Foreword

The idea of parallel transport along a path in a Riemannian manifold gave birth to the concept of a linear connection on M at the end of 19th century. Subsequently, it was extended to arbitrary vector bundles and much later, at the time of the Second War, to general bundles. According to the now standard approach, which is mainly due to Ch. Ehresmann, a connection in a fiber bundle is just a distribution of ‘horizontal planes’ on its total space. Duly specified to various types of fiber bundles this approach leads to connections of a particular interest, such as affine or linear. Geometrical clarity and apparent simplicity is an important advantage of Ehresmann’s approach, which, unfortunately, is well balanced by a not negligible disadvantage. Namely, it gives almost no constructive indications on the operative machinery to work with. In particular, it does not allow an immediate natural extension of the theory to some recently emerged situations of a noteworthy importance such as supermanifolds (graded commutative algebras) or secondary calculus (see [Vinogradov (2001)]). Indeed, it would be hardly possible even to imagine what is a secondary (‘quantized’) connection in terms of a distribution of horizontal planes. Moreover, in field theory one deals directly with fields which may be, or not be interpreted as sections of a vector bundle but not with the bundle as such. So, in this context a connection *must* be defined as a construction which is pertinent to the fields ‘in person’. This kind considerations and the fact that differential calculus is, in reality, an aspect of commutative algebra (see [Nestruev]) plainly indicate that a natural framework for the theory of linear connections is differential calculus in the category of modules over a (graded) commutative ground algebra. This point of view combines naturally with the idea to treat a vector bundle as a ‘fat’ manifold composed of ‘fat’ points that are its fibers. By using the term ‘fat point’ we refer to an object possessing an ‘inner structure’ whose constituents, nevertheless, can not be directly observed, *i.e.*, something like an elementary particle. In the theory of gauge fields one deals, as a matter of fact, with fat points. In this context unobservability of the constituents is formalized by means of a suitable symmetry group that produce the necessary inseparable mixture.

These and other similar considerations leads to suppose existence of a ‘fat’ analogue of differential calculus on a fat manifold well adopted to treat various questions concerning a given vector bundle(s) and, in particular, connections in it. Such an analogue positively exists and the gauge freedom is an inherent feature of it. On the other hand, connections in the context of this ‘fat’ calculus play the role of a mechanism naturally effecting interrelations among fat points.

In these notes we present some basic elements of the fat calculus and then, on its basis, develop the theory of linear connections. In a sense this text may be viewed as a translation of the classical theory of linear connections in smooth vector bundles into its native language. An extension of the domain of the theory of linear connections much beyond its traditional differential geometry frames is one of results of this translation. For instance, this way one discovers that families of vector spaces over a smooth manifold different from vector bundles can also possess connections as well as vector bundles over manifolds with singularities. Another advantage of this new language is that it simplifies noteworthy working techniques and manipulations with connections by offering simple algebraic computations as a substitute for non infrequently ponderous geometrical constructions. In addition, it makes much easier to perceive more delicate aspects of the theory. An instance of that is the notion of compatibility of two connections along a morphism of vector bundles, introduced and studied in these notes for the first time.

These notes are structured along the following lines. The introductive zeroth chapter contains an algebraic interpretation of some basic facts of differential calculus on smooth manifolds that are brought to the form allowing a direct 'fat' generalization. Materials gathered in this chapter make the subsequent exposition self-contained and accessible for graduate students.

Fat manifolds and first elements of 'fat calculus' are introduced and discussed in the 1-st chapter. A fat manifold is simply a pair composed of a smooth manifold and a vector bundle on it. This notion, synonymous by itself to that of vector bundle, acquires, nevertheless, a new meaning in the context of fat calculus. This subtle but important difference is similar to that between 'just a particle' and a charged particle. A general algebraic counterpart of fat manifolds is a pair composed of a commutative algebra and a module over it. A good deal of fat calculus can be developed in this algebraic context and we do that as much as possible. In the 1-st chapter we discuss only simplest elements of fat calculus such as fat tangent vectors, fat vector fields, *etc.*, simultaneously, with their algebraic counterparts. Other fat notions are introduced as required in the course the exposition.

A fat manifold may be viewed as the result of a 'thickening' of the underlying ordinary manifold, say, M. A natural question is whether this thickening can be extended to other geometrical structures on M. In particular, the problem of a simultaneous thickening of vector fields on M leads to discover the notion of a linear connection in the corresponding vector bundle. In chapter 2 the theory of linear connections is build on the basis of this idea. The main tools in doing that are fat differential calculus on M and its algebraic counterpart. Among other things, here we construct some exotic examples of connections already mentioned above and describe basic operations of linear algebra with connections.

More fine elements of the theory of connections are developed in the 3-rd chapter. Covariant differential, duly interpreted, is the conceptual center of our exposition here. In particular, we show that a connection can be understood as a *cd-module structure* in the graded algebra of *thickened differential forms.* This fact makes possible to introduce the concept of *compatibility* of two connections and the concept of a *connection along a fat map.* From one side, this enriches the standard theory of connections with morphisms and relative objects and, from the other side, allows to develop a more satisfactory theory of the covariant Lie derivation.

The covariant differential of a flat connection transforms the algebra of thickened form into a complex. This kind of cohomology is studied at the beginning of the concluding 4-th chapter. The main result here is the fat homotopy formula, which is surprisingly valid even for cd-modules. As a curiosity we show that the parallel translation along a curve is described naturally by the 'fat Newton-Leibniz formula'.

A cd-module associated with a connection is not, generally, a complex. Nevertheless, there are naturally related with it differential complexes furnishing connections with cohomological invariants. We interpret Maxwell's equations as dynamics of gauge equivalence classes of connections over the fat Minkowski space-time in order to illustrate importance of this aspect in the theory of connections. The theory of characteristic classes of gauge structures is the final accord of these notes. Indeed, many elements of the previously developed theory are here shown in a common action.

Linear connections appear naturally in many areas of mathematics by starting from abstract algebra and up to mathematical physics. A finite separable extension of an algebraic field is supplied canonically with a flat connection. This elementary fact is easily seen from the point of view presented in these notes. On the other hand, the cohomology of the associated with this connection de Rham like complex is an invariant of the extension and a natural question is what are these and, in particular, how to compute them. Some hints about one can extract from differential geometry where flat connection cohomology appears as the de Rham cohomology with 'twisted coefficients'. Moreover, this kind cohomology appears in some situations in (physical) field theory, *etc.* This simple example illustrates why a unified point of view on connections could be of interest and our hopes are that these notes would be useful as a reference point to the subject.