm{Hom}_A(P, Q)$ is an *epimorphism*, if it is surjection, that is for any $q \in Q$ there exists $p \in P$ such that $f(p) = q$. In this case we also say that the sequence $P \xrightarrow{f} Q \to 0$ is exact. We say that f is an *embedding* (or *monomorphism*), if $f(p) = 0$ implies $p = 0$ and represent this fact by the exact sequence $0 \to P \xrightarrow{f} Q$. Epimorphic embeddings are called *isomorphisms*.

If $P = Q$, elements of module $\mathrm{Hom}_A(P, P)$ are called *endomorphisms* and the notation

$$\mathrm{End}_A P \stackrel{\text{def}}{=} \mathrm{Hom}_A(P, P)$$

is used. Note that $\mathrm{End}_A P$ is an associative A-algebra with respect to composition.

Now let P be an A-module and $P' \subset P$ a subset of P. We say that P' is a *submodule* in P if

$$p_1 + p_2 \in P', \qquad ap \in P'$$

for all $p, p_1, p_2 \in P'$ and $a \in A$.

EXAMPLE 22. Let $P = A$. Then submodules of A are just its ideals.

Let P' be a submodule of P. Then $p \sim q \Leftrightarrow p - q \in P'$ is an equivalence relation and we can consider the quotient P/P'. Evidently, the operations

$$(p + P') + (q + P') \stackrel{\text{def}}{=} (p + q) + P'$$
$$a(p + P') \stackrel{\text{def}}{=} ap + P'$$

determine a well-defined module structure in P/P' and one has a natural epimorphism $P \to P/P'$, $p \mapsto p + P'$.

Let P, Q be A-modules and $f\colon P \to Q$ be an A-module homomorphism. Consider the sets

$$\begin{aligned} \ker f &= \{p \in P \mid f(p) = 0\} \\ \operatorname{im} f &= \{q \in Q \mid q = f(p) \text{ for some } p \in P\}\,. \end{aligned}$$

Both are submodules (of P and Q respectively) called *kernel* and *image* of f. We can consider the quotient $P/\ker f$ and it is easy to check that $P/\ker f$ is isomorphic to $\operatorname{im} f$.

Now given two modules P and Q we can consider their Cartesian product

$$P \times Q = \{(p, q) \mid p \in P,\ q \in Q\}$$

and introduce an A-module structure to it by

$$\begin{aligned} (p, q) + (p', q') &\stackrel{\text{def}}{=} (p + p', q + q') \\ a(p, q) &\stackrel{\text{def}}{=} (ap, aq). \end{aligned}$$

The module obtained is denoted by $P \oplus Q$ and is called the *direct sum* of P and Q. The simplest A-module is A itself, so we can construct the module

$$A^n = \underbrace{A \oplus A \oplus \cdots \oplus A}_{n \text{ times}}.$$

which is called a *free module* with n generators (or of rank n). In A^n we can consider the *free generators*

$$\begin{aligned} e_1 &= (1_A, 0, \ldots, 0) \\ e_2 &= (0, 1_A, \ldots, 0) \\ &\ldots\ldots\ldots\ldots\ldots \\ e_n &= (0, 0, \ldots, 1_A) \end{aligned}$$

and any element $a \in A^n$ can be uniquely written in the form

$$a = a_1 e_1 + \cdots + a_n e_n.$$

The following result explains why the modules and generators above are called free.

Proposition 6. *Let P be an arbitrary A-module and A^n be the free module of rank n with free generators $e_1, \ldots, e_n$. Then for any elements $p_1, \ldots, p_n \in P$ there exists a unique homomorphism $f\colon A^n \to P$ such that $f(e_i) = p_i$, $i = 1, \ldots, n$.*

For an arbitrary A-module P, we say that elements $p_1, \ldots, p_n$ are *generators* of p, if any element $p \in P$ can be represented as a linear combination $p = \sum_{i=1}^{n} a_i p_i$, $a_i \in A$ (of course, this representation may not be unique). From Proposition 6 we obtain the following

Corollary 7. *Any A-module P with n generators can be represented as a quotient of the free module A^n of rank n.*

Thus, given a module P, we have $P \simeq A^n/R$ for some submodule $R \subset A^n$. If $r_1, \ldots, r_k$ are generators of R, we say that P is described by relations $r_1 = 0, \ldots, r_k = 0$.

Example 23. The module $\mathbb{Z}_m$ is the quotient of $\mathbb{Z}$ described by the relation $m = 0$.

Now let A be a commutative ring and P, Q, R be A-modules. Consider the set $P \times Q$ and a mapping $f\colon P \times Q \to R$. We say that f is *bilinear* if

1. $f(ap, q) = af(p, q)$,
2. $f(p, aq) = af(p, q)$,
3. $f(p + p', q) = f(p, q) + f(p', q)$,
4. $f(p, q + q') = f(p, q) + f(p, q')$.

for any $a \in A$, $p, p' \in P$, $q, q' \in Q$. In other words, the mappings $f_p\colon q \mapsto f(p, q)$ and $f_q\colon p \mapsto f(p, q)$ are A-homomorphisms from Q to R and from P to R respectively.

Let now T be an A-module and $t\colon P \times Q \to T$ be a bilinear map. We say that T is the *tensor product* of P and Q if for any bilinear map $f\colon P \times Q \to R$ to an arbitrary A-module R, there exists a unique homomorphism f' such that the diagram

$$\begin{tikzcd} P \times Q \arrow[r, "f"] \arrow[d, "t"'] & R \arrow[dl, dashed, "f'"] \\ T & \end{tikzcd} \tag{28}$$

is commutative.

The tensor product T of P and Q is denoted by $P \otimes_A Q$. If it exists, it is unique up to isomorphisms.

Consider the free module generated by the formal expressions $p \otimes q$ with $p \in P$ and $q \in Q$ and denote it by $L(P, Q)$. Let $R \subset L(P, Q)$ be the submodule generated by the elements

$$\begin{aligned} &(p + p') \otimes q - p \otimes q - p' \otimes q \\ &p \otimes (q + q') - p \otimes q - p \otimes q' \\ &ap \otimes q - p \otimes aq. \end{aligned}$$

One can show that the quotient module $L(P, Q)/R$ satisfies property (28) and thus we have $L(P, Q)/R \simeq P \otimes_A Q$. This proves existence of tensor product. Its main properties are formulated in the following

Proposition 8. *For any A-modules the following isomorphisms take place:*

1. $P \otimes_A (Q \otimes_A R) \simeq (P \otimes_A Q) \otimes_A R$,
2. $A \otimes_A Q \simeq Q \otimes_A A \simeq Q$,
3. $P \otimes_A (Q \oplus R) \simeq P \otimes_A Q \oplus P \otimes_A R$,

4. $(P \oplus Q) \otimes_A R \simeq P \otimes_A R \oplus Q \otimes_A R$,
5. $\mathrm{Hom}_A(P \otimes_A Q, R) \simeq \mathrm{Hom}_A(P, \mathrm{Hom}_A(Q, R))$.

Now we shall introduce an important class of modules.

DEFINITION 3. An A-module P is called a *projective* if for any A-modules M and M', any homomorphism $f \in \mathrm{Hom}_A(P, M')$, and any epimorphism $h\colon M \to M'$ there exists a homomorphism $f' \in \mathrm{Hom}_A(P, M)$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "f'"'] \arrow[d, "f"] & \\ M \arrow[r, "h"] & M' \arrow[r] & 0 \end{tikzcd}$$

is commutative.

THEOREM 9. *An A-module P is projective if and only if there exists a free module F and a submodule $P' \subset F$ such that $F = P \oplus P'$*

In particular, all free modules are projective, but not vice versa.

Now let us consider a commutative ring A and the set of all A-modules, which we denote by $\mathcal{M}od(A)$. We can define some nice operations in this set; for example,

$$P, Q \in \mathcal{M}od(A) \Longrightarrow \mathrm{Hom}_A(P, Q) \in \mathcal{M}od(A).$$

What are the properties of this correspondence? First fix P and consider the correspondence $\mathrm{Hom}_A(P, \cdot)\colon Q \mapsto \mathrm{Hom}_A(P, Q)$. Then you have a correspondence from $\mathcal{M}od(A)$ to itself

$$Q \in \mathcal{M}od(A) \Longrightarrow \mathrm{Hom}_A(P, Q) \in \mathcal{M}od(A).$$

Now, consider another module Q' and a homomorphism $g \in \mathrm{Hom}_A(Q, Q')$. Then to any $f \in \mathrm{Hom}_A(P, Q)$ we can put into correspondence the composition $g \circ f \in \mathrm{Hom}_A(P, Q')$. Denoting $g \circ f$ by $\mathrm{Hom}_A(P, g)(f)$, we obtain the mapping

$$\mathrm{Hom}_A(P, g)\colon \mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_A(P, Q').$$

So, for a fixed P, $\mathrm{Hom}_A(P, \cdot)$ is a correspondence taking A-modules to A-modules and A-homomorphisms to A-homomorphisms. Moreover, if we have a sequence

$$Q \xrightarrow{g} Q' \xrightarrow{g'} Q'',$$

then it can be easily seen that

$$\mathrm{Hom}_A(P, g' \circ g) = \mathrm{Hom}_A(P, g') \circ \mathrm{Hom}_A(P, g) \tag{29}$$

and

$$\mathrm{Hom}_A(P, \mathrm{id}_Q) = \mathrm{id}_{\mathrm{Hom}_A(P,Q)}. \tag{30}$$

Let us consider another example of a similar nature. Fix a module P construct the correspondence

$$T_P\colon \mathcal{M}od(A) \to \mathcal{M}od(A)$$

by setting $T_P(Q) = P \otimes_A Q$. If we have a homomorphism $g \in \mathrm{Hom}_A(Q, Q')$, we can define $T_P(g)\colon P \otimes_A Q \to P \otimes_A Q'$ by

$$T_P(g)(p \otimes q) \stackrel{\mathrm{def}}{=} p \otimes g(q).$$

Similar to the previous case, we have $T_P(g' \circ g) = T_P(g') \circ T_P(g)$, if the composition $g' \circ g$ is defined, and $T_P(\mathrm{id}_Q) = \mathrm{id}_{P\otimes Q}$.

Let us now fix a module Q and consider the correspondence

$$\mathrm{Hom}_A(\cdot, Q)\colon \mathcal{M}od(A) \to \mathcal{M}od(A)$$

taking a module P to the module $\mathrm{Hom}_A(P, Q)$. Then for any homomorphism $f \in \mathrm{Hom}_A(P', P)$ we have the homomorphism $\mathrm{Hom}_A(f, Q)\colon\ \mathrm{Hom}_A(P, Q) \to \mathrm{Hom}_A(P', Q)$ defined by $\mathrm{Hom}_A(f, Q)(g) \stackrel{\mathrm{def}}{=} f \circ g$, where $g\colon P \to Q$. In this case we have

$$\mathrm{Hom}_A(f \circ f', Q) = \mathrm{Hom}_A(f', Q) \circ \mathrm{Hom}_A(f, Q), \tag{31}$$

whenever the composition $f \circ f'$ is defined. Note that the order in the right-hand side is reversed with respect to (29).

Let us generalize these constructions. Consider "something" (a sort of *Universum*) denoted by $\mathcal{C}$, which consists of the collection $\mathcal{O}$ of some *objects* such that, given a pair of objects $O_1, O_2 \in \mathcal{O}$, we can put it into correspondence with a set $\mathcal{M}or(O_1, O_2)$, whose elements are called *morphisms* from O_1 to O_2. Suppose that the correspondence between the pair O_1, O_2 and this set $\mathcal{M}or(O_1, O_2)$ satisfies the following properties:

1. For any object O there exists a morphism $\mathrm{id}_O \in \mathcal{M}or(O, O)$;
2. Given a triple O_1, O_2, O_3, one has the *composition* mapping

$$\circ\colon \mathcal{M}or(O_1, O_2) \times \mathcal{M}or(O_2, O_3) \to \mathcal{M}or(O_1, O_3)$$

   such that:
   (a) For any $\varphi \in \mathcal{M}or(O_1, O_2)$ one has $\mathrm{id}_{O_2} \circ \varphi = \varphi$ and $\varphi \mathrm{o} \mathrm{id}_{O_1} = \varphi$;
   (b) If $\varphi \in \mathcal{M}or(O_1, O_2)$, $\psi \in \mathcal{M}or(O_2, O_3)$, and $\zeta \in \mathcal{M}or(O_3, O_4)$, the associative rule holds: $\zeta \circ (\psi \circ \varphi) = (\zeta \circ \psi) \circ \varphi$.

Then $\mathcal{C}$ is called a *category*. Let us consider some examples.

Example 24. Consider the category whose objects are sets. If we have two sets S_1 and S_2, and we define

$$\mathcal{M}or(S_1, S_2) = \mathcal{M}aps(S_1, S_2),$$

i.e., objects are mappings of sets. We have a category of sets.

Example 25. Consider all finite groups as objects. If we have two finite groups G_1 and G_2 and define the set of morphisms as the set of all group homomorphisms between G_1 and G_2 we get a category. The same can be done with Abelian groups or with all groups.

EXAMPLE 26. Consider all modules over a commutative ring A and, as above, denote them by $\mathcal{M}od(A)$. In this case

$$\mathcal{M}or(P_1, P_2) = \mathrm{Hom}_A(P_1, P_2).$$

EXAMPLE 27. In the case of vector spaces over a field $\Bbbk$, morphisms will be linear maps between these spaces.

How to establish relations between different categories? Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. Consider a correspondence

$$F : \mathcal{C}_1 \Longrightarrow \mathcal{C}_2 \tag{32}$$

taking any object of $\mathcal{C}_1$ to an object of $\mathcal{C}_2$,

$$O \mapsto F(O)$$

and if we have a pair of objects O and O' in $\mathcal{C}_1$, then F takes

$$\mathcal{M}or(O, O') \Longrightarrow \mathcal{M}or(F(O), F(O')).$$

Assume that three objects of $\mathcal{C}_1$ are given, O_1, O_2, O_3, together with two morphisms $\varphi \in \mathcal{M}or(O_1, O_2)$ and $\psi \in \mathcal{M}or(O_2, O_3)$. Then F is called a *covariant functor* between $\mathcal{C}_1$ and $\mathcal{C}_2$, if for all morphisms φ, ψ

$$F(\varphi \circ \psi) = F(\varphi) \circ F(\psi).$$

whenever the composition $\varphi \circ \psi$ is defined. If F takes

$$\mathcal{M}or(O, O') \Longrightarrow \mathcal{M}or(F(O'), F(O)).$$

and

$$F(\varphi \circ \psi) = F(\psi) \circ F(\varphi),$$

then it is called is called a *contravariant functor*.

EXAMPLE 28. The correspondence $T_P : Q \mapsto P \otimes_A Q$ is a covariant functor from the category $\mathcal{M}od(A)$ to itself.

EXAMPLE 29. For a fixed P, the correspondence $Q \mapsto \mathrm{Hom}_A(P, Q)$ is also a covariant functor from the category $\mathcal{M}od(A)$ to itself.

EXAMPLE 30. If we fix Q, the correspondence $P \mapsto \mathrm{Hom}_A(P, Q)$ is a contravariant functor.

EXAMPLE 31. Let G be a finite group. Consider the normal subgroup generated by $g_1 g_2 g_1^{-1} g_2^{-1}$ with $g_1, g_2 \in G$, and denoted by $[G, G]$:

$$[G, G] = \{g_1 g_2 g_1^{-1} g_2^{-1}\} \subset G.$$

Then $G/[G, G]$ is an Abelian group while the correspondence

$$G \mapsto G/[G, G]$$

is a functor from the category of finite groups to the category of Abelian groups.

EXAMPLE 32. Smooth manifolds make a category with smooth maps playing the role of morphisms. Then the correspondence $M \mapsto C^\infty(M)$ is a contravariant functor from this category to the category of $\mathbb{R}$-algebras.

Let now $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories and $F, F \colon \mathcal{C}_1 \Longrightarrow \mathcal{C}_2$ be two (say, covariant) functors. Assume that for any object O of $\mathcal{C}_1$ a morphism $T_O \colon F(O) \to G(O)$ is given. Then the correspondence $O \mapsto T_O$ is called a *natural transformation* of functors F and G if the diagram

$$\begin{tikzcd} F(O) \arrow[r, "F(\varphi)"] \arrow[d, "T_O"'] & F(O') \arrow[d, "T_{O'}"] \\ G(O) \arrow[r, "G(\varphi)"] & G(O') \end{tikzcd}$$

is commutative for all objects O, O' and all morphisms $\varphi \in \mathcal{M}or(O, O')$. For example, the isomorphism $\mathrm{Hom}_A(P \otimes_A Q, R) \simeq \mathrm{Hom}_A(P, \mathrm{Hom}(Q, R))$ gives a natural transformation of the functors $R \mapsto \mathrm{Hom}_A(P \otimes_A Q, R)$ and $R \mapsto \mathrm{Hom}_A(P, \mathrm{Hom}(Q, R))$. We shall see other examples of natural transformations in next lectures.

Lecture 3

Using a geometrical analogy now, we shall start to construct differential calculus in the category $\mathcal{M}od(A)$.

Let U be a domain in $\mathbb{R}^n$. Let us fix a point $x \in U$; a tangent vector ξ_x in x is a map

$$\xi_x \colon C^\infty(U) \to \mathbb{R}$$

which is linear and such that for all $f, g \in C^\infty(U)$ one has

$$\xi_x(fg) = \xi_x(f)g(x) + f(x)\xi_x(g). \tag{33}$$

Suppose we have a tangent vector at any point of the domain. Then we can consider the mapping

$$\xi \colon C^\infty(U) \to C^\infty(U)$$

such that $\xi(f)(x) \overset{\text{def}}{=} \xi_x(f)$ smoothly depending on x. Then (33) acquires the from

$$\xi(fg) = \xi(f)g + f\xi(g). \tag{34}$$

It is easy to check that for all $f, g \in C^\infty(U)$ and $\alpha \in \mathbb{R}$ one has also

1. $\xi(fg) = \xi(f)g + f\xi(g)$,
2. $\xi(\alpha f) = \alpha\xi(f)$.

Thus, smooth family of tangent vectors is a $\mathbb{R}$-linear map from the ring of smooth functions to itself that satisfies the Leibniz rule (34).

Now let us try to generalize this concept. Let K be a commutative unitary ring and A be a K-algebra. A map $X \colon A \to A$ is called a *derivation*, if it is K-linear and satisfies the Leibniz rule, that is

1. $X(\kappa a) = \kappa X(a)$;
2. $X(a + b) = X(a) + X(b)$;
3. $X(ab) = X(a)b + bX(a)$.

for all $a, b \in A$ and $W\kappa \in K$. The set of all derivations will be denoted by $D(A)$. The classical geometric case corresponds to $K = \mathbb{R}$ and $A = C^\infty(\mathbb{R}^n)$.

For any $X, Y \in D(A)$ and $a, b \in A$, we can set

$$(X + Y)(a) \overset{\text{def}}{=} X(a) + Y(a)$$

$$(aX)(b) \overset{\text{def}}{=} aX(b),$$

and it is easy to check that $X+Y$ and aX are derivations as well. So the set of derivations $D(A)$ in an A-module. In fact it is more than just a module.

For any $X, Y \in D(A)$ we can consider the compositions $X \circ Y$ and $Y \circ X$ and define

$$[X, Y] \stackrel{\text{def}}{=} X \circ Y - Y \circ X. \tag{35}$$

It is easy to check that it is a derivation; it is called the *Lie bracket* of X and Y. One can show that for all $X, Y, Z \in D(A)$ and $\kappa \in K$ the following identities hold

1. $[X, Y] + [Y, X] = 0$;
2. $[X + Y, Z] = [X, Z] + [Y, Z]$;
3. $[\kappa X, Y] = \kappa[X, Y]$;
4. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$.

Thus, $D(A)$ is a Lie algebra over K.

Consider two domains $W \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and let $F\colon W \to V$ be a smooth mapping. Then this mapping generates a ring homomorphism $F^*\colon C^\infty(V) \to C^\infty(W)$ which is defined by

$$(F^*(f))(b) = f(F(b)). \tag{36}$$

So every function f on V is pulled back to a smooth function on W in this way. Note that F^* is also a homomorphism of $\mathbb{R}$-algebras.

We can generalize this concept. Let us denote $A = C^\infty(V)$ and $B = C^\infty(W)$. Having a mapping $F\colon W \to V$, we can construct F^* and so, having a derivation ξ in A we can define a mapping $\overline{\xi}\colon A \to B$ such that for all $f \in A$ one has

$$\overline{\xi}(f) \stackrel{\text{def}}{=} F^*(\xi(f)).$$

One can check that $\overline{\xi}$ satisfies the Leibniz rule, that is

$$\overline{\xi}(fg) = \overline{\xi}(f)F^*(g) + F^*(f)\overline{\xi}(g)$$

for any $f, g \in A$ (action by F^* on B determines an A-algebra structure in B).

Let A be a K-algebra and P be an A-module. We can define

$$D(P) = \{f \in \mathrm{Hom}_K(A, P) \mid f(ab) = af(b) + bf(a)\, \forall a, b \in A\}. \tag{37}$$

If we have such an $f \in D(P)$ and $a \in A$, then

$$(af)(b) \stackrel{\text{def}}{=} a(f(b))$$

and it is easy to check that $af \in D(P)$, so $D(P)$ is an A-module. However it is not a Lie algebra, since there is no natural way to define a commutator.

On the other hand, if we have two A-modules P, Q, an A-module homomorphism $\varphi\colon P \to Q$ and a derivation with values in P, we can consider the composition $D(\varphi) \stackrel{\text{def}}{=} \varphi \circ f\colon A \to Q$. It is K-linear and it satisfies the same identities, so $\varphi \circ f \in D(Q)$. Then to any module P we put into correspondence the module $D(P)$. Moreover, having two homomorphisms $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$, it results that

$$D(\psi \circ \varphi) = D(\psi) \circ D(\varphi),$$

so $D(\cdot)$ a covariant functor. We say that elements of $D(P)$ are *P-valued derivation* of A.

Fix a commutative ring A which is an algebra over a ring K. Let P be an A-module and X be a derivation, that is $X \in \mathrm{Hom}_K(A, P)$ and

$$X(ab) = aX(b) + bX(a). \tag{38}$$

for any $a, b \in A$. Any element X of $D(P)$ can be multiplied by an element of A in two ways, from the left, that is

$$(aX)(b) \stackrel{\text{def}}{=} aX(b) \tag{39}$$

or from the right

$$(a^+X)(b) \stackrel{\text{def}}{=} X(ab). \tag{40}$$

Now, $(a^+X)(b) = X(ab) = aX(b) + X(a)b = (aX)(b) + X(a)b$, so we can write

$$(a^+X - aX)(b) = X(a)b$$

and this suggests a way to define a commutator

$$[a, X](b) \stackrel{\text{def}}{=} X(a)b. \tag{41}$$

Now let a be fixed; then if we have another element $\overline{a}$, we can easily check that for all b one has $[\overline{a}, [a, X]](b) = 0$, that is

$$[\overline{a}, [a, X]] = 0. \tag{42}$$

Let now $X : A \to P$ be such that $[\overline{a}, [a, X]] = 0$ for all $a, \overline{a} \in A$. Then the difference $X - X(1)$ satisfies (38) provided X itself satisfies (42). Then, if we put $X - X(1) = \overline{X}$, then $X = \overline{X} + f$, where $\overline{X}$ is a derivation and $f = X(1)$ is a constant. We conclude that any element satisfying (42) is the sum of a derivation and a constant. Standard first order differential operators (on $\mathbb{R}$) are of this form,

$$\Delta = g\frac{\partial}{\partial x} + f.$$

Let A be a K-algebra and P, Q be A-modules. Then we can consider the bimodule $\mathrm{Hom}_K(P, Q)$. Let $\Delta \in \mathrm{Hom}_K(P, Q)$ and consider

$$[a, \Delta] = a^+\Delta - \Delta a = \Delta a - a\Delta.$$

This difference belongs to $\mathrm{Hom}_K(P, Q)$ so we can take $\overline{a} \in A$ and consider $[\overline{a}, [a, \Delta]]$. We say that Δ is a *first order differential operator* acting from P to Q if for all $a, \overline{a} \in A$ one has

$$[\overline{a}, [a, \Delta]] = 0.$$

Consider two A-modules P, Q and a homomorphism $\Delta \in \mathrm{Hom}_K(P, Q)$. We say that Δ is a *differential operator* of order $\leq k$ if for all $a_0, a_1, \ldots, a_k \in A$ it results

$$[a_0, [a_1, \ldots [a_k, \Delta] \ldots]] = 0.$$

Let us denote the set of all such maps as $\widetilde{\mathrm{Diff}}_k(P,Q)$, $k \in \mathbb{N}$. What are the structures living in this set? If we have two differential operators $\Delta, \nabla \in \widetilde{\mathrm{Diff}}_k(P,Q)$, then it is easy to check that $\Delta+\nabla \in \widetilde{\mathrm{Diff}}_k(P,Q)$, so $\widetilde{\mathrm{Diff}}_k(P,Q)$ is an Abelian group. Moreover, given $a \in A$, it results that

$$[a_k, a\Delta] = a[a_k, \Delta]$$

and so $\widetilde{\mathrm{Diff}}_k(P,Q)$ is also an A-module, which will be denoted by $\mathrm{Diff}_k(P,Q)$. If we consider the right multiplication, we get the second module structure denoted by $\mathrm{Diff}_k^{+}(P,Q)$, and finally $\mathrm{Diff}_k^{(+)}(P,Q)$ will denote the bimodule structure.

Consider three modules and two linear maps $P \xrightarrow{\Delta} Q \xrightarrow{\nabla} R$. If you take $a \in A$ and consider $[a, \nabla \circ \Delta]$, you will have

$$[a, \nabla \circ \Delta] = [a, \nabla] \circ \Delta + \nabla \circ [a, \Delta].$$

Let Δ be a K-linear map. You can prove by induction or directly from the definition that

$$\underbrace{[a,[a,\dots[a,\Delta]\dots]]}_{k+1 \text{ times}} = \sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} a^i \nabla a^{k+1-i}.$$

Consider $f \in C^\infty(\mathbb{R})$ and $x_0 \in \mathbb{R}$. Then that f can be represented in the form

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \cdots + \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + (x-x_0)^{k+1} g(x)$$

with $g \in C^\infty(\mathbb{R})$. Now let $\Delta \colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ satisfy the algebraic definition of a differential operator of order $\leq k$ and apply Δ to the function f. Because of linearity, we have

$$\begin{aligned}\Delta(f) = f(x_0)\Delta(1) + f'(x_0)\Delta(x-x_0) + \cdots \\ \cdots + \frac{f^{(k)}(x_0)}{k!}\Delta\left[(x-x_0)^k\right] + \Delta\left[(x-x_0)^{k+1}g(x)\right].\end{aligned}$$

If we compute this expression at $x = x_0$ it suffices to notice that the last term in the right-hand side vanishes and so the value of $\Delta(f)$ at the point x_0 is determined by the set of coefficients $\Delta_0 = \Delta(1)$, $\Delta_1 = \Delta(x-x_0), \dots, \Delta_k = \Delta(x-x_0)^k$ and

$$\Delta = \Delta_0 + \Delta_1 \frac{d}{dx} + \cdots + \frac{\Delta_k}{k!}\frac{d^k}{dx^k}.$$

A similar proof is valid for arbitrary differential operators $\Delta \colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. So from the algebraic definition in the case of smooth functions we obtain the usual expression and, vice versa, if we have

$$\Delta = \sum \Delta_\sigma \frac{\partial^\sigma}{\partial x^\sigma},$$

where $\Delta_\sigma \in C^\infty(\mathbb{R})$, this Δ satisfies our algebraic definition of a differential operator. Thus we can think of differential operators as of purely algebraic objects.

Consider the bimodule $\mathrm{Diff}_k^{(+)}(P,Q)$. If we have a differential operator of order k, i.e.,

$$[a_0,[a_1,\ldots[a_k,\Delta]\ldots]]=0,$$

then we can add an arbitrary a' and obtain

$$[a',[a_0,[a_1,\ldots[a_k,\Delta]\ldots]]]=0,$$

so Δ is a differential operator of order $k+1$ as well and consequently

$$\mathrm{Diff}_k^{(+)}(P,Q)\hookrightarrow \mathrm{Diff}_{k+1}^{(+)}(P,Q).$$

Operators of zero order are those which satisfy $[a_0,\Delta]=0$ for all $a_0\in A$, that is $a_0\Delta=\Delta a_0$ and so elements of $\mathrm{Diff}_0(P,Q)$ are just A-homomorphisms and the two module structures coincide in this case.

Having the sequence of embeddings

$$\mathrm{Hom}_A(P,Q)=\mathrm{Diff}_0(P,Q)\hookrightarrow\cdots\hookrightarrow \mathrm{Diff}_k^{(+)}(P,Q)\hookrightarrow \mathrm{Diff}_{k+1}^{(+)}(P,Q)\hookrightarrow\cdots,$$

we can define a new object

$$\mathrm{Diff}_*^{(+)}(P,Q)\stackrel{\mathrm{def}}{=}\bigcup_{k\geq 0}\mathrm{Diff}_k^{(+)}(P,Q).$$

It is a bimodule consisting of all differential operators of any finite order.

Let us fix a module P. Then, having a homomorphism $f\colon Q\to Q'$ and an operator $\Delta\in \mathrm{Diff}_k^{(+)}(P,Q)$, we can consider the composition $P\xrightarrow{\Delta}Q\xrightarrow{f}Q'$ and associate to any differential operator $\Delta\in\mathrm{Diff}_k^{(+)}(P,Q)$ the operator $f\circ\Delta\in\mathrm{Diff}_k^{(+)}(P,Q')$. This map, which depends of P and on f, will be denoted by

$$\mathrm{Diff}_k^{(+)}(P,f)\colon \Delta\in \mathrm{Diff}_k^{(+)}(P,Q)\mapsto f\circ\Delta\in\mathrm{Diff}_k^{(+)}(P,Q'),$$

and it defines a functor from the category of A-modules to the category of A-bimodules.

Fix Q now. Then, given a homomorphism $f\colon P\to P'$, to any operator $\Delta\in\mathrm{Diff}_k^{(+)}(P',Q)$ we can associate the operator $P\xrightarrow{f}P'\xrightarrow{\Delta}Q$. If $P\xrightarrow{f}P'\xrightarrow{g}P''$ are two homomorphisms, then

$$\mathrm{Diff}_k^{(+)}(f\circ g,Q)=\mathrm{Diff}_k^{(+)}(g,Q)\circ\mathrm{Diff}_k^{(+)}(f,Q)$$

and so it is a contravariant functor.

Consider the modules $\mathrm{Diff}_k(P,Q)$ and $\mathrm{Diff}_k^{+}(P,Q)$. They are isomorphic as Abelian groups, but they are different as far as module structures are concerned. Then we can consider the identical mappings

$$\mathrm{Diff}_k(P,Q)\xrightarrow{\mathrm{id}^+}\mathrm{Diff}_k^{+}(P,Q)$$

and

$$\mathrm{Diff}_k(P,Q) \xleftarrow{\mathrm{id}_+} \mathrm{Diff}_k^+(P,Q).$$

The mappings id_+ and id^+ are differential operators of order $\le k$, but there compositions $\mathrm{id}_+ \circ \mathrm{id}^+$ and $\mathrm{id}^+ \circ \mathrm{id}_+$ are operators of zero order (identical homomorphisms).

What happens in general if we consider the compositions of differential operators $P \xrightarrow{\Delta} Q \xrightarrow{\nabla} R$ with $\Delta \in \mathrm{Diff}_k^{(+)}(P,Q)$ and $\nabla \in \mathrm{Diff}_l^{(+)}(Q,R)$? Generally speaking, $\nabla \circ \Delta \in \mathrm{Hom}_K(P,R)$. Let us prove that it is a differential operator. To this end, take $a \in A$ and consider the identity

$$[a, \nabla \circ \Delta] = [a, \nabla] \circ \Delta + \nabla \circ [a, \Delta].$$

Hence, for $m = k + l$ one has

$$[a_0, [a_1, \dots [a_m, \nabla \circ \Delta] \dots]] = \sum [a_{i_0}, [a_{i_1}, \dots [a_{i_s}, \nabla] \dots]] \circ [a_{j_0}, [a_{j_1}, \dots [a_{j_t}, \Delta] \dots]].$$

The whole number of a's is $k+l+1$; then either in the first term the number of commutators is $\ge l$ or in the second one it is $\ge k$. In any case, one of the two components vanishes and hence $\nabla \circ \Delta \in \mathrm{Diff}_{k+l}^{(+)}(P,R)$, just as in the standard case.

Consider a particular case $P = Q$. Taking the composition of two differential operators from P to itself, we obtain a new differential operator from P to itself. Composition is obviously associative so $\mathrm{Diff}_*^{(+)}(P,P)$ is an associative algebra with respect to composition (but not commutative!).

To deal with differential operators from A (as A-module over itself) to an A-module Q, we shall use the notation

$$\mathrm{Diff}_k^{(+)}(Q) \stackrel{\mathrm{def}}{=} \mathrm{Diff}_k^{(+)}(A,Q).$$

Derivations with values in Q are first order differential operators so we have $D(Q) \hookrightarrow \mathrm{Diff}_1^{(+)}(Q)$. Note that the embedding $D(Q) \hookrightarrow \mathrm{Diff}_1^+(Q)$ ia a first order monomorphic differential operator, while $D(Q) \hookrightarrow \mathrm{Diff}_1(Q)$ is an A-module monomorphism.

Let us consider the sequence $0 \to D(Q) \to \mathrm{Diff}_1(Q)$. If we take the quotient of $\mathrm{Diff}_1(Q)$ by the image, we obtain Q (we are killing the differential part), so we have the exact sequence of homomorphisms

$$0 \to D(Q) \to \mathrm{Diff}_1(Q) \to Q \to 0. \tag{43}$$

On the other hand, $\mathrm{Diff}_0^{(+)}(Q) \hookrightarrow \mathrm{Diff}_1^{(+)}(Q)$ and, since we established that $\mathrm{Diff}_0^{(+)}(Q) \equiv \mathrm{Diff}_0^{(+)}(A,Q) = \mathrm{Hom}_A(A,Q) = Q$, we have the exact sequence of differential operators

$$0 \to Q \to \mathrm{Diff}_1^+(Q) \to D(Q) \to 0. \tag{44}$$

From these two sequences, (43) and (44), we get two important theories: *Spencer cohomology* and *algebraic model of Hamiltonian mechanics*.

Lecture 4

In the previous lecture we established that

$$Q \mapsto \mathrm{Diff}_l^{(+)}(P, Q), \tag{45}$$

$$P \mapsto \mathrm{Diff}_l^{(+)}(P, Q) \tag{46}$$

are covariant and contravariant functors from $\mathcal{M}od(A)$ to the category of A-bimodules.

Let us fix the second module structure and consider the mapping

$$\mathit{Д}_l \colon \mathrm{Diff}_l^{+} Q \to Q$$

defined by

$$\mathit{Д}_l(\Delta) \stackrel{\mathrm{def}}{=} \Delta(1), \quad \Delta \in \mathrm{Diff}_l^{+} Q. \tag{47}$$

Then $\mathit{Д}_l$ is a differential operator of order l (note that in the first module structure it is a homomorphism). In fact, for any $a \in A$ one has $\mathit{Д}_l(a^{+}\Delta) = \Delta(a)$.

Consider an arbitrary differential operator of order $\Delta \in \mathrm{Diff}_l^{+}(P, Q)$ and let us establish a correspondence between this module and the module $\mathrm{Hom}_A(P, \mathrm{Diff}_l^{+} Q)$. Let us construct a mapping $\varphi \colon A \to Q$ by setting

$$(\varphi(\Delta)(p))(a) = \Delta(ap), \quad a \in A, p \in P.$$

One can easily check that it is a homomorphism and, moreover, it is an isomorphism. In fact if $f \in \mathrm{Hom}_A(P, \mathrm{Diff}_l^{+} Q)$ then the inverse mapping is $f \mapsto \mathit{Д}_l \circ f$. This means that for any differential operator Δ of order l acting from P to Q, there exists a homomorphism $\varphi(\Delta)$ such that the diagram

$$\begin{tikzcd}
P \arrow[rr, "\Delta"] \arrow[dr, "\varphi(\Delta)"'] & & Q \\
& \mathrm{Diff}_l^{+} Q \arrow[ur, "\mathit{Д}_l"'] &
\end{tikzcd}$$

is commutative.

In fact, we established that the functors

$$P \mapsto \mathrm{Diff}_l^{+}(P, Q) \text{ and } P \mapsto \mathrm{Hom}_A(P, \mathrm{Diff}_l^{+} Q)$$

act in the same way, they are identical to each other. In category theory this means that the functor $\mathrm{Diff}_l^+(\cdot, Q)$ is *representable*. The module $\mathrm{Diff}_l^+ Q$ is called the *representative object* for this functor.

Consider now the operator $\Delta_l\colon \mathrm{Diff}_l^+(Q) \to Q$. Then we can also consider the composition

$$\mathrm{Diff}_s^+(\mathrm{Diff}_l^+(Q)) \xrightarrow{\Delta_s} \mathrm{Diff}_l^+(Q) \xrightarrow{\Delta_l} Q,$$

which is a differential operator of order $l+s$. Hence, by the universal property of the operator $\mathrm{Diff}_{l+s}^+ Q \xrightarrow{\Delta_{l+s}} Q$ there exists a unique homomorphism $\varphi(\Delta_l \circ \Delta_s) \overset{\mathrm{def}}{=} c_{l,s}$ such that the following diagram

$$\begin{tikzcd}
\mathrm{Diff}_s^+(\mathrm{Diff}_l^+ Q) \arrow[r, "\Delta_s"] \arrow[d, "c_{l,s}"'] & \mathrm{Diff}_l^+ Q \arrow[d, "\Delta_l"] \\
\mathrm{Diff}_{s+l}^+ Q \arrow[r, "\Delta_{s+l}"] & Q
\end{tikzcd}$$

is commutative. The mapping $c_{l,s}$ is called the *universal composition*. Universal composition is *associative* in the sense that the diagram

$$\begin{tikzcd}
\mathrm{Diff}_l^+(\mathrm{Diff}_s^+(\mathrm{Diff}_m^+ Q)) \arrow[r, "c_{l,s}"] \arrow[d, "\mathrm{Diff}_l^+(c_{s,m})"'] & \mathrm{Diff}_{l+s}^+(\mathrm{Diff}_m^+ Q) \arrow[d, "c_{l+s,m}"] \\
\mathrm{Diff}_l^+(\mathrm{Diff}_{s+m}^+ Q) \arrow[r, "c_{l,s+m}"] & \mathrm{Diff}_{s+l+m}^+ Q
\end{tikzcd}$$

is commutative.

Note also that the universal composition is related to any module Q and, strictly speaking, we must write $c_{l,s} = c_{l,s}(Q)$. Let $\varphi\colon Q \to Q'$ be a homomorphism. Then the diagram

$$\begin{tikzcd}
\mathrm{Diff}_l^+(\mathrm{Diff}_s^+ Q) \arrow[r, "c_{l,s}(Q)"] \arrow[d, "\mathrm{Diff}_l^+(c_{s,m})"'] & \mathrm{Diff}_{l+s}^+ Q \arrow[d, "\mathrm{Diff}_{l+s}^+(\varphi)"] \\
\mathrm{Diff}_l^+(\mathrm{Diff}_s^+ \varphi) \arrow[r, "c_{l,s}(Q')"] & \mathrm{Diff}_{l+s}^+ Q'
\end{tikzcd}$$

is commutative. It means that $c_{l,s}\colon \mathrm{Diff}_l^+ \mathrm{Diff}_s^+ \Rightarrow \mathrm{Diff}_{l+s}^+$ is a natural transformation of functors.

Now consider the functor

$$Q \mapsto \mathrm{Diff}_l(P, Q)$$

for a fixed P and let us try to find the representative object for it. We need to find something, $?(P)$, which possesses the property

$$\mathrm{Diff}_l(P,Q) \simeq \mathrm{Hom}_A(?(P),Q).$$

Consider the tensor product $A \otimes_K P$ and let us introduce two A-module structures into this object. Namely, we set

$$b(a \otimes p) = (ba) \otimes p, \qquad b_+(a \otimes p) = a \otimes (bp).$$

In such a way we obtain bimodule since, given b, b', we have

$$b(b'_+(a \otimes p)) = b'_+(b(a \otimes p)).$$

Let us measure the difference between these two multiplications. If $\theta \in A \otimes_K P$, we set

$$[a, \theta] \stackrel{\mathrm{def}}{=} a\theta - a_+\theta.$$

Taking arbitrary elements $a_0, a_1, \dots a_l \in A$, we can consider the iterated commutator

$$[a_0, [a_1, \dots [a_l, \theta] \dots]] \in A \otimes_K P$$

and generate, for a fixed l, the submodule

$$\mu_l(P) \stackrel{\mathrm{def}}{=} \{\, [a_0, [a_1, \dots [a_l, \theta]]] \,\}$$

taking all $a_i \in A$ and $\theta \in A \otimes_K P$. The quotient module

$$\mathcal{J}^l(P) = A \otimes_K P/\mu_l(P).$$

is called the *module of l-jets.*

Let us take an element $a \otimes p \in A \otimes_K P$ and denote its coset in $\mathcal{J}^l(P)$ by $[a \otimes p]$. Having an element $p \in P$, we can consider the element $1 \otimes p$ and the coset $[1 \otimes p]$ thus defining the mapping

$$j_l \colon P \to \mathcal{J}^l(P), \qquad p \mapsto j_l(p) \stackrel{\mathrm{def}}{=} [1 \otimes p].$$

The element $j_l(p)$ is called the *l-jet* of p.

The set $\mathcal{J}^l(P)$ is an A-module with respect to multiplication $b[a \otimes p] \stackrel{\mathrm{def}}{=} [ba \otimes p]$. Note that $\mathcal{J}^0(P) = P$. It can be easily proved that j_l is a differential operator of order l.

THEOREM 10. *Let $\Delta \colon P \to Q$ be a differential operator of order $\leq l$. Then there exists a uniquely defined homomorphism $\psi(\Delta)$ such that the diagram*

$$\begin{tikzcd} P \arrow[rr, "\Delta"] \arrow[dr, "j_l"'] & & Q \\ & J^l(P) \arrow[ur, "\psi(\Delta)"'] & \end{tikzcd}$$

is commutative.

From this theorem we obtain the isomorphism

$$\mathrm{Diff}_l(P,Q) \simeq \mathrm{Hom}_A(\mathcal{J}^l(P),Q)$$

which means that the module $\mathcal{J}^l(P)$ is the representative object for the functor $\mathrm{Diff}_l(P,\cdot)$.

Consider the operator j_l. Being of order $\le l$, it is also of order $\le l+1$ and consequently we have the commutative diagram

$$\begin{tikzcd}
\mathcal{J}^l(P) & \mathcal{J}^{l+1}(P) \arrow[l, "\nu_{l+1,l}"'] \\
P \arrow[u, "j_l"] \arrow[ur, "j_{l+1}"'] &
\end{tikzcd}$$

The mapping $\nu_{l+1,l}$ is dual to the embedding $\mathrm{Diff}_l(P,Q) \hookrightarrow \mathrm{Diff}_{l+1}(P,Q)$ and, in fact, is an epimorphism. So, we have the sequence of mappings

$$P \equiv \mathcal{J}^0(P) \leftarrow \mathcal{J}^1(P) \leftarrow \cdots \leftarrow \mathcal{J}^l(P) \leftarrow \mathcal{J}^{l+1}(P) \leftarrow \cdots \tag{48}$$

The kernel of the projection $\nu_{l+1,l}$ coincides with the quotient $\mu_l(P)/\mu_{l+1}(P)$ and thus we have the exact short sequence of modules

$$0 \to \mu_l(P)/\mu_{l+1}(P) \to \mathcal{J}^{l+1}(P) \xrightarrow{\nu_{l+1,l}} \mathcal{J}^l(P) \to 0.$$

Now consider (48) and a sequence of elements θ_l, $l=0,1,\cdots$, such that $\theta_l \in \mathcal{J}^l(P)$ and $\nu_{l+1,l}(\theta_{l+1}) = \theta_l$. These sequences may be added to each other and multiplied by elements $a \in A$ component-wise. So they form an A-module which is denoted by $\mathcal{J}^\infty(P)$ and called the *module of infinite jets*. If we know the sequence as a whole, we, in particular, know its l-th term and thus get the mapping

$$\nu_{\infty,l} \colon \mathcal{J}^\infty(P) \to \mathcal{J}^l(P).$$

From our definition it follows also that we can construct the diagram

$$\begin{tikzcd}
J^\infty(P) \arrow[d, "\nu_{\infty,l}"'] \arrow[dr, "\nu_{\infty,l+1}"] & \\
J^l(P) & J^{l+1}(P) \arrow[l, "\nu_{l+1,l}"']
\end{tikzcd}$$

and it is commutative.

In particular, if $p \in P$, then the sequence $\{j_l(p)\}_{l\ge 0}$, is a element of $\mathcal{J}^\infty(P)$. It is denoted by $j_\infty(p)$ and is called th *infinite jet* of the element p. Obviously, $\nu_{\infty,l}(j_\infty(p)) = j_l(p)$. Formally speaking, the mapping $j_\infty \colon P \to \mathcal{J}^\infty(P)$ is not a differential operator, that is there always exist a sequence $a_0,\ldots,a_l$ of an arbitrary high length such that

$$[a_0,[a_1,\ldots[a_l,j_\infty]\ldots]] \neq 0,$$

but if we restrict ourselves to a finite part, it becomes a differential operator.

The correspondence $P \mapsto \mathcal{J}^l(P)$ is a covariant functor. In fact, if we consider a homomorphism $f\colon P \to Q$, then the composition $f \circ j_l$ is a differential operator of order l, so there exists a unique homomorphism from $\mathcal{J}^l(P)$ to $\mathcal{J}^l(Q)$, denoted by $J^l(f)$, such that the diagram

$$\begin{tikzcd} P \arrow[r, "j_l"] \arrow[d, "f"'] & \mathcal{J}^l(P) \arrow[d, "\mathcal{J}^l(f)"] \\ Q \arrow[r, "j_l"] & \mathcal{J}^l(Q) \end{tikzcd}$$

is commutative. Due to uniqueness, we have

$$\mathcal{J}^l(f \circ g) = \mathcal{J}^l(f) \circ \mathcal{J}^l(g)$$

whenever it makes sense.

Now consider a module P and for any l the operator $j_l\colon P \to \mathcal{J}^l(P)$. Since $\mathcal{J}^l(P)$ is an A-module as well, we can take the composition

$$P \xrightarrow{j_l} \mathcal{J}^l(P) \xrightarrow{j_s} \mathcal{J}^s(\mathcal{J}^l(P)),$$

which is a differential operator of order $l+s$. By the universal property, we can construct the commutative diagram

$$\begin{tikzcd} P \arrow[r, "j_{l+s}"] \arrow[d, "j_l"'] & \mathcal{J}^{l+s}(P) \arrow[d, "c^{l,s}"] \\ \mathcal{J}^l(P) \arrow[r, "j_s"] & \mathcal{J}^s(\mathcal{J}^l(P)) \end{tikzcd}$$

and this mapping $c^{l,s}$ is dual to the universal composition map $c_{l,s}$. The homomorphisms $c^{l,s}$ and $c_{l,s}$ are adjoint in categorical sense and $c^{l,s}$ is called the *universal co-composition*. This operation possesses the *co-associativity* property expressed by commutativity of the diagram

$$\begin{tikzcd} \mathcal{J}^{l+s+m}(P) \arrow[r, "c^{s,l+m}"] \arrow[d, "c^{l+s,m}"'] & \mathcal{J}^s(\mathcal{J}^{l+m}(P)) \arrow[d, "\mathcal{J}^s(c^{l,m})"] \\ \mathcal{J}^{s+l}(\mathcal{J}^m(P)) \arrow[r, "c^{s+l,m}"] & \mathcal{J}^s(\mathcal{J}^l(\mathcal{J}^m(P))) \end{tikzcd}$$

Consider now the case $l = 1$ and take the module $\mathcal{J}^1(A)$. Then we have the mapping

$$i_1\colon A \to \mathcal{J}^1(A)$$

defined by $i_1(a) \stackrel{\text{def}}{=} [a \otimes 1]$. It is a homomorphism, so its image is a submodule. Let us set

$$\Lambda^1(A) \stackrel{\text{def}}{=} \mathcal{J}^1(A)/\operatorname{im} i_1. \tag{49}$$

Then we obtain the exact sequence of modules

$$0 \to A \xrightarrow{i_1} \mathcal{J}^1(A) \xrightarrow{\pi} \Lambda^1(A) \to 0,$$

where π is the natural projection. The module $\Lambda^1(A)$ is called the *module of* 1*-forms* of algebra A. The composition

$$d \equiv \pi \circ j_1 \colon A \to \Lambda^1(A)$$

is called the first *de Rham differential* and is a derivation of A with values in $\Lambda^1(A)$. It is easy to check that $\Lambda^1(A)$ is generated by the elements d_a, $a \in A$, with the relations

$$d_{ab} = ad_b + bd_a, \qquad d_{\alpha a+\beta b} = \alpha d_a + \beta d_b,$$

where $a, b \in A$, $\alpha, \beta \in K$.

We continue to deal with differential forms. An immediate consequence of Theorem 10 is

THEOREM 11. *For any derivation $X\colon A \to P$ there exists a unique homomorphism $\psi(X)\colon \Lambda^1(A)$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "X"'] & \Lambda^1(A) \arrow[dl, "\psi(X)"] \\ P & \end{tikzcd}$$

is commutative.

Hence, we have the isomorphism

$$\operatorname{Hom}_A(\Lambda^1(A), P) = D(P),$$

and the module of 1-forms is the representative object of the covariant functor $D(\cdot)$.

Now let us define the module $\Lambda^l(A)$ of differential l-forms as the l-th exterior power of $\Lambda^1(A)$ (see Example 11), that is

$$\Lambda^l(A) = \bigwedge^l \Lambda^1(A)$$

(when A is fixed, we shall skip it as argument and use a simpler notation $\Lambda^l = \Lambda^l(A)$). Thus, we have a series of modules

$$\Lambda^0 = A,\ \Lambda^1,\ \Lambda^2, \dots, \Lambda^l, \dots$$

The module Λ^l is generated by the elements $a_0, da_1 \wedge da_2 \wedge \cdots \wedge da_l$ and we set by definition

$$d(a_0\, da_1 \wedge da_2 \wedge \cdots \wedge da_l) = da_0 \wedge da_1 \wedge da_2 \wedge \cdots \wedge da_l. \qquad (50)$$

We take (50) for a definition of d and thus obtain the sequence of first order differential operators

$$0 \to A \xrightarrow{d} \Lambda^1 \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^{l-1} \xrightarrow{d} \Lambda^l \xrightarrow{d} \cdots \qquad (51)$$

We call the mapping d the *de Rham differentials* and the sequence (51) is called the *de Rham complex* of the algebra A.

REMARK 2. This is a "bad" definition of the de Rham complex. A "good" one refers to rather complicated categorial construction, which are beyond the scope of these lectures.

The basic properties of the de Rham differential are as follows:

1. As it was already mentioned, it is a first order differential operator.
2. It differentiates the wedge product, i.e.,

$$d(\omega \wedge \theta) = d(\omega) \wedge \theta + (-1)^l \omega \wedge d(\theta) \qquad (52)$$

   where l is the degree of ω.
3. $d \circ d = 0$

From the last property it follows that

$$\operatorname{im} d \subset \ker d$$

at all terms in the chain (48). Hence, we can introduce the K-modules

$$H^l(A) \stackrel{\text{def}}{=} \ker d / \operatorname{im} d,$$

which are called the *de Rham cohomology* of the algebra A.

REMARK 3. Note that if you have an arbitrary sequence of homomorphisms

$$P_0 \xrightarrow{\partial_0} P_1 \xrightarrow{\partial_1} \cdots \to P_{l-1} \xrightarrow{\partial_l} P_l \xrightarrow{\partial_{l+1}} \cdots$$

with the property $\partial_{l+1} \circ \partial_l = 0$, you can realize the same construction. Such sequences are called *complexes* of modules and the modules of the form $\ker \partial_{l+1} / \operatorname{im} \partial_l$ are called *homologies* (or *cohomologies*) associated to these complexes.

Now consider the composition $j_s \circ d \colon \Lambda^{l-1} \to \mathcal{J}^s(\Lambda^l)$. It is a differential operator of order $s+1$. Due to the universal property of operators j_s, we obtain the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda^{l-1} \arrow[r, "d"] \arrow[d, "j_{s+1}"'] & \Lambda^{l} \arrow[r, "d"] \arrow[d, "j_s"'] & \Lambda^{l+1} \arrow[r] \arrow[d, "j_{s-1}"'] & \cdots \\
\cdots \arrow[r] & \mathcal{J}^{s+1}(\Lambda^{l-1}) \arrow[r, "\mathcal{S}"] & \mathcal{J}^{s}(\Lambda^{l}) \arrow[r, "\mathcal{S}"] & \mathcal{J}^{s-1}(\Lambda^{l+1}) \arrow[r] & \cdots
\end{tikzcd}$$

A simple exercise is to prove that the sequence

$$0 \to A \xrightarrow{j_k} \mathcal{J}^k(A) \xrightarrow{\mathcal{S}} \mathcal{J}^{k-1}(\Lambda^1) \to \dots \mathcal{J}^1(\Lambda^{k-1}) \xrightarrow{\mathcal{S}} \Lambda^k \to 0$$

is a complex. It is called the *Spencer complex* of A and its cohomologies are called the *Spencer cohomologies*.

We have established already that the module Λ^1 is the representative object for the functor $D(\cdot)$, i.e., $D(P) = \mathrm{Hom}_A(\Lambda^1, P)$. Let us now try to understand what functors are represented by the modules Λ^l. Let us set $\mathrm{Hom}_A(\Lambda^l, P) \overset{\text{def}}{=} D_l(P)$ and start with the case $l = 2$. Thus,

$$D_2(P) = \mathrm{Hom}_A(\Lambda^2, P).$$

Note that there exists a natural homomorphism from this module to the module $\mathrm{Hom}_A(\Lambda^1, \mathrm{Hom}_A(\Lambda^1, P))$; let us denote it by η. To construct this η, we set

$$[\eta(f)](\omega)(\theta) \overset{\text{def}}{=} f(\omega \wedge \theta),$$

where $f \in \mathrm{Hom}_A(\Lambda^2, P)$, $\omega, \theta \in \Lambda^1$. Hence, $D_2(P)$ lies in $D_1(D_1(P))$. Consequently, we can take an element $\nabla \in D_2(P)$ and consider $\nabla(a)$, obtaining a new derivation lying in $D_1(P)$. Obviously,

$$\nabla(a)(b) = -\nabla(b)(a).$$

and thus we have

$$D_2(P) = \{\nabla\colon A \to D_1(P) \mid \nabla(a, b) = -\nabla(b, a)\}.$$

In a similar way, for an arbitrary l, we have

$$\mathrm{Hom}_A(\Lambda^l, P) \to \mathrm{Hom}_A\left(\Lambda^1, \underbrace{\mathrm{Hom}_A(\Lambda^{l-1}, P)}_{D_l(P)}\right),$$

which means that we have a mapping from $D_l(P)$ to $D_1(D_{l-1}(P))$. It means that any element $\nabla \in D_l(P)$ can be evaluated at $a_1, \dots, a_l \in A$ and by simple induction on l we obtain that $\nabla\colon A \times \dots \times A \to P$ lies in $D_l(P)$ if and only if

1. It is a derivation with respect to all arguments,

$$\begin{aligned}\nabla(a_1, \dots, a_{i-1}, ab, a_{i+1}, \dots, a_l) = a\nabla(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_l)\\ + b\nabla(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_l)\end{aligned}$$

2. It is skew-symmetric,
$$\nabla(a_1,\dots,a_l)=(-1)^{|\sigma|}\nabla(a_{\sigma(1)},\dots,a_{\sigma(l)}),$$
where $|\sigma|$ denotes the parity.

So we have two ways to describe the module $D_l(P)$: the first one is to identify $D_l(P)$ with $\mathrm{Hom}_A(\Lambda^l,P)$ thus obtaining the coupling $\langle\nabla,\omega\rangle\in P$; the second one is to describe $D_l(P)$ as skew-symmetric derivations of the form $A\times\dots\times A\to P$. The relation between the two interpretation is
$$\nabla(a_1,\dots,a_l)=\langle\nabla, da_1\wedge\dots\wedge da_l\rangle.$$

Let us now define inductively the *wedge product* operation
$$\wedge\colon D_l(A)\otimes D_s(P)\to D_{l+s}(P).$$
When $l=s=0$, we have $D_0(A)=A$, $D_0(P)=P$ and we set $a\wedge p=ap$. Now, by induction on $l+s$ we set for $\Delta\in D_l(A)$ and $\nabla\in D_s(P)$
$$(\Delta\wedge\nabla)(a)=\Delta\wedge\nabla(a)+(-1)^{|\nabla|}\Delta(a)\wedge\nabla. \tag{53}$$
We formulate the final result in the following

PROPOSITION 12. *Let A be a commutative K-algebra and P be an A-module. Then*:

1. *The wedge product $\wedge\colon D_l(A)\otimes D_s(P)\to D_{l+s}(P)$ is well defined by* (53).
2. *The module $D_*(A)=\bigoplus_{l\geq 0}D_l(A)$ is a graded associative commutative algebra with respect to the wedge product, i.e.,*
$$\Delta\wedge\nabla=(-1)^{|\Delta|\cdot|\nabla|}\nabla\wedge\Delta.$$
3. *The module $D_*(P)=\bigoplus_{l\geq 0}D_l(P)$ is a graded module over $D_*(A)$.*

Consequently, the correspondence $P\mapsto D_*(P)$ is a covariant functor from the category of all A-modules to the category of all graded $D_*(A)$-modules.

We shall now define a coupling between the modules $D_1(A)$ and Λ^l (the *inner product*) Namely, if $\omega\in\Lambda^l$ and $X\in D_1(A)$, then for any $\nabla\in D_{l-1}(P)$ we set
$$\langle\nabla, X\lrcorner\,\omega\rangle=(-1)^{|\nabla|}\langle X\wedge\nabla,\omega\rangle.$$
Thus, the inner product $\lrcorner$ is the adjoint to the wedge product $\wedge$ of poly-derivations. We shall also use the notation $i_X\omega$ instead of $X\lrcorner\,\omega$. The main properties of the inner product are formulated in the following

PROPOSITION 13. *For any $X,Y\in D(A)$ and $\omega,\theta\in\Lambda^*$ one has*:

1. *The mapping $i_X\colon\Lambda^l\to\Lambda^{l-1}$ is A-linear.*
2. *The action of $\lrcorner$ is anticommutative, i.e.,*
$$i_X\circ i_Y=-i_Y\circ i_X$$
(*in particular $i_X\circ i_X=0$*).
3. *For all $a\in A$ one has $i_X(da)=X(a)$.*

4. *Finally, it is a derivation of* $\Lambda^* = \bigoplus_{l\geq 0}\Lambda^l$, *i.e.,*

$$i_X(\omega\wedge\theta) = (i_X\omega)\wedge\theta + (-1)^{|\omega|}\omega\wedge i_X\theta.$$

Now, we shall write the action of the de Rham differential in terms of inner product. Introduce the notation

$$\omega(X_1,\dots,X_l) = i_{X_l}(\dots(i_{X_1}\omega)\dots)\in A.$$

Let ω be a form of degree l; then

$$\begin{aligned}(d\omega)(X_1,\dots,X_{l+1}) = &\sum_{\alpha=1}^{l+1}(-1)^{\alpha-1}X_\alpha\omega(X_1,\dots,\widehat{X}_\alpha,\dots,X_{l+1})\\ &+\sum_{\alpha<\beta}(-1)^{\alpha+\beta}\omega([X_\alpha,X_\beta],X_1,\dots,\widehat{X}_\alpha,\dots\widehat{X}_\beta,\dots,X_{l+1}),\end{aligned}$$

and it is the usual form of the de Rham differential.

Finally, we define the *Lie derivative* L_X of a form $\omega\in\Lambda^l$ with respect to a derivation $X\in D_1(A)$. Namely, we set

$$L_X(\omega) \overset{\text{def}}{=} d(L_X\omega) + L_X(d\omega) = [d, L_X]\omega,$$

It is easy to check that $L_X\colon\Lambda^l\to\Lambda^l$ satisfies the following properties:

Proposition 14. *For any* $X, Y\in D_1(A)$, $\omega,\theta\in\Lambda^*$, *and* $a\in A$ *one has*

1. $L_X\colon\Lambda^*\to\Lambda^*$ *is a* K*-linear mapping.*
2. L_X *is a derivation of* Λ^*, *i.e.,*

$$L_X(\omega\wedge\theta) = (L_X\omega)\wedge\theta + \omega\wedge(L_X\theta).$$

3. $L_X(d\omega) = d(L_X\omega)$, *i.e.,* $[L_X, d] = 0$.
4. $L_{aX}(\omega) = aL_X\omega + da\wedge L_X\omega$.
5. $L_{[X,Y]} = [L_X, L_Y]$.
6. $[L_X, i_Y] = i_{[X,Y]}$.

Lecture 5

Our last step is to construct a bridge between algebraic theory of linear differential operators and geometry of nonlinear differential equations. We start with general considerations.

Consider an abstract mechanical system. To observe properties of the system, you need the notion of observables and these observables are real functions on the manifold M of positions. But the set of initial positions is not sufficient to describe system dynamics: we must also know, for example, momenta or velocities of the points, which adds to the picture cotangent or tangent bundle of M. If particles under consideration possess additional properties (e.g., spin),we should also add new variables corresponding to these observables.

Let us begin with simplest situation. Suppose we have a smooth manifold M (which can be understood as a set of states of a physical system). What are smooth functions on this manifold? Of course, there is an analytical definition, but how can it be understood in a geometric way? A function is a smooth map from M to $\mathbb{R}$. But one can consider the Cartesian product $M \times \mathbb{R}$ and the canonical projection $\pi \colon M \times \mathbb{R} \to M$. Then to say that f is a smooth function on M is the same as to say that $f \colon M \to M \times \mathbb{R}$ is a smooth mapping satisfying

$$\pi \circ f = \mathrm{id}_M. \tag{54}$$

This definition can be generalized. Consider an additional manifold N and the Cartesian product $M \times N$. An N-valued function f is a smooth mapping from M to $M \times N$ satisfying the same identity (54). For instance if $N = \mathbb{R}^s$, we obtain the concept of smooth vector-valued functions on our manifold M, i.e., functions represented by vectors $(f_1, \dots, f_s)$, whose components are just smooth real-valued functions.

Consider now a more complicated situation. Let M be a smooth manifold. Then, by definition, it is locally diffeomorphic to $\mathbb{R}^n$. If a point moves in the neighborhood $\mathcal{U} \subset M$ and $x = (x_1, \dots, x_n)$ are local coordinates of this point in $\mathcal{U}$, the velocity is just a tangent vector $v_1\partial/\partial x_1 + \cdots + v_n\partial/\partial x_n$. So at one point all possible velocities are described by points of $\mathbb{R}^n$ with coordinates $(v_1, \dots, v_n)$. If we consider two intersecting local charts $\mathcal{U}$ and $\mathcal{V}$ and a point in their intersection, we can express the velocities in both local coordinates. Then we have the sets of all possible velocities $\mathbb{R}^n \times \mathcal{U}$ and $\mathbb{R}^n \times \mathcal{V}$ and these two objects, if $\mathcal{U} \cap \mathcal{V}$ is not void, are related to each other by the Jacobi matrix. In this way, spaces of "local velocities" in different neighborhoods are glued together to a new manifold TM called the *tangent manifold*. There is a natural projection $\pi\colon TM \to M$ and locally the projection looks as the projection $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Any smooth mapping $f\colon M \to TM$ satisfying (54) may be understood as a *field of velocities* on M.

This construction is a particular case of the following one. Consider three smooth manifolds M, O, and F let $\pi\colon O \to M$ be a surjective smooth mapping. Suppose any point $x \in M$ possesses a neighborhood $\mathcal{U}_x = \mathcal{U}$ such that there exists a diffeomorphism $\varphi = \varphi_{\mathcal{U}}\colon \pi^{-1}(\mathcal{U}_x) \to \mathcal{U}_x \times F$ satisfying $\pi\big|_{\pi^{-1}(\mathcal{U}_x)} = \mathrm{pr}_{\mathcal{U}_x} \circ \varphi$, where $\mathrm{pr}_{\mathcal{U}_x}\colon \mathcal{U}_x \times F \to \mathcal{U}_x$ is the projection to the first component. In this case we say that this π is a *locally trivial fiber bundle* over M, M is called the *base*, F the *fiber*, and O is called the *total space* of the bundle π. The set $F_x = \pi^{-1}(x)$ is called the *fiber over the point x*. From the definition it follows that for any $x \in M$ there exists a diffeomorphism

$$\varphi_x\colon F_x \to F. \tag{55}$$

EXAMPLE 33. For any two manifolds, M and F, the projection $M \times F \to M$ is a fiber bundle.

EXAMPLE 34. The Möbius band is the total space of the bundle over the circle S^1 with the fiber $\mathbb{R}$.

EXAMPLE 35. The natural projection $TM \to M$ is a fiber bundle with the fiber $\mathbb{R}^n$, $n = \dim M$.

EXAMPLE 36. The natural projection $T^*M \to M$, where T^*M is the cotangent manifold, is a fiber bundle with the fiber $\mathbb{R}^n$, $n = \dim M$.

Now fix M and consider two fiber bundles π and π'. A *morphism* of π to π' is a smooth mapping $\varphi\colon O \to O'$ such that the diagram

$$\begin{tikzcd} O \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & O' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd} \tag{56}$$

is commutative. A morphism of bundles which is a diffeomorphism is called an *isomorphism*. A bundle isomorphic the Cartesian product $M \times F$ is called *trivial*. If the mapping $\varphi$is an embedding, we say that π is a *subbundle* in π'.

Fiber bundles over a manifold M together with their morphisms form a category. To any bundle $\pi\colon O \to M$ one can put into correspondence the set smooth mappings

$$\Gamma(\pi) \overset{\text{def}}{=} \{f\colon M \to O \mid \pi \circ f = \mathrm{id}_M\}.$$

The elements of $\Gamma(\pi)$ are called *sections*. The correspondence $\pi \mapsto \Gamma(\pi)$ is a functor from the category of fiber bundles to the category of sets. In fact, if φ is a morphism of the form (56), we can set $\Gamma(\varphi)\colon f \mapsto \varphi \circ f \in \Gamma(\pi')$ for any $f \in \Gamma(\pi)$.

We can generalize definition of a morphism in a following way. Let $\pi\colon O \to M$ and $\pi'\colon O' \to M'$ be two fiber bundles and $\psi\colon M \to M'$ be a smooth mapping. Then $\varphi\colon O \to O'$ is a morphism, if the diagram

$$\begin{tikzcd} O \arrow[r, "\varphi"] \arrow[d, "\pi"'] & O' \arrow[d, "\pi'"] \\ M \arrow[r, "\psi"] & M' \end{tikzcd}$$

is commutative.

Now let us consider a smooth mapping $g\colon M' \to M$ and the subset in Cartesian product $M' \times O$, which consists of pairs defined by

$$g^*(O) = \{(x', o) \mid g(x') = \pi(o)\}$$

It is a submanifold in $M' \times O$ and can be mapped both to O and to M':

$$\begin{tikzcd} (x', o) \in g^*(O) \arrow[r, "\pi^*(g)"] \arrow[d, "g^*(\pi)"'] & O \ni o \arrow[d, "\pi"] \\ x' \in M' \arrow[r, "g"] & M \end{tikzcd}$$

By definition of the space g^*O, it is a commutative diagram and the mapping $g^*(\pi)\colon g^*(O) \to M'$ is a fiber bundle. Is called the *pullback* (or *induced bundle*). In particular, if g is an embedding, the bundle $g^*(\pi)$ is called the *restriction* of π to the submanifold M'.

When g is a fiber bundle too, the diagonal arrow is a fiber bundle as well:

$$\begin{tikzcd} g^*(O) \arrow[r] \arrow[d, "g^*(\pi)"'] \arrow[dr, "W"] & O \arrow[d, "\pi"] \\ M' \arrow[r, "g"] & M \end{tikzcd}$$

and is called the *Whitney product* of bundles g and π. Note that if F is the fiber of the bundle π and G is that of the bundle g, then the fiber of Whitney product will be $F \times G$.

In what follows, we shall deal with a special type of fiber bundle.

DEFINITION 4. A fiber bundle $\pi\colon O \to M$ is called a *vector bundle*, if

1. The fiber F is a vector space.
2. For any $\mathcal{U}, \mathcal{V} \subset M$ such that $\varphi_\mathcal{U} \colon \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ and $\varphi_\mathcal{V} \colon \pi^{-1}(\mathcal{V}) \to \mathcal{V} \times F$ the mappings
$$\varphi_\mathcal{U}\Big|_{\pi^{-1}(\mathcal{U}\cap\mathcal{V})} \circ \varphi_\mathcal{V}^{-1}\Big|_{(\mathcal{U}\cap\mathcal{V})\times F} \colon \mathcal{U} \cap \mathcal{V}) \times F \to (\mathcal{U} \cap \mathcal{V}) \times F$$
are fiber-wise linear.

We shall consider the case when F is an $\mathbb{R}$-vector space. The dimension of F as a vector space is called the *dimension* of a fiber bundle.

Let $\pi \colon O \to M$ and $\pi' \colon O' \to M$ be two vector bundles and $\varphi \colon O \to O'$ be a morphism. We say that φ is a *morphism of vector bundles* if it is fiber-wise linear, i.e., $\varphi(a + b) = \varphi(a) + \varphi(b)$ for any $a, b \in F_x$ and $x \in M$. Obviously, vector bundles over M together with vector bundle morphisms form a category which we denote by $\mathcal{V}ect(M)$.

Consider now two sections $f, g \in \Gamma(\pi)$ of a vector bundle π. Then, due to vector space structure in the fibers, we can set
$$(f + g)(x) \overset{\text{def}}{=} f(x) + g(x), \qquad x \in M.$$
Moreover, by the same reason, for any smooth function $r \in C^\infty(M)$ we set
$$(rf)(x) \overset{\text{def}}{=} r(x)f(x), \qquad x \in M.$$
Now, we see that the correspondence $\Gamma \colon \pi \mapsto \Gamma(\pi)$ is a functor from the category $\mathcal{V}ect(M)$ to the category $\mathcal{M}od(C^\infty(M))$ of $C^\infty(M)$-modules. An amazing fact is that for any finite-dimensional manifold M and finite-dimensional vector bundle π the module $\Gamma(\pi)$ is projective with finite number of generators. Moreover, any such a module over the algebra $C^\infty(M)$ can be realized as $\Gamma(\pi)$ for some vector bundle π! A detailed discussion of these matters can be found in [**9**].

Let us fix now a smooth manifold M and establish relations between calculus in the category $\mathcal{M}od(C^\infty(M))$ and analytical constructions over M needed for the geometrical theory of differential equations.

Consider two vector bundles, π and ξ, over M. Then a *linear differential operator* of order k acting from π to ξ is an element of the module $\mathrm{Diff}_k(\Gamma(\pi), \Gamma(\xi))$. Note that this definition is in full agreement with the classical one. In a similar way, π-valued derivation on M are derivations $C^\infty(M) \to \Gamma(\pi)$ and, in particular, $C^\infty(M)$-valued derivations coincide with vector fields on M, or with section of the tangent bundle TM. We shall meet no problem in carrying over the theory of functors $\mathrm{Diff}_k^{(+)}$, D_i, etc. to the geometric situation.

The problem will arise when passing to representative objects. Consider a projective module P with finite number of generators over $C^\infty(M)$. Then, as we already know, a locally trivial vector bundle π over M corresponds to P such that $\Gamma(\pi) = P$. Consider a point $x \in M$ and the ideal $\mu_x \overset{\text{def}}{=} \{f \in C^\infty(M) \mid f(x) = 0\}$. The value of an element $p \in P$, as of a section of π, at x is its coset $p(x)$ in the quotient $C^\infty(M)/\mu_x$, the latter being a finite-dimensional space over the field $C^\infty(M) = \mathbb{R}$. Projective modules over $C^\infty(M)$ are characterized by the fact that their elements are completely determined by the values $p(x)$ at all points $x \in M$. It follows from Theorem 9 and the fact that the bundles corresponding to free modules are trivial.

EXAMPLE 37. Let $M = \mathbb{R}$ and $A = C^\infty(M)$. The module $\Lambda^1(A)$ contains, for example, the element $d(e^x) - e^x\,dx$. This element vanishes at all points of $\mathbb{R}$, but is nontrivial in $\Lambda^1(A)$. It is easy to find similar examples for the modules of jets. Thus, in representative objects a sort of "ghost" elements arise, which are not observed geometrically.

This example shows that the category of projective $C^\infty(M)$-modules is not closed with respect to construction of representative objects for the basic functors of differential calculus and a straightforward approach does not lead to geometrical constructions parallel to these objects.

To kill the "ghost" similar to the one arising in Example 37, we shall apply the following procedure. For a smooth manifold M and a $C^\infty(M)$-module P, let us set $\mathfrak{M}(P) \stackrel{\text{def}}{=} \bigcap_{x\in M} \mu_x \cdot P$. Let us say that a module is *geometrical*, if $\mathfrak{M}(P) = 0$. The we have the *functor of geometrization* $\mathfrak{G}\colon P \mapsto P/\mathfrak{M}(P)$ acting from the category of all $C^\infty(M)$-modules to the category of geometrical modules over the same algebra.

PROPOSITION 15. *The functors $D_i(\cdot)$ and $\mathrm{Diff}_k(P, \cdot)$ are representable in the category of geometrical modules over $C^\infty(M)$. Moreover, for any geometrical module Q one has*

$$D_i(Q) = \mathrm{Hom}_{C^\infty(M)}(\mathfrak{G}(\Lambda^i(C^\infty(M)), Q)$$

and

$$\mathrm{Diff}_k(P, Q) = \mathrm{Hom}_{C^\infty(M)}(\mathfrak{G}(\mathcal{J}^k(P)), Q).$$

COROLLARY 16. *For any smooth manifold M one has*

$$\Lambda^i(M) = \mathfrak{G}(\Lambda^i(C^\infty(M)).$$

To formulate a result similar to Corollary 16, we need a new geometrical construction.

Let $\pi\colon E \to M$ be locally trivial vector bundle. Consider a point $\theta \in E$, $\pi(\theta) = x \in M$, and a section $f \in \Gamma(\pi)$ whose graph passes through θ: $f(x) = \theta$. Denote by $[f]^k_x$ the set of sections whose graphs are tangent to the graph of f at θ with order k. This class is called the *k-jet* of f at x. Obviously, it is completely determined by the Taylor expansion of f at x of order k. Let us choose a coordinate neighborhood $\mathcal{U} \ni x$ in such a way that the bundle π becomes trivial over $\mathcal{U}$. Let $x_1, \dots, x_n$ be local coordinates in $\mathcal{U}$ and $e_1, \dots, e_m$ be a basis in the fiber of π. Then any section is locally represented as $f = f^1 e_1 + \cdots + f^m e_m$, where $f^j = f^j(x_1, \dots, x_n)$ are smooth functions on $\mathcal{U}$. Hence, the class $[f]^k_x$ is completely determined by the values of all partial derivatives $\partial^{|\sigma|} f^j / \partial x_\sigma$ at x, $j = 1, \dots, m$, $0 \le |\sigma| \le k$. Denote the number of these partial derivatives by $N = N(n, m, k)$.

Now, let us take an atlas $\{\mathcal{U}_\alpha\}$ in M consisting of coordinate neighborhoods of the above described type. Consider the set

$$J^k(\pi) \stackrel{\text{def}}{=} \{[f]_x^k \mid x \in M,\ f \in \Gamma(\pi)\}. \tag{57}$$

It is covered by the subsets

$$\tilde{\mathcal{U}}_\alpha \stackrel{\text{def}}{=} \{[f]_x^k \mid x \in \mathcal{U}_\alpha,\ f \in \Gamma(\pi)\}. \tag{58}$$

Define coordinate functions x_i and p_σ^j in $\mathcal{U}_\alpha$ by

$$x_i([f]_x^k) = x_i, \quad p_\sigma^j([f]_x^k) = \frac{\partial^{|\sigma|} f^j}{\partial x_\sigma}(x). \tag{59}$$

Then functions (59) establish a one-to-one correspondence between $\tilde{\mathcal{U}}_\alpha$ and the space $\mathcal{U}_\alpha \times \mathbb{R}^N$.

Proposition 17. *The system of sets $\tilde{\mathcal{U}}_\alpha$ together with coordinate functions* (59) *constitute an atlas in the set $J^k(\pi)$. Thus $J^k(\pi)$ becomes a smooth manifold. Moreover, the projection $\pi_k \colon J^k(\pi) \to M$, $\pi_k([f]_x^k) \stackrel{\text{def}}{=} x$, is a smooth locally trivial vector bundle.*

Definition 5. The manifold $J^k(\pi)$ is called the *manifold of k-jets* emphmanifold of jets for the bundle π. The bundle π_k is called the *bundle of k-jets*emphbundle of jets for the bundle π.

We can now formulate a statement similar to Corollary 16:

Proposition 18. *Let M be a smooth manifold and $\pi \colon E \to M$ be a locally trivial vector bundle. Let $P = \Gamma(\pi)$. Then*

$$\Gamma(\pi_k) = \mathfrak{G}(\mathcal{J}^k(P)).$$

Manifolds of jets is a natural environment for geometrical theory of differential equations. This theory will be discussed in the next lecture course.

Exercises

Exercise 1. Let A be a ring. Prove that $0 \cdot a = a \cdot 0 = 0$ for all $a \in A$.

Exercise 2. Let A be a set and 2^E the set of all subsets of A. Two natural binary operations exist in 2^E: the union $\cup$ and the intersection $\cap$ operations.

1. Show that 2^E is not a ring with respect to these operations.
2. Prove that if we define

$$a \cdot b = a \cap b \tag{60}$$

$$a + b = (\overline{a} \cap b) \cup (a \cap \overline{b}) \tag{61}$$

where the " bar" denotes complement in E, then 2^E is a commutative unitary ring.

Exercise 3. Consider the set $\mathbb{Z}_2 = \{0, 1\}$. Check that it is a commutative ring with the following rules of summation

$$0 + 0 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0$$

and the usual ones for multiplication.

Let now E be a set. The *characteristic functions* of a subset $a \in 2^E$ is defined as

$$\chi_a(x) = \begin{cases} 1 & x \in a \\ 0 & x \notin a \end{cases}$$

Since there exists a one-to-one correspondence between 2^E and the set of characteristic functions $X(E)$, we can introduce the structure of a ring in the latter by setting

$$\chi_a + \chi_b = \chi_{a+b}, \quad \chi_a \chi_b = \chi_{a \cdot b},$$

where $a, b \in 2^E$ and sum and product of sets are defined by (60) and (61).

Exercise 4. Prove that the correspondence $0 \mapsto \chi_\emptyset$, $1 \mapsto \chi_E$ is a homomorphism of $\mathbb{Z}_2$ to $X(E)$ and thus $X(E)$ is a $\mathbb{Z}_2$-algebra.

Exercise 5. Let A and B be two rings and $f\colon A \to B$ be a ring homomorphism. Prove that $\ker f$ is a two-sided ideal in A.

Exercise 6. Let A be a ring and I be a two-side ideal of A. Consider the quotient A/I and prove that the operations

$$[a] + [b] \stackrel{\text{def}}{=} [a + b], \quad [a] \cdot [b] \stackrel{\text{def}}{=} [ab]$$

determine a well-defined ring structure in A/I. What structure will be carried by A/I, if I is a left or right ideal?

EXERCISE 7. Let $A = C^\infty(\mathbb{R})$ and $x \in \mathbb{R}$. Consider the set

$$\mu_x = \left\{ f \in A \mid f(x) = 0 \right\}.$$

Prove that μ_x is an ideal and that A/μ_x is isomorphic to $\mathbb{R}$.

EXERCISE 8. Let again $A = C^\infty(\mathbb{R})$ and $x, y \in \mathbb{R}$ be two different points. Consider the set

$$\mu_{xy} = \{ f \in A \mid f(x) = f(y) = 0 \}.$$

Prove that μ_{xy} is an ideal and that A/μ_{xy}, as an $\mathbb{R}$-vector space is isomorphic to $\mathbb{R}^2$ and describe the ring structure in terms of a basis in $\mathbb{R}^2$.

This problem has an interesting prolongation. Of course, if we set $x = y$, the result will the same as in Exercise 7. But we can act in a different way. Namely, let $d = d(x, y)$ be the distance between the points x and y. Then, as $d \to 0$, the ideal I_{xy} becomes

$$I = \left\{ f \in C^\infty(\mathbb{R}) \mid f(x) = \frac{df}{dx}(x) = 0 \right\}$$

and you can prove that A/I is still isomorphic to $\mathbb{R}^2$ as a vector space, but sum and product will be defined as follows

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$
$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_1 b_2 + a_2 b_1).$$

Now we shall need the following

DEFINITION 6. Let A be a ring and $I \subset A$ an ideal. The ideal I is called *principal*, if there exists an element $a \in A$ such that $I = \{aA\}$.

If all ideals of A are principal ideals then A is called *principal ideal domain.*

We shall denote by I_n the principal ideals of $\mathbb{Z}$ (that is ideals of the form $I_n = n\mathbb{Z}$ with $n \in \mathbb{Z}$).

EXERCISE 9. Prove that $\mathbb{Z}$ is a principal ideal domain.

EXERCISE 10. Let $p \in \mathbb{Z}$ Prove that $\mathbb{Z}_p \stackrel{\text{def}}{=} \mathbb{Z}/I_p$ is a field if and only if p is prime.

Let us, for example, consider the case $p = 3$. Then

$$\mathbb{Z}/3\mathbb{Z} = \{[0], [1], [2]\}$$

and we can write the tables of sum and multiplication:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

EXERCISE 11. Prove that $n\mathbb{Z} \subset m\mathbb{Z}$ if and only if m divides n.

For an arbitrary ring A and its ideals I and I' we can define the ideal $I + I'$ by setting

$$I + I' \stackrel{\text{def}}{=} \{a + b \mid a \in I, b \in I'\}.$$

EXERCISE 12. Consider the ideals $n\mathbb{Z}$ and $m\mathbb{Z}$ with $m, n \in \mathbb{Z}$. Then $n\mathbb{Z} + m\mathbb{Z} = l\mathbb{Z}$ for some $l \in \mathbb{Z}$. Prove that l is maximal common divisor (mcd) of m and n.

EXERCISE 13. Prove also that $n\mathbb{Z} \cap m\mathbb{Z} = s\mathbb{Z}$ with s being the minimal common multiple (mcm) of m and n.

EXERCISE 14. Let A be a field. Prove that $I \subset A$ is an ideal if and only if $I = \{0\}$ or $I = A$. Thus, fields, and they only, have no nontrivial ideals.

EXERCISE 15. Let A be a (commutative unitary) ring. Prove $I \subset A$ is a maximal ideal if and only if A/I is a field.

EXERCISE 16. Prove that an ideal $I \subset A$ is prime if and only if the quotient A/I possesses no zero divisors.

EXERCISE 17. Prove that any maximal ideal is prime.

EXERCISE 18. Prove that any ideal is contained in a maximal ideal. (*Hint*: Use the Zorn Lemma.)

EXERCISE 19. Let A be a ring and $I \subset A$ be its ideal. Consider the natural projection $\pi\colon A \to A/I$, $\pi(a) = [a]$. Prove that:

1. if $J \subset A/I$ is an ideal then $\pi^{-1}(J) \subset A$ is an ideal in A containing I;
2. if $I' \subset A$ is an ideal, then $\pi(J) \subset A/I$ is an ideal too.

EXERCISE 20. Construct an example of projective module which is not free.

EXERCISE 21. Let $x \in \mathbb{R}$ and $\mu_x \subset C^\infty(\mathbb{R})$ be the ideal of functions vanishing at x. Prove that μ_x is a projective $C^\infty(\mathbb{R})$-module. Is it true for the case $\mathbb{R}^n$?

EXERCISE 22. Let $x \in S^1$ and $\mu_x \subset C^\infty(S^1)$ be the ideal of functions vanishing at x. Prove that μ_x is a projective $C^\infty(S^1)$-module.

EXERCISE 23. Prove that P is projective if and only if the exists P' such that $P' \oplus P$ is a free module.

EXERCISE 24. Prove that P is a projective module if and only if for any A-modules Q and R an exact sequence

$$0 \to Q \xrightarrow{i} R \xrightarrow{\pi} P \to 0$$

splits, i.e., there exists a homomorphism $j\colon P \to R$ such that $\pi \circ j = \mathrm{id}_P$.

EXERCISE 25. Let

$$0 \to R \to S \to T \to 0$$

Be an exact sequence of modules. Prove that P is projective if and only if the sequence

$$0 \to \mathrm{Hom}_A(P, R) \to \mathrm{Hom}_A(P, S) \to \mathrm{Hom}_A(P, T)$$

is an exact as well.

EXERCISE 26. Consider $C^\infty(\mathbb{R})$ and vector valued functions (i.e., rows consisting of smooth functions)

$$C^\infty(\mathbb{R}) \oplus \cdots \oplus C^\infty(\mathbb{R}).$$

Compute derivations of $C^\infty(\mathbb{R})$ with values in this module.

EXERCISE 27. Let $A = C^\infty(\mathbb{R})$, $P = A \oplus A$ be the set of vector valued functions with two components. Describe first order differential operators acting from P to itself.

EXERCISE 28. Consider the cross $M = \{(x, y) \mid xy = 0\} \subset \mathbb{R}^2\}$ and let us define the algebra $A = C^\infty(M)$ by

$$C^\infty(M) \stackrel{\text{def}}{=} \{f \colon M \to \mathbb{R} \mid f = f'|_M,\ f' \in C^\infty(\mathbb{R}^2)\}$$

1. Describe all tangent vectors at the point 0.
2. Describe all derivations $A \to A$.
3. Describe $\mathrm{Diff}_*(A)$. Show that this algebra is not generated by functions $\mathrm{Diff}_0(A) = A$ and derivations $D(A)$ (contrary to the case of smooth manifolds).

EXERCISE 29. Let $A = C(\mathbb{R})$ the algebra of all continuous functions on $\mathbb{R}$. Prove that $D(A) = 0$.

EXERCISE 30. Let $A = \mathbb{R}[x]/(x^n)$. Describe $D(A)$, $\mathrm{Diff}_*(A)$.

EXERCISE 31. Let $A = \mathbb{Z}_m[x]/(x^n)$. Describe $D(A)$, $\mathrm{Diff}_*(A)$.

For any commutative K-algebra A, let us introduce the quotient modules

$$\mathrm{Smbl}_k A \stackrel{\text{def}}{=} \mathrm{Diff}_k^{(+)} A / \mathrm{Diff}_{k-1}^{(+)} A$$

and denote by $\mathrm{smbl}_k \Delta$ the coset of $\Delta \in \mathrm{Diff}_k^{(+)} A$ in $\mathrm{Smbl}_k A$. This coset is called the *symbol* of the operator Δ.

EXERCISE 32. Prove that the module structures inherited by $\mathrm{Smbl}_k A$ from $\mathrm{Diff}_k^+ A$ and from $\mathrm{Diff}_k A$ coincide.

Let $f = \mathrm{smbl}_k \Delta$ and $g = \mathrm{smbl}_l \nabla$, $\Delta \in \mathrm{Diff}_k A$, $\nabla \in \mathrm{Diff}_l A$. Let set $f \cdot g \stackrel{\text{def}}{=} \mathrm{smbl}_{k+l}(\Delta \circ \nabla)$. Then $\mathrm{Smbl}_* A = \bigoplus_{i \geq 0} \mathrm{Smbl}_i A$ becomes an algebra, which is call the *algebra of symbols* for A. Let us also set $\{f, g\} \stackrel{\text{def}}{=} \mathrm{smbl}_{k+l-1}(\Delta \circ \nabla - \nabla \circ \Delta)$.

EXERCISE 33. Prove that $\mathrm{Smbl}_* A$ is a commutative algebra with respect to the multiplication $(f, g) \mapsto f \cdot g$.

EXERCISE 34. Prove that $\mathrm{Smbl}_* A$ is a Lie K-algebra with respect to the multiplication $(f,g) \mapsto \{f,g\}$ and

$$\{f, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\}$$

for any $f,g,h \in \mathrm{Smbl}_* A$.

EXERCISE 35. Write down the action of $c_{l,s}$ in terms of elements and prove commutativity of all the diagrams above.

EXERCISE 36. Prove the associativity law for the universal composition $c_{l,s}$.

EXERCISE 37. Prove that the universal composition $c_{l,s}$ is a natural transformation of functors.

EXERCISE 38. Let $\Delta \in \mathrm{Diff}_l^{(+)}(A)$ with $A = C^\infty\mathbb{R})$. Prove in coordinate form that $\nabla = [a,\Delta]$ lies in $\mathrm{Diff}_{l-1}^+ A$.

EXERCISE 39. Prove that id_+ and id^+ are differential operators.

EXERCISE 40. Consider the operator $\mathcal{A}_l\colon \mathrm{Diff}_l^+ A \to A$ defined in (47). Check in coordinate form that it is a differential operator of order l (in algebraic form it has already been proved) for $A = C^\infty(\mathbb{R}^n)$.

EXERCISE 41. Prove the isomorphism between the modules $\mathrm{Diff}_l^+(P,Q)$ and $\mathrm{Hom}_A(P, \mathrm{Diff}_l^+ Q)$.

EXERCISE 42. Let $P = Q = \left\{\binom{f_1}{f_2} \mid f_2 \in C^\infty(\mathbb{R}) \equiv A\right\}$. Describe the homomorphism $\varphi(\Delta)\colon P \to \mathrm{Diff}^+ P$ for an arbitrary differential operator $\Delta\colon P \to P$.

EXERCISE 43. Prove that $\mathcal{J}^k(P)$ is the representative object for the functor $\mathrm{Diff}_k(P,\cdot)$ (Theorem 10).

EXERCISE 44. Prove the embeddings $\mu_{l+1} \subset \mu_l$ for the submodules $\mu_l \subset A\otimes_K P$ defining $\mathcal{J}^l(P)$.

EXERCISE 45. Prove that $\mu_l/\mu_{l+1} = \ker \nu_{l+1,l}$, where the projections $\nu_{l+1,l}\colon \mathcal{J}^{l+1}(P) \to \mathcal{J}^l(P)$ is defined by the universal property of $\mathcal{J}^l(P)$.

EXERCISE 46. Prove co-associativity of the co-composition $c^{k,l}\colon \mathcal{J}^{k+l} \to \mathcal{J}^k(\mathcal{J}^l(P))$.

EXERCISE 47. Prove that the co-composition $c^{k,l}$ is a natural transformation of functors $\mathcal{J}^{k+l}$ and $\mathcal{J}^k(\mathcal{J}^l(P))$.

EXERCISE 48. Prove that Λ^1 is the representative object for the functor $D_1(\cdot)$ (Theorem 11).

Let P be an A-module and $N \subset P$ be a *subset* in P. Introduce the notation

$$D(N) \stackrel{\mathrm{def}}{=} \{\Delta \in D(P) \mid \mathrm{im}\,\Delta \subset N\}.$$

Of course, in general, $D(N)$ does not possess an A-module structure.

Exercise 49. Prove that the modules $D_i(P)$ may be defined in the following inductive way: $D_0(P) = P$, $D_1(P) = D(P)$ and

$$D_{i+1}(P) = D(D_i(P) \subset (\mathrm{Diff}_1^+)^i(P))$$

Exercise 50. Prove the basic properties inner product (Proposition 13):

1. The mapping $i_X \colon \Lambda^l \to \Lambda^{l-1}$ is A-linear.
2. The action of $\lrcorner$ is anticommutative, i.e.,

$$i_X \circ i_Y = -i_Y \circ i_X$$

(in particular $i_X \circ i_X = 0$).
3. For all $a \in A$ one has $i_X(da) = X(a)$.
4. Finally, it is a derivation of $\Lambda^* = \bigoplus_{l \geq 0} \Lambda^l$, i.e.,

$$i_X(\omega \wedge \theta) = (i_X \omega) \wedge \theta + (-1)^{|\omega|} \omega \wedge i_X \theta.$$

Exercise 51. Prove the basic properties of the de Rham differential d:

1. It is a first order differential operator.
2. It differentiates the wedge product, i.e.,

$$d(\omega \wedge \theta) = d(\omega) \wedge \theta + (-1)^l \omega \wedge d(\theta)$$

3. $d \circ d = 0$

Exercise 52. Prove the formula

$$\begin{aligned} d\omega(X_1, \ldots, X_n) = \sum_i X_i(\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_n)) \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots \widehat{X}_j, \ldots, X_n). \end{aligned}$$

Exercise 53. Prove the basic properties of Lie derivatives (Proposition 14):

1. $L_X \colon \Lambda^* \to \Lambda^*$ is a K-linear mapping.
2. L_X is a derivation of Λ^*, i.e.,

$$L_X(\omega \wedge \theta) = (L_X \omega) \wedge \theta + \omega \wedge (L_X \theta).$$

3. $L_X(d\omega) = d(L_X \omega)$, i.e., $[L_X, d] = 0$.
4. $L_{aX}(\omega) = aL_X \omega + da \wedge L_X \omega$.
5. $L_{[X,Y]} = [L_X, L_Y]$.
6. $[L_X, i_Y] = i_{[X,Y]}$.

Exercise 54. Prove the formula

$$(L_X \omega)(X_1, \ldots, X_n) = X\omega(X_1, \ldots, X_n) + \sum_{i=1}^n \omega(X_1, \ldots, [X, X_i], \ldots, X_n).$$

A K-algebra A is called *smooth* if (a) K is an algebra over the field $\mathbb{Q}$ of rational numbers and (b) $\Lambda^1(A)$ is a projective module with finite number of generators.

Exercise 55. Prove that when A is a smooth algebra, then one has the isomorphism

$$\ker \nu_{l,l-1} = S^{\otimes k}\Lambda^1(A),$$

where $S^{\otimes k}$ denotes the k-th symmetric power of a module.

Exercise 56. Prove that when A is a smooth algebra and P is a projective module, then $\mathcal{J}^k(P)$ is projective as well.

Exercise 57. Prove that for smooth algebras the algebra $\mathrm{Diff}_* A$ is multiplicatively generated by elements of $\mathrm{Diff}_1 A$ (cf. Exercise 28).

Let A be a commutative ring. The set $\operatorname{Spec} A$ consisting of all its prime ideals is called the *spectrum* of A. Let $a \in A$. Let us define

$$U_a \stackrel{\text{def}}{=} \{I \in \operatorname{Spec} A \mid a \notin I\}$$

and take the system $\{U_a\}_{a\in A}$ for a base of open sets in $\operatorname{Spec} A$. Thus, $\operatorname{Spec} A$ becomes a topological space and the corresponding topology in $\operatorname{Spec} A$ is called the *Zarissky topology.* If $f \colon A \to B$ is an algebra homomorphism, then for any $I \in \operatorname{Spec} B$ the set $f^{-1}I \subset A$ is a prime ideal in A. Thus we obtain the mapping $f^* \colon \operatorname{Spec} B \to \operatorname{Spec} A$ which is continuous in Zarissky topology. It is easy to see that the correspondence $A \mapsto \operatorname{Spec} A$ determines a contravariant functor from the category of commutative rings to that of topological spaces.

The set $\operatorname{Spec}_m A$ of maximal ideals contains in $\operatorname{Spec} A$, but the correspondence $A \mapsto \operatorname{Spec}_m A$ is not functorial. Nevertheless, we can consider the following construction. Fix a field $\Bbbk$ and call a homomorphism $\varphi \colon A \to \Bbbk$ a $\Bbbk$-*point.* Denote the set of $\Bbbk$-points by $\operatorname{Spec}_{\Bbbk} A$. Let $a \in A$ and $U \stackrel{\text{def}}{=} \{\varphi \in \operatorname{Spec}_{\Bbbk} A \mid \varphi(a) \neq 0$. Taking $\{U_a\}_{a\in A}$ for a base of open sets, we make a topological space of $\operatorname{Spec}_{\Bbbk} A$. The correspondence $A \mapsto \operatorname{Spec}_{\Bbbk} A$ determines a functor and there exists natural continuous mappings $\operatorname{Spec}_{\Bbbk} A \to \operatorname{Spec}_m A \to \operatorname{Spec} A$ (the composition being a natural transformation of functors $\operatorname{Spec}_{\Bbbk} A$ and $\operatorname{Spec} A$!).

Exercise 58. Consider the polynomial algebra $A = \mathbb{R}[x]$. Describe $\operatorname{Spec} A$, $\operatorname{Spec}_{\mathbb{R}} A$ and $\operatorname{Spec}_{\mathbb{C}} A$.

Exercise 59. Consider the ring of periodic functions

$$A = \{f \in C^\infty(\mathbb{R}) \mid f(x) = f(x+1)\, \forall x \in \mathbb{R}\}.$$

Describe $\operatorname{Spec}_{\mathbb{R}} A$.

Exercise 60. Let

$$A = \{f \in C^\infty(\mathbb{R}^2) \mid f(x,y) = f(x,y+1)\, \forall (x,y) \in \mathbb{R}^2\}.$$

Prove that $\operatorname{Spec}_{\mathbb{R}} A \simeq S^1 \times \mathbb{R}$.

Exercise 61. Let

$$A = \{f \in C^\infty(\mathbb{R}^2) \mid f(x+1,y) = f(x,y) = f(x,y+1)\, \forall (x,y) \in \mathbb{R}^2\}.$$

Prove that $\operatorname{Spec}_{\mathbb{R}} A \simeq S^1 \times S^1$.

EXERCISE 62. Let
$$A = \{f \in C^\infty(\mathbb{R}^2) \mid f(x+1, y) = f(x, y) = f(x, -y)\, \forall (x, y) \in \mathbb{R}^2\}.$$
Prove that $\operatorname{Spec}_{\mathbb{R}} A$ is the Klein bottle.

EXERCISE 63. Let
$$A = \{f \in C^\infty(\mathbb{R}^2) \mid f(x, y) = f(x+1, -y)\, \forall (x, y) \in \mathbb{R}^2\}.$$
Prove that $\operatorname{Spec}_{\mathbb{R}} A$ is the Möbius band.

EXERCISE 64. Let
$$A = \{f \in C^\infty(\mathbb{R}3 \setminus \{0\}) \mid f(x, y, z) = f(\lambda x, \lambda y, \lambda z)\, \forall \lambda > 0\}.$$
Prove that $\operatorname{Spec}_{\mathbb{R}} A \simeq S^2$.

EXERCISE 65. Let
$$A = \{f \in C^\infty(\mathbb{R}^3 \setminus \{0\}) \mid f(x, y, z) = f(\lambda x, \lambda y, \lambda z)\, \forall \lambda \neq 0\}.$$
Prove that $\operatorname{Spec}_{\mathbb{R}} A \simeq \mathbb{RP}^2$.

EXERCISE 66. Let
$$A = C^\infty(\mathbb{R}^3)/(x^2 + y^2 + z^2).$$
Prove that $\operatorname{Spec}_{\mathbb{R}} A \simeq S^2$.

EXERCISE 67. Consider the ring of characteristic functions (see Exercise 3) and describe its spectrum and Zarissky topology.

EXERCISE 68. Consider the ring of smooth bounded functions on $\mathbb{R}^n$ and describe its spectrum and Zarissky topology.

EXERCISE 69. Prove that functions (59) on p. 38 determine a smooth manifold structure on $J^k(\pi)$ (Proposition 17).

EXERCISE 70. Prove that $\pi_k\colon J^k(\pi) \to M$ is a smooth locally trivial vector bundle.

EXERCISE 71. Let M be a smooth manifold and $\tau^*\colon T^*M \to M$ be its cotangent bundle. Prove that $\Gamma(\tau^*) = \mathfrak{G}(\Lambda^i(C^\infty(M))$, where $\mathfrak{G}$ is the geometrization functor.

EXERCISE 72. Let $\pi\colon E \to M$ be a smooth finite-dimensional vector bundle and $\pi_k\colon J^k(\pi) \to M$ be the bundle of its k-jets. Prove that $\Gamma(\pi_k) = \mathfrak{G}(\mathcal{J}^k(P))$, where $P = \Gamma(\pi)$.

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